NOW WE TURN TO FUNCTIONS, ESPECIALLY LIMITS OF FUNCTIONS. CERTAINLY YOU KNOW THAT THE DERIVATIVE IS DESCRIBED IN TERMS OF A LIMIT.

INTERESTINGLY, THE DERIVATIVE WAS DEVELOPED AND USED IN THE 1600s (AS EARLY AS 1629 BY FERMAT) BUT QUESTIONS OF CONTINUITY OR EVEN THE RIGOROUS DEFINITION OF A FUNCTION DIDN'T ARISE UNTIL ABOUT 1820 OR SO.

CERTAINLY YOU ALL "KNOW" THE DEFINITION OF A FUNCTION $f: A \to B$. (DISCUSSION)

<u>DEF</u>: A <u>FUNCTION</u> f IS A PAIRING OF ELEMENTS FROM A SET A (THE DOMAIN) WITH A SET B (THE CO-DOMAIN OR RANGE) SO THAT: • FOR EACH $x \in A$, $f(x) \in B$ IS UNIQUELY SPECIFIED

BY "UNIQUELY SPECIFIED" I MEAN IF $f(x) = a \in B$ AND $f(x) = b \in B$ THEN $a = b$.

[DISCUSS "VERTICAL LINE TEST"]

COMMON ISSUES:

- FORGETTING THAT THE DOMAIN AND RANGE ARE PART OF THE DEFINITION.
- TAKING THE 18^{TH} CENTURY APPROACH THAT
  FUNCTION = FORMULA
  OR EVEN: THE ONLY FUNCTIONS ARE POLYNOMIALS, OR (MAYBE) TRIG FUNCTIONS ... AND ALWAYS SMOOTH AND CONTINUOUS ON DOMAIN.

GOAL: GIVEN $f: A \to \mathbb{R}$ (WITH $A \subseteq \mathbb{R}$),

WE WANT TO CAREFULLY DEFINE THE IDEA OF CONTINUITY WITHOUT VAGUE NOTIONS LIKE "NO JUMPS" "NO HOLES", ETC.

"SCHOOL DEFINITION": f IS CONTINUOUS IF WE CAN DRAW THE GRAPH OF f WITH NO GAPS OR JUMPS.

HW: WHATS WRONG WITH THIS?

ONE ISSUE: WHAT IS "THE GRAPH OF f"?

~~DESCR~~ (DISCUSS).

(GIVEN A FUNCTION $f: A \to \mathbb{R}$ (WITH $A \subseteq \mathbb{R}$), THE GRAPH OF f IS THE SET OF ALL POINTS $(x, f(x)) \in A \times \mathbb{R} \subseteq \mathbb{R}^2$)

IN ORDER TO ACHIEVE THE GOAL ABOVE, WE NEED TO DEFINE

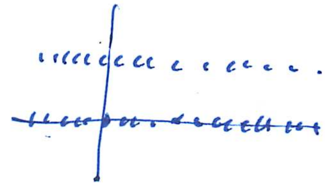
$\lim_{x \to c} f(x)$. THEN f IS CONTINUOUS AT c IF $\lim_{x \to c} f(x) = f(c)$.

- WHAT DO WE WANT CONTINUOUS TO MEAN?

(JIGGLE THE INPUT A LITTLE, THE OUTPUT CHANGES A LITTLE. JIGGLE INPUT LESS, OUTPUT CHANGES LESS.)

CONSIDER THE FOLLOWING FUNCTION, INTRODUCED BY DIRICHLET (1829):

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

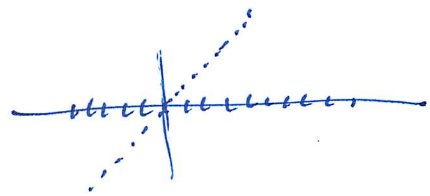


"GRAPH"

- SHOULD NOT BE CONTINUOUS ANYWHERE, SINCE ANY TINY CHANGE IN x CAN CAUSE A BIG CHANGE IN $f(x)$.

