LAST TIME:

- A SET $\mathcal{O} \subseteq \mathbb{R}$ IS OPEN IF FOR EVERY $a \in \mathcal{O}$, THERE IS AN ε-NEIGHBORHOOD $V_\varepsilon(a) \subseteq \mathcal{O}$.
- $x \in A$ IS A LIMIT POINT OF A IF EVERY ε-NEIGBORHOOD OF x IS CONTAINED IN A
- A SET $E \subseteq \mathbb{R}$ IS CLOSED IF IT CONTAINS ALL OF ITS LIMIT POINTS.
- A SET $K \subseteq \mathbb{R}$ IS COMPACT IF EVERY SEQUENCE OF POINTS IN K HAS A CONVERGENT SUBSEQUENCE WITH A LIMIT IN K

DEF: GIVEN ANY $A \subseteq \mathbb{R}$, LET L BE THE SET OF ALL LIMIT POINTS OF A. THEN THE CLOSURE OF A IS $\overline{A} = A \cup L$.

EX:
- LET $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. THEN $\overline{A} = A \cup \{0\}$.
- $\overline{(0,1)} = [0,1]$
- $\overline{[0,1]} = [0,1]$
- $\overline{\mathbb{Q}} = \mathbb{R}$
- $\overline{\mathbb{R}} = \mathbb{R}$

THM: FOR ANY $A \subseteq \mathbb{R}$, $\overline{A}$ IS CLOSED AND $\overline{A}$ IS THE SMALLEST CLOSED SET CONTAINING A

PF IN HW

~~T~~DEF GIVEN $A \subseteq \mathbb{R}$, THE <u>COMPLEMENT OF A</u> IS

$$A^c = \{ x \in \mathbb{R} \mid x \notin A \}.$$

THM:

$$\boxed{\begin{array}{l} \bullet \text{ A SET } \mathcal{O} \text{ IS OPEN } \Longleftrightarrow \mathcal{O}^c \text{ IS CLOSED.} \\ \bullet \text{ A SET } E \text{ IS CLOSED } \Longleftrightarrow E^c \text{ IS OPEN} \end{array}}$$

PF/ $\Rightarrow$/ SPOZE $\mathcal{O}$ IS OPEN. TO SEE $\mathcal{O}^c$ IS CLOSED, WE HAVE TO SHOW IT CONTAINS ITS LIMIT POINTS. LET x BE A LIMIT POINT OF $\mathcal{O}^c$, SO FOR EVERY $\varepsilon > 0$, $V_\varepsilon(x) \cap \mathcal{O}^c \neq \emptyset$.

BUT THEN $x \notin \mathcal{O}$, FOR IF SO, EVERY POINT OF $V_\varepsilon(x)$ IS IN $\mathcal{O}$, IE $V_\varepsilon(x) \cap \mathcal{O}^c = \emptyset$.

$\Leftarrow$/ SPOZE $\mathcal{O}^c$ IS CLOSED. LET x BE ANY POINT OF $\mathcal{O}$. SINCE $\mathcal{O}^c$ IS CLOSED x IS <u>NOT</u> A LIMIT POINT OF $\mathcal{O}^c$. BUT THEN THERE MUST BE A NEIGHBORHOOD $V_\varepsilon(x)$ THAT IS DISJOINT FROM $\mathcal{O}^c$. THAT IS, $V_\varepsilon(x) \subseteq \mathcal{O}$.

SINCE $(E^c)^c = E$, THIS GIVES THE SECOND STATEMENT.

THM: • IF E_i ARE CLOSED, $\bigcup_{i=1}^{K} E_i$ IS A CLOSED SET.

• IF $\{E_\lambda\}$ IS A ~~AND ARB~~ COLLECTION OF CLOSED SETS $\bigcap_{\lambda \in \Lambda} E_\lambda$ IS CLOSED.

THIS FOLLOWS FROM $\left(\bigcup_{\lambda \in \Lambda} E_\lambda \right)^c = \bigcap_{\lambda \in \Lambda} E_\lambda^c$

AND $\left(\bigcap_{\lambda \in \Lambda} E_\lambda \right)^c = \bigcup_{\lambda \in \Lambda} E_\lambda^c$

THM: $K \subseteq \mathbb{R}$ IS COMPACT $\iff$ K IS CLOSED AND BOUNDED

Pf/ $\Rightarrow$ SPOZE K IS COMPACT. LET'S SHOW IT IS BOUNDED.

IF IT IS NOT BOUNDED, THEN IT HAS ARBITRARILY LARGE ELEMENTS. IE, FOR EACH $n \in \mathbb{N}$, WE CAN FIND (NOT NECESSARILY DISTINCT) $X_n \in K$ WITH $|X_n| > n$.

