LAST TIME:

- BOLZANO-WEIERSTRASS: (EVERY BDD SEQ HAS CONVERGENT SUBSEQ)
- CAUCHY SEQUENCES.

LETS RETURN TO INFINITE SERIES FOR A BIT. MOSTLY WE'VE DEALT WITH SERIES $\sum_1^\infty a_n$ WITH $a_n \geq 0$.

EXAMPLE

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

JUST LOOKING AT SEQUENCE OF PARTIAL SUMS, WE GET

$$S_1 = 1, \quad S_2 = 1/2, \quad S_3 = 5/6, \quad S_4 = 7/12, \quad \ldots$$

NOTE THAT $\{S_{2n}\}$ IS DECREASING

$\{S_{2n+1}\}$ IS INCREASING.

LIMIT LOOKS LIKE ABOUT 0.69. (LIMIT IS $\ln 2$)

BUT!

$$\frac{1}{2}S = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots$$

$$+ \quad S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} + \cdots$$

$$\frac{3}{2}S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \cdots$$

NOTE THAT IF $S = \ln 2$, REARRANGING TERMS SOMEHOW GIVES $\frac{3 \ln 2}{2}$.

ANOTHER EXAMPLE:

LOOK AT GEOMETRIC SERIES

$$\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - = \frac{1}{1-(-\frac{1}{2})} = \frac{2}{3}.$$

IF WE REARRANGE TERMS AS

$$1 + \frac{1}{4} - \frac{1}{2} + \frac{1}{16} + \frac{1}{64} - \frac{1}{8} + + \cdots$$

THE PARTIAL SUMS SEEM TO CONVERGE TO $\frac{2}{3}$

(30 TERMS GIVES $2/3$ TO 6 PLACES).

SO THIS IS OK?

WHEN ~~CAN WE USE~~ IS THIS "INFINITE ADDITION" COMMUTATIVE?

LET'S LOOK AT SOME SERIES WITH $\pm$ TERMS.

THM: ABS CONVERGENCE.

$\boxed{\text{IF } \sum |a_n| \text{ CONVERGES, THEN } \sum a_n \text{ ALSO CONVERGES}}$

NOTE $\Leftarrow$ IS FALSE: $\sum \frac{(-1)^n}{n}$ CONVERGES, BUT $\sum \frac{1}{n}$ DIVERGES.

PROOF SINCE $\sum |a_n|$ CONVERGES, $\{S_K\} = \sum_{n=0}^{K} |a_n|$ IS A CAUCHY SEQUENCE.

THAT IS, WE CAN MAKE $|S_m - S_n|$ AS SMALL AS WE LIKE FOR $m > n > N$, N BIG.

THAT IS, GIVEN $\varepsilon > 0$, THERE IS AN N SO THAT

$$|a_{N+1}| + |a_{N+2}| + \cdots + |a_m| < \varepsilon$$

BUT

$$|S_m - S_n| = |a_{n+1} + a_{n+2} + \cdots + a_m| \leq |a_{n+1}| + |a_{n+2}| + \cdots + |a_m| < \varepsilon$$

THAT IS $\{S_n\}$ IS CAUCHY, SO $\sum_1 a_n$ CONVERGES. $\blacksquare$

PUT HERE

THM ALTERNATING SERIES TEST:

SUPPOSE
- $a_1 \geq a_2 \geq a_3 \geq a_4 \cdots \geq 0$ (ie a_n IS DECREASING POSITIVE SEQ.)
- $\lim_{n \to \infty} a_n = 0$

THEN
$$\sum_{n=1}^{\infty} (-1)^n a_n \text{ CONVERGES.}$$

PROOF IN HW. (SHOW IT IS CAUCHY, OR USE NESTED INTERVALS).

~~SO NOW BAC~~

DEF: A SERIES $\sum_1 a_n$ CONVERGES ABSOLUTELY IF $\sum_1 |a_n|$ CONVERGES

IF $\sum_1 a_n$ CONVERGES, BUT $\sum_1 |a_n|$ DOES NOT, THEN $\sum_1 a_n$ CONVERGES CONDITIONALLY

EXAMPLES: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

CONVERGES COND.

$\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$ CONV ABS.

ANY SEQUENCE WITH ALL (OR ALL BUT FINITELY MANY) POSITIVE TERMS CONVERGES ABSOLUTELY.

BACK TO REARRANGING:

DEF: LET $\sum_1 a_k$ BE A SERIES. THEN $\sum_1 b_k$ IS A REARRANGEMENT OF $\sum_1 a_k$ IF THERE IS A BIJECTION (ONE-TO-ONE, ONTO FUNCTION) $f : \mathbb{N} \to \mathbb{N}$ SO THAT $b_{f(k)} = a_k$ FOR ALL $k \in \mathbb{N}$.

THM: IF $\sum_1 a_n$ CONVERGES ABSOLUTELY, THEN ANY REARRANGEMENT OF $\sum_1 a_n$ CONVERGES TO THE SAME LIMIT.

ABSOLUTELY

PROOF: SPOZE $A = \sum_{j=1}^{\infty} a_j$ AND LET $\sum_{j=1}^{\infty} b_j$ BE A REARRANGEMENT OF $\sum a_j$.

SWITCH TO a_j, b_j!

LET $\{S_n\}$ BE THE SEQUENCE OF PARTIAL SUMS OF $\sum a_j$

IE $S_n = \sum_{j=1}^{n} a_j$.

$\{t_m\}$ THE PARTIAL SUMS OF $\sum b_j$,

IE. $t_m = \sum_{j=1}^{m} b_j$

MUST SHOW $t_m \to A$.

FIX $\varepsilon > 0$

- SINCE $S_n \to A$, THERE IS A K_1 SO THAT $|S_n - A| < \varepsilon/2$ FOR ALL $n > K_1$

- BECAUSE WE HAVE ABS. CONV., $\sum |a_j|$ IS CAUCHY, SO WE CAN FIND K_2 SO THAT $\sum_{j=m+1}^{n} |a_j| < \varepsilon/2$ FOR $n > m > K_2$

NOW TAKE $K \geq \max\{K_1, K_2\}$.

EACH OF THE TERMS $\{a_1, a_2, \ldots a_K\}$ APPEAR IN $\sum b_j$, AND LETS PICK M SO THAT IT IS WHERE THE LAST IS,

IE $M = \max\{f(n) \mid 1 \leq n \leq K\}$

NOW, IF $m \geq M$, THEN $\{t_m - S_K\}$ HAS A FINITE NUMBER OF TERMS, ALL IN THE TAIL OF $\sum_{j=K+1}^{\infty} |a_j|$. BUT THE CHOICE OF K_2 ENSURES $|t_m - S_K| < \varepsilon/2$

SO WE HAVE

$$
\begin{aligned}
|t_m - A| &= |t_m - s_k + s_k - A| \\
&\le |t_m - s_k| + |s_k - A| \\
&\le \varepsilon/2 + \varepsilon/2 = \varepsilon
\end{aligned}
$$