- NOW CONSIDER THE "MODIFIED DIRICHLET FUNCTION"

$$h(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$



HOW ARE THESE DIFFERENT?

~~NOTE THAT FOR $f(x)$~~,

~~FOR $f(x)$, WE~~

CONSIDER A SEQUENCE $a_n \to 0$

WE NOW FORM $\{f(a_n)\}_{n=0}^{\infty}$

CAN WE PREDICT $\lim_{n\to\infty} f(a_n)$ IF ALL WE KNOW IS $a_n \to 0$?

- NO. IT COULD BE 0, IT COULD BE 1, IT COULD NOT EXIST.

- NOW WHAT ABOUT $\lim_{n\to\infty} h(a_n)$?

  THIS IS 0, NO MATTER WHAT a_n IS, AS LONG AS $a_n \to 0$.

CONSIDER NOW THOMAE'S FUNCTION (1875)

$$t(x) = \begin{cases} 1 & \text{IF } x=0 \\ 1/n & \text{IF } x = m/n \text{ IN LEAST TERMS}, n>0 \ (x \in \mathbb{Q} \setminus \{0\}) \\ 0 & \text{IF } x \notin \mathbb{Q}. \end{cases}$$

NOTE THAT IF $c \in \mathbb{Q}$, WE CAN FIND A SEQUENCE OF IRRATIONALS $y_n \to c$, BUT $\lim_{n\to\infty} (t(y_n)) = 0 \neq t(c)$

WHAT IF $c \notin \mathbb{Q}$? SAY, $c = \sqrt{2}$.

SPOZE $x_n \to \sqrt{2}$ ~~IS IRRATIONAL~~ WITH $x_n \in \mathbb{Q}$.

SAY $x_n = 1, 1.4, 1.41, 1.414, 1.4142, \ldots$

EVENTUALLY, THE DENOMINATOR OF x_n MUST GET VERY LARGE, SO $t(x_n) \to 0 = t(\sqrt{2})$.

SO ~~WHAT~~ LET $f(x)$ BE A FUNCTION FROM $A \to \mathbb{R}$.

WE WANT TO DEFINE $\lim_{x \to c} f(x) = L$

IN A WAY THAT IS COMPATIBLE WITH

$$\lim_{n\to\infty} y_n = L.$$

AS A FIRST PASS (NOT THE BEST WAY):

DEF: LET $f: A \to \mathbb{R}$, AND LET $c \in \overline{A}$ (IE c IS A LIMIT POINT OF A)

IF ~~THEN~~ FOR EVERY SEQUENCE $\{x_n\}$ WITH

$$x_n \in A, \quad \lim x_n \to c$$

WE HAVE

$$\lim_{n \to \infty} f(x_n) \to L$$

THEN WE SAY $\lim_{x \to c} f(x) = L$

PROBLEM HERE IS "EVERY SEQUENCE $\{x_n\}$" PART.

HOW CAN WE ENSURE THAT?

$x_n \to c$ MEANS WE HAVE FOR ANY $\delta > 0$,

THERE IS K SO THAT $|x_n - c| < \delta$ FOR ALL $n > K$.

AND WE WANT TO ENSURE $f(x_n) \to L$, THAT IS FOR $\varepsilon > 0$, EVENTUALLY $|f(x_n) - L| < \varepsilon$ (IE, FOR $n > K$).

[~~WE NEED TO HAVE δ DEPEND ON ε TO DO AWAY W~~

BUT WE DON'T ACTUALLY CARE ABOUT THE ORDER OF THE x_n, SO WE CAN SWITCH FROM ACTUAL SEQUENCES TO NEIGHBORHOODS.