BUT IF K IS COMPACT, $\{X_n\}$ HAS A CONVERGENT SUBSEQUENCE $\{X_{n_j}\}$. $\Rightarrow\Leftarrow$ BECAUSE $\{X_{n_j}\}$ IS UNBOUNDED AND CONVERGENT SEQS ARE BDD.

K IS ALSO CLOSED: SPOZE $x_n \to x$ WITH $x_n \in K$.

WE MUST SHOW $x \in K$.

BUT SINCE K IS COMPACT, THERE IS A CONVERGENT SUBSEQ OF $\{x_n\}$ WHOSE LIMIT IS IN K.

BUT EVERY SUBSEQUENCE OF A CONVERGENT SEQUENCE HAS THE SAME LIMIT. THUS $x \in K$.

$\Leftarrow$/ HOMEWORK.

THERE ARE MANY PARALLELS BETWEEN CLOSED INTERVALS AND COMPACT SETS.

FOR EXAMPLE:

<u>NESTED COMPACT SET PROP</u>.

SPOZE K_n IS COMPACT AND NONEMPTY FOR EACH n, WITH

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$$

THEN $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

Pf/ FOR EACH n, LET $x_n \in K_n$.

SINCE THE SETS ARE NESTED, $x_n \in K_j$ FOR ALL $j \leq n$.

THIS MEANS $\{x_n\} \subseteq K_1$

NOW SINCE K_1 IS COMPACT, $\{x_n\}$ HAS A CONVERGENT SUBSEQUENCE WITH A LIMIT IN K_1, IE $x_{n_j} \to x \in K_1$

BUT ALSO, $x \in K_n$ FOR <u>EVERY</u> n.

~~[illegible]~~ GIVEN ANY $n_0 \in \mathbb{N}$, WE HAVE $x_{n_0} \in K_{n_0}$ AND ALSO $x_n \in K_{n_0}$ FOR ALL $n > n_0$. BUT THIS MEANS THAT $x \in K_{n_0}$ SINCE THE LIMIT OF THE SUBSEQ $\{x_{n_j}\}_{n_j > n_0}$ IS ALSO x.

SOMETIMES COMPACT SET IN $\mathbb{R}$ IS DEFINED AS CLOSED AND BOUNDED ; THESE DEFINITIONS ARE EQUIVALENT (ALTHOUGH THE DEF WE TOOK IS MORE GENERAL.)

THERE IS ANOTHER ~~DEF~~ EQUIVALENT OF COMPACT IN TERMS OF OPEN SETS ~~&~~ BUT WE NEED ANOTHER IDEA FIRST.

DEF: LET $A \subseteq \mathbb{R}$. AN OPEN COVER OF A IS A COLLECTION OF OPEN SETS $\{\mathcal{O}_\lambda \mid \lambda \in \Lambda\}$ [COULD BE INFINITE, EVEN UNCOUNTABLE] SO THAT

$$A \subset \bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda .$$

A FINITE SUBCOVER IS A FINITE COLLECTION OF THESE $\mathcal{O}_{\lambda_j}$ SO THAT $A \subset \bigcup_{j=1}^{n} \mathcal{O}_{\lambda_j}$

EXAMPLE: CONSIDER $(0,1)$.

FOR EACH $x \in (0,1)$, LET $\mathcal{O}_x = (x/2, 1)$.

NOTE THAT THIS IS AN OPEN COVER, SINCE

$x \in (x/2, 1)$ FOR EVERY $x \in (0,1)$.

BUT THERE IS NO FINITE SUBCOVER. IF THERE WERE, IT WOULD LOOK LIKE $\{\mathcal{O}_{x_1}, \mathcal{O}_{x_2}, \dots, \mathcal{O}_{x_n}\}$.

LET $x_0 = \min(x_1, x_2, \dots, x_n)$ AND LET $0 < y \leq x_0/2$.

THEN $y \notin \mathcal{O}_{x_i}$ FOR $i = 1 \dots n$.

NOW LETS TRY THE SAME FOR $[0,1]$.

NEITHER 0 NOR 1 ARE IN ANY OF THE SETS $\mathcal{O}_x$, SO WE MUST COVER THEM SOMEHOW.

ANY OPEN SET CONTAINING 0 MUST HAVE A SUBSET $(-\varepsilon, \varepsilon)$ FOR SOME $\varepsilon > 0$; SIMILARLY, $(1-\varepsilon, 1+\varepsilon)$ IS IN ANY OPEN AROUND 1.

LET $\mathcal{O}_0 = (-\varepsilon, \varepsilon)$, $\mathcal{O}_1 = (1-\varepsilon, 1+\varepsilon)$.

THEN $\{\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_x \mid x \in (0,1)\}$ COVERS $(0,1)$.

