LAST TIME, DEFINED THE NOTION OF A SUBSEQUENCE. INFORMALLY, GIVEN A SEQUENCE $a_1, a_2, a_3, a_4 \cdots$, WE JUST PICK OUT (IN ORDER) AN INFINITE LIST OF THE a_i AND DENOTE THEM $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$

- ALL SUBSEQUENCES OF A CONVERGENT SEQUENCE MUST CONVERGE (TO THE SAME LIMIT),

BUT EVEN DIVERGENT SEQUENCES CAN HAVE CONVERGENT SUBSEQUENCES

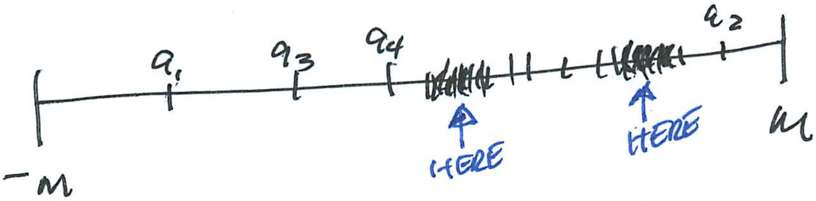
(EASY EXAMPLE: $\{(-1)^n\} = 1, -1, 1, -1, \cdots$)

IN FACT:

THM: (THE BOLZANO-WEIERSTRASS THM)

EVERY BOUNDED SEQUENCE CONTAINS A CONVERGENT SUBSEQUENCE.

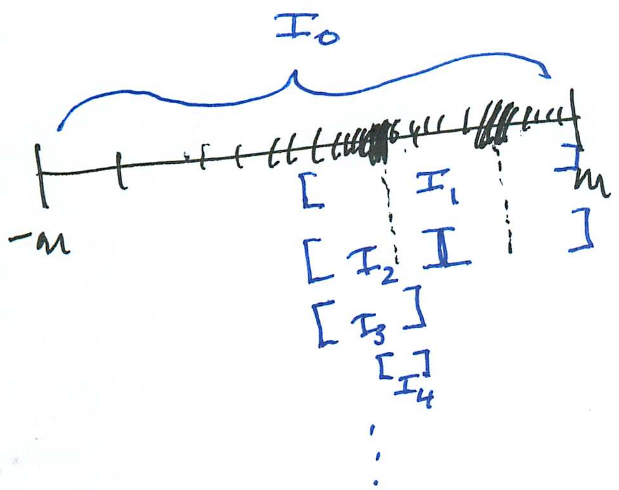
~~HOW TO PR~~ WHY IS THIS TRUE? SINCE THERE ARE INFINITELY MANY TERMS, THEY MUST PILE UP IN (AT LEAST) ONE PLACE, (MAYBE MORE)



LETS MAKE THIS MORE FORMAL.

Pf/ SINCE $\{a_n\}$ IS BOUNDED, THERE IS AN INTERVAL $I_0 = [-M, M]$ WHICH CONTAINS ALL THE TERMS. LET $b_0 = a_0$

SPLIT I_0 INTO $I_0^- = [-M, 0]$ AND $I_0^+ = [0, M]$. AT LEAST ONE OF THESE CONTAINS INFINITELY MANY TERMS OF $\{a_n\}$. LET I_1 BE THAT INTERVAL. LET $b_1 = a_{n_1}$ WITH $a_{n_1} \in I_1$



NOW DIVIDE I_1 INTO TWO HALVES, AND LET I_2 BE ~~ONE OF THE~~ AN INTERVAL WITH ∞-MANY TERMS.

IN GENERAL, LET I_{n+1} BE ONE OF THE HALVES OF I_n THAT CONTAINS ∞-MANY TERMS OF $\{a_n\}$, AND SET $b_{j+1} = a_{n_{j+1}}$ WITH $a_{n_{j+1}} \in I_{n_{j+1}}$.

BY THE NESTED INTERVALS THEOREM,

$$I_0 \supset I_1 \supset I_2 \supset \cdots \quad \text{AND} \quad \bigcap_{j=1}^{\infty} I_j \neq \emptyset.$$

BUT SINCE THE LENGTH OF I_j IS $\frac{2M}{2^j} = \frac{M}{2^{j-1}}$, $\bigcap I_j = \{L\}$.

SO WE HAVE CONSTRUCTED A SUBSEQUENCE

$$a_0, a_{n_1}, a_{n_2}, \ldots \quad \text{WITH } a_{n_j} \to L.$$

∎

ONE ISSUE WITH THE DEFINITION OF A CONVERGENT SEQUENCE IS THAT WE MUST KNOW WHAT THE LIMIT IS, IN ORDER TO SHOW CONVERGENCE.

[$\pi = 3.141593589\ldots$]

EXAMPLE:

DEFINE THE FOLLOWING "NUMBER".

LET $\{P_n\}$ BE GIVEN BY THE n^{th} DIGIT OF π.

IE $P_0 = 3,\ P_1 = 1,\ P_2 = 4,\ P_3 = 1,\ P_4 = 5,\ \ldots$

AND LET γ HAVE A DECIMAL EXPANSION OF 7 FOLLOWED BY P_k ZEROS ...

$$\gamma = 7.\underbrace{000}_{P_0}7\underbrace{0}_{P_1}7\underbrace{0000}_{P_2}7\underbrace{0}_{P_3}7\underbrace{00000}_{P_5}7\underbrace{00\cdots}_{P_6}$$

DOES THIS CONVERGE? SURE. BUT WHAT IS γ?

WORSE: IS $\pi^{\sqrt{2}}$ A NUMBER? SURE. WHAT IS IT?