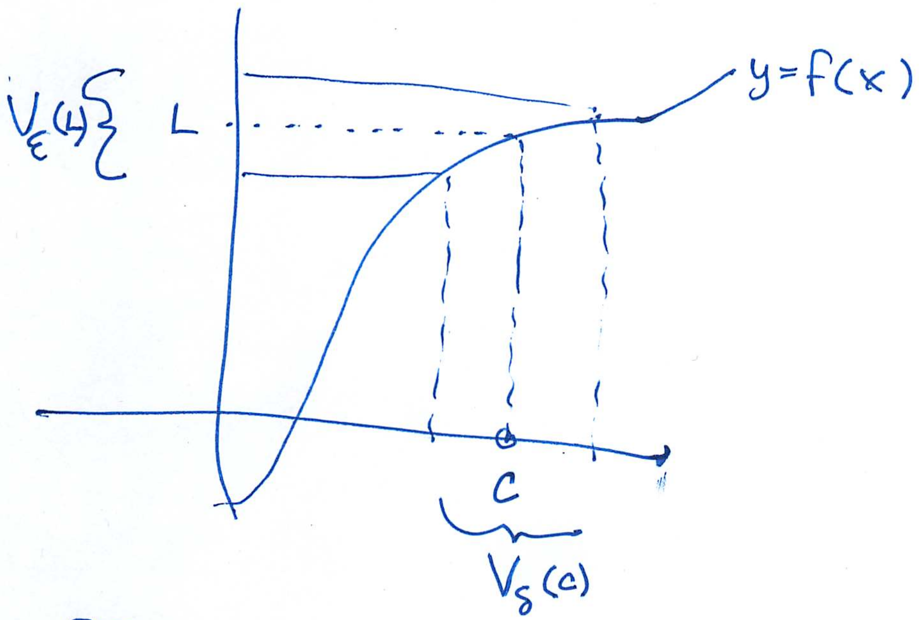
WE NEED δ TO DEPEND ON ε TO DO AWAY WITH THE K AND GET THE IMPLICATION.

THAT IS,

DEF LET $f: A \to \mathbb{R}$, $c \in \bar{A}$. THEN

$$\lim_{x \to c} f(x) = L \quad \text{WHENEVER}$$

FOR EVERY (~~[illegible]~~) ε-NEIGHBORHOOD $V_\varepsilon(L)$,
THERE IS A δ-NEIGHBORHOOD $V_\delta(c)$ SO
THAT FOR ALL $x \in V_\delta(c)$ (WITH $x \neq c$, $x \in A$)
WE HAVE $f(x) \in V_\varepsilon(L)$



NOTE $V_\varepsilon(L) = \{y \mid |L - y| < \varepsilon\}$

AND

$V_\delta(c) = \{x \mid |x - c| < \delta\}$

SO

DEF LET $f: A \to \mathbb{R}$, $c \in \bar{A}$. THEN $\lim_{x \to c} f(x) = L$ MEANS

FOR EVERY $\varepsilon > 0$, THERE IS A δ SO THAT

$0 < |x - c| < \delta$ (WITH $x \in A$)

$\Rightarrow |f(x) - L| < \varepsilon$

EASY EXAMPLE!

LET $f(x) = 2x+1$. LET'S SHOW $\lim_{x \to 3} f(x) = 7$.

TAKE $\varepsilon > 0$. WE NEED TO FIND $\delta > 0$ SO THAT

$$0 < |x-3| < \delta \Rightarrow |f(x)-7| < \varepsilon.$$

BUT $|f(x)-7| = |(2x+1)-7| = |2x-6| = 2|x-3|$.

SO IF WE TAKE $\delta \leq \varepsilon/2$, IT SHOULD WORK:

FIX $\varepsilon > 0$: IF $0 < |x-3| < \delta \leq \varepsilon/2 \Rightarrow 2|x-3| < \varepsilon$

ie $|f(x)-7| < \varepsilon$ $\boxtimes$.

I DOUBT
WE WILL GET
THIS
FAR....