3/7/2022

he keeps referring to proving the Alternate series Test
(so study that!)

Prove that the ratio test works ← he almost put this on the midterm, not bad to do for practice though

$\sum_{n=0}^{\infty} a_n$ satisfies $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < r < 1$

Hint: consider geometric $\sum_{n=0}^{\infty} r^n$

At least one will be a definition or a theorem
Not going to ask the made up ones/long ones
- No need to know how to write things in binary, etc.

Recap

- Reps of reals, decimals blah, blah
- Def of lim of a sequence (i.e. convergence)
  $\forall \varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N}$ s.t. $|a_n - L| < \varepsilon\ \forall n > N_\varepsilon$.
- Stuff about sequences (bounded, monotone, divergent) | squeeze theorem
- Bolzano-Weirestrass thm - every bounded sequence has a convergent sequence

Subsequence always infinite

- Completeness of $\mathbb{R}$, (Archimedian Property), ← If we assume this, they're equivalent
  Nested Interval Theorem, Cauchy sequences converge, monotone convergence
  every infinite decimal is a real #
  $.99\bar{9} = 1$

Should understand these

Inf series = inf sums

Definition - $\sum_{n=0}^{\infty} a_n$ converges to L → This means the sequence of partial sums converges to L

Let $S_n = a_0 + a_1 + \ldots + a_n = \sum_{n=0}^{n} a_n$

↓

just means $\lim_{n\to\infty} S_n = L$ e.g. $1 + \frac{1}{2} + \frac{1}{9} + \frac{1}{16} + \ldots = \frac{\pi^2}{2}$

.33333....

$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} +$

gets so small so fast that it converges

$\lim a_n \neq 0$ diverges

$\lim a_n = 0$ not enough

Need them to go to 0 fast enough so it doesn't pile up → This is why harmonic series ~~seems~~ like it converges but it doesn't

wants to ask us about continuous functions not so much the different ways to determine convergence/divergence

★ If it's absolutely convergent, you can rearrange the terms, if it doesn't you can't

Can give us geometric series $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \ldots$

Does this converge? What's the limit?

$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16}$ converges b/c $|-\frac{1}{2}| < 1$

(it's a geometric series so $|r| < 1$)

★ good to know ★

Yes b/c Alt series $\underbrace{|a_n| > |a_{n+1}|}_{\text{Dec } a_i}$ or , $\lim a_n = 0$

But doesn't tell us what it converges to

> since geometric, converges to $\frac{1}{1-(-\frac{1}{2})} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$

$S_N = 1 + r + r^2 + \ldots + r^n$

How do we know what N is? Multiply by r and subtract

$-rS_N = r + r^2 + \ldots + r^n + r^{n+1}$

$(1-r)S_N = 1 - r^{n+1}$

$S_N = \frac{1 - r^{n+1}}{1-r}$

then this goes to 0

if $|r|<1$

$\lim_{N \to \infty} S_N = \frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$

★ Could also do comparison test or

p-series $\frac{1}{n^p}$ $p>1$ it converges

if $p=1$, harmonic, diverges

eg $\frac{1}{n^2}$ $\frac{n^2}{n^3}$ { sequence converges but not the series

harder to converge than a sequence b/c you're adding

series

Comparison test Example

- If $0 \le a_n \le b_n$ $\sum^{\infty} b_n$ converges, so does $\sum^{\infty} a_n$
  $\sum a_n$ diverges, so does $\sum^{\infty} b_n$

Squeeze theorem - sequences

three sequences $x_n \le y_n \le z_n$

If $\lim x_n = \lim z_n = L$ then $\lim y_n = L$

NOT TRUE -1 1

If $\lim x_n$ and $\lim z_n$ exist,

then $\lim y_n$ exists

b/c if $y_n = \frac{(-1)^n}{2}$

$\{a_n\} = \{1, 1, 1, 1 \ldots\}$

but series doesn't converge b/c your just adding

$S_1 = 1$

$S_2 = 2$

$S_3 = 3$

$S_n = n$

Show that

$\lim_{n \to \infty} \left(\frac{1}{n^3 + \sin(n)} \right)$ converges

Can say we know $\frac{1}{n^p}$ converges $\Leftrightarrow$ $p > 1$ w/o proving it

$\frac{1}{n^3 + \sin(n)} \sim \frac{1}{n^3}$ ← we know this by the definition of a limit

$\frac{1}{n^3 + \sin(n)} < \frac{2}{n^3}$ so this certainly must be true

$\lim \left(\frac{1}{n^3 + \sin(n)} \right) \leq 2 \lim \frac{1}{n^3} = 0$

For series, show partial sums

$\sum a_n \quad 0 \le a_n \le b_n \quad \sum b_n$

$\Rightarrow a_0 + a_1 + \ldots + a_n < b_0 + b_1 + \ldots + b_n$

a_n pos, bounded by b_n

Partial sums increasing & bounded, so a_n converges

increasing, bounded, Monotone convergence Theorem done.

Now w/ series $\quad b_n$ converges $\Rightarrow a_n$ converges

$\sum \frac{1}{n^3 + \sin(n)}$ (a_n) for $n \ge 2$ $\quad n^3 + \sin(n) > (n-1)^3 - \sin(n-1)$

aka if n is big, $\sin n$ doesn't matter

so replace w/ biggest it can be

$\sum \frac{1}{n^3 - 1} \longrightarrow b_n$

$< \sum \frac{1}{n^2} = \frac{\pi^2}{6}$

Comparing this to $\frac{1}{n}$ is not a good comparison.