BUT IF WE TAKE x^* SO THAT $x^*/2 < \varepsilon$, THE COLLECTION

$\{\mathcal{O}_0, \mathcal{O}_{x^*}, \mathcal{O}_1\}$ COVERS $[0,1]$ AND IS A FINITE SUBCOVER.

THE HEINE-BOREL THEOREM

LET $K \subseteq \mathbb{R}$. THE FOLLOWING ARE EQUIVALENT:

(i) K IS COMPACT

(ii) K IS CLOSED AND BOUNDED

(iii) EVERY OPEN COVER OF K ADMITS A FINITE SUBCOVER.

Pf/ WE'VE SEEN (i)$\Leftrightarrow$(ii).

(iii)$\Rightarrow$(ii)/ TO SEE THAT K IS BOUNDED,

LET $\mathcal{O}_x = V_1(x) = (x-1, x+1)$.

CERTAINLY $\{\mathcal{O}_x \mid x \in K\}$ IS AN OPEN COVER.

BUT IF (iii) HOLDS, THERE IS A SUBCOVER

$\{\mathcal{O}_{x_1}, \mathcal{O}_{x_2}, \dots, \mathcal{O}_{x_n}\}$ WHICH COVERS.

SINCE EACH $\mathcal{O}_{x_i}$ IS BOUNDED, SO IS THE UNION. THUS K IS BOUNDED.

NOW TO SEE K CLOSED, SPOZE NOT. LET $y_n \in K$ WITH $\{y_n\}$ CAUCHY ~~BUT~~ (SO $y_n \to y$) AND $y \notin K$.

SINCE $y \notin K$, $|x-y| > 0$ FOR EVERY $x \in K$.

NOW LET $\mathcal{O}_x = \left(x - \frac{|x-y|}{2}, x + \frac{|x-y|}{2}\right)$ FOR EACH $x \in K$. $\{\mathcal{O}_x\}$ IS AN OPEN COVER OF K.

(iii) $\Rightarrow$ $\{\mathcal{O}_x\}$ HAS A FINITE SUBCOVER, $\{\mathcal{O}_{x_1}, \mathcal{O}_{x_2} \dots \mathcal{O}_{x_n}\}$.

LET $\varepsilon_0 = \min \left\{\frac{|x_i - y|}{2} \mid 1 \le i \le n\right\}$. SINCE $y_n \to y$, THERE IS $\underset{\in K}{y_N}$ WITH $|y_N - y| < \varepsilon_0$. BUT THEN $y_N \notin \bigcup_{i=1} \mathcal{O}_{x_i}$ $\Rightarrow\!\Leftarrow$.

(ii) ⇒ iii | SPOZE K IS CLOSED AND BOUNDED (AND HENCE COMPACT)

AND LET $\{ \mathcal{O}_\lambda \mid \lambda \in \Lambda \}$ BE AN OPEN COVER OF K.
LETS ASSUME NO FINITE SUBCOVER EXISTS.
SINCE K IS BOUNDED, THERE IS A CLOSED INTERVAL

$[-M, M] = I_0 \supseteq K$.

NOW LET $I_1^- = [-M, 0]$, $I_1^+ = [0, M]$, AT LEAST
ONE OF THESE IS NONEMPTY (INTERSECTION WITH K) AND DOES NOT ADMIT
A FINITE SUBCOVER; CALL THIS I_1.
NOW SUBDIVIDE I_1 TO GET I_2 WITH $|I_2| = \frac{1}{2}|I_1|$, $I_2 \cap K \neq \emptyset$
AND I_2 NOT ADMITTING A FINITE SUBCOVER.

NOW WE HAVE $I_0 \supset I_1 \supset I_2 \supset \cdots$
WITH $I_n \cap K$ NOT ADMITTING A FINITE SUBCOVER FOR EVERY n

BUT SINCE $|I_n| \leq \frac{|I_0|}{2^n}$ AND $I_n \cap K \neq \emptyset$,
THERE MUST BE ~~$x \in I_n$ FOR ALL n~~
$x \in K$ WITH $x \in I_n$ FOR ALL n (RECALL K IS CLOSED)
SINCE $\{ \mathcal{O}_\lambda \}$ WAS A COVER OF K,
THERE IS AN ELEMENT $\mathcal{O}_{\lambda_0}$ WITH $x \in \mathcal{O}_{\lambda_0}$
AND HENCE (SINCE IT IS OPEN) $\mathcal{O}_{\lambda_0} \supset I_m$ FOR m SUFF. LARGE.

BUT I_m IS SUPPOSED TO NOT ADMIT A
FINITE SUBCOVER AND IT IS COVERED BY ONE ELT.

$\Rightarrow\Leftarrow$