SO HERE'S SOMETHING WE CAN USE WITHOUT BEING ABLE TO KNOW THE LIMIT.

DEF: A SEQUENCE $\{a_n\}$ IS A CAUCHY SEQUENCE IF FOR EVERY $\varepsilon > 0$, THERE IS A $K_\varepsilon \in \mathbb{N}$ SO THAT WHENEVER $m, n > K_\varepsilon$, WE KNOW THAT

$$|a_m - a_n| < \varepsilon.$$

WAIT LONG ENOUGH, AND THE TERMS GET CLOSE TOGETHER (AND STAY CLOSE).

THM: EVERY CONVERGENT SEQUENCE IS A CAUCHY SEQUENCE

Pf/ SPOZE $\{x_n\} \to L$. ~~THEN~~ CHOOSE $\varepsilon > 0$. WE MUST SHOW THAT THERE IS A POINT K_ε AFTER WHICH $|a_m - a_n| < \varepsilon$ FOR ALL $m, n > K_\varepsilon$. BUT SINCE $x_n \to L$, THERE IS N_ε SO THAT $|a_j - L| < \varepsilon/2$ FOR ALL $j > N_\varepsilon$.

~~THEN WE HAVE~~ LET $m, n > N_\varepsilon$

THEN
$$|a_m - a_n| \leq |a_m - L| + |L - a_n| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

THE OTHER DIRECTION HOLDS, TOO

THM EVERY CAUCHY SEQUENCE CONVERGES

Pf/ FIRST, LET'S SHOW THAT EVERY CAUCHY SEQ. IS BOUNDED.

TAKE $\varepsilon = 1$, SO THERE IS A K SO THAT
$$|x_m - x_n| < 1 \quad \text{FOR ALL } m, n \geq K.$$

BUT THEN $|x_n| \leq |x_K| + 1$ (TAKE $m = K$) FOR ALL $n \geq K$.

LET
$$M = \max\{|x_1|, |x_2|, \ldots, |x_{K-1}|, |x_K| + 1\}$$

AND
$$-M < x_n < M \quad \text{FOR ALL } x_n \in \{x_n\}.$$

NOW WE CAN USE BOLZANO-WEIRSTRASS:

- FIRST, PICK OUT SOME CONVERGENT SUBSEQUENCE $\{X_{n_j}\}$ AND LET $L = \lim X_{n_j}$. (KNOW WE CAN DO THIS BY B-W)
- THEN SHOW THAT $X_n \to L$.

TO SEE THAT $X_n \to L$, FIX $\varepsilon > 0$. MUST SHOW THAT $\exists K$ SO THAT $|X_n - L| < \varepsilon$ FOR $n > K$.

BUT, SEQ. IS CAUCHY, SO $\exists K_1$ WITH
$|X_n - X_{n_j}| < \varepsilon/2$ FOR ALL $n, n_j > K_1$.

ALSO $X_{n_j} \to L$ SO $\exists K_2$ WITH
$|X_{n_j} - L| < \varepsilon/2$ FOR ALL $n_j > K_2$.

LET $K > \max(K_1, K_2)$. THEN

$$|X_n - L| \leq |X_n - X_{n_j}| + |X_{n_j} - L| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

☺

(CONVERGENT)

EXAMPLE OF CAUCHY SEQUENCE

LET $\{y_n\}$ BE GIVEN BY

$$y_1 = 1$$
$$y_2 = 1 - \frac{1}{2!}$$
$$y_3 = 1 - \frac{1}{2!} + \frac{1}{3!}$$
$$\vdots$$
$$y_n = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots \frac{(-1)^{n+1}}{n!}$$

SEQUENCE IS NOT MONOTONE.
BUT IT IS CAUCHY: FOR $m > n$,

$$|y_m - y_n| = \frac{(-1)^{n+2}}{(n+1)!} + \frac{(-1)^{n+3}}{(n+2)!} + \cdots + \frac{(-1)^{m+1}}{m!}$$

$$< \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{m!}$$

(USING FACT THAT $2^{r-1} < \frac{1}{r!}$) $$< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^m} < \frac{1}{2^{n-1}}$$

SO THE SEQUENCE IS CAUCHY AND THEREFORE CONVERGES.

(THE LIMIT CAN BE SHOWN TO BE $1 - 1/e \approx 0.632120\ldots$)

ANOTHER PROOF THAT THE HARMONIC SERIES DIVERGES. (IT IS NOT CAUCHY).

PF/ LET $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, CONSIDER $\{S_n\}_{n=1}^{\infty}$.

LET $m > n$, WE HAVE

$$S_m - S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m}$$
$$> \underbrace{\frac{1}{m} + \frac{1}{m} + \cdots + \frac{1}{m}}_{m-n \text{ TERMS}} = \frac{m-n}{m} = 1 - \frac{n}{m}.$$

OBSERVE THAT $S_{2n} - S_n = 1 - \frac{n}{2n} = 1/2$,
SO $\{S_n\}$ IS NOT CAUCHY AND DOES NOT CONVERGE.

WE'VE SEEN:

AoC (AXIOM OF COMPLETENESS)

$\Downarrow$ NIP (NESTED INTERVALS) | $\Downarrow$ MCT (MONOTONE CONVERGENCE THM)

$\Downarrow$ (BW) BOLZANO WEIERSTRASS

$\Downarrow$ (CC). CAUCHY CRITERIA

BUT IN FACT, IF WE ALSO ASSUME THE ARCHIMEDIAN PROPERTY (IMPLIED BY AoC AND MCT NOT OTHERS.)

$$\text{AoC} \Leftrightarrow \text{NIP} \Leftrightarrow \text{MCT} \Leftrightarrow \text{BW} \Leftrightarrow \text{CC}$$