LAST TIME [QUIZ ON MONDAY]

$\boxed{\text{DEF: } \lim_{n\to\infty} a_n = L \text{ MEANS FOR EVERY } \varepsilon>0, \text{ THERE IS AN } N_\varepsilon \text{ SO THAT } |a_n - L| < \varepsilon \text{ FOR ALL } n > N_\varepsilon}$

"YOU TELL ME HOW CLOSE TO L YOU WANT IT (ε), THEN I TELL YOU HOW LONG YOU HAVE TO WAIT (N_ε) SO THAT ALL FOLLOWING TERMS ARE AT LEAST THAT CLOSE.

EXAMPLE ~~THE~~ LET $a_n = \frac{n+1}{2n-1} \to \overset{1}{2}, \overset{2}{\frac{3}{3}}=1, \overset{3}{\frac{4}{5}}, \frac{5}{7}, \frac{6}{9}=\frac{2}{3},$

SHOW $\lim_{n\to\infty} \frac{n+1}{2n-1} = \frac{1}{2}$.

THINK: WANT $\left|\frac{n+1}{2n-1} - \frac{1}{2}\right| < \varepsilon$.

ALWAYS >0, SO $\frac{n+1}{2n-1} - \frac{1}{2} = \frac{(2n+2)-(2n-1)}{2(2n-1)}$

$= \frac{3}{4n-2} < \varepsilon$ WANT

IE $\frac{3}{\varepsilon} < 4n-2 < 4n$

$$\boxed{\frac{3}{4\varepsilon} < n}$$

OR $\frac{2\varepsilon+3}{4\varepsilon} < n$ BUT YUCK!

NOW DO

FIX $\varepsilon > 0$. WE MUST FIND N SO THAT IF $n > N$,

WE HAVE $\left| \frac{n+1}{2n-1} - \frac{1}{2} \right| < \varepsilon$.

SPOZE $n > \frac{3}{4\varepsilon}$.

CAN DROP.

THEN $\left| \frac{n+1}{2n-1} - \frac{1}{2} \right| = \left| \frac{3}{4n-2} \right| < \left| \frac{3}{4n} \right| < \left| \frac{3}{4\left(3/4\varepsilon\right)} \right| = |\varepsilon|$

$\boxtimes$

PROP: THE LIMIT OF A SEQUENCE IS UNIQUE, WHEN IT EXISTS

NON-EXAMPLE: $\left\{ (-1)^n \right\}_{n=0}^{\infty}$ HAS NO LIMIT. $-1, +1, -1, +1, \ldots$

Pf/ SPOZE $\lim a_n = L$ AND $\lim a_n = K$, WITH $L \neq K$.

BUT THEN SINCE $L \neq K$, $|L - K| > 0$.

CHOOSE $\varepsilon < |L - K|$ CANT WORK.

WHAT DOES $a = b$ MEAN?

IT MEANS $|a - b| = 0$

IE $\forall \varepsilon > 0$, $|a - b| < \varepsilon$

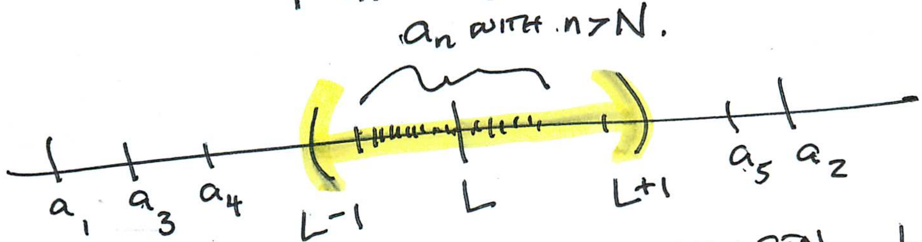
PROP: SPOZE $\{a_n\}$ CONVERGES. THEN THE SEQUENCE IS BOUNDED

RECALL DEF. OF **BOUNDED** SEQUENCE

(THERE IS $M \in \mathbb{R}$ SO THAT $a_n \in [-M, M]$ FOR ALL $n \in \mathbb{N}$.

PROOF: DISCUSS.

IDEA: $\lim \{a_n\} = L$, SO $\exists N$ SO THAT $|a_n - L| < 1$ FOR $n > N$.



IGNORE $a_1 \cdots a_N$. REST ARE BETWEEN $L-1$ AND $L+1$, SO BOUNDED BY $|L|+1$.

JUST "A FEW" $a_1, \dots, a_N$, SO HAS A MAX.

SET $M = \max\{ |a_1|, |a_2|, |a_3|, \cdots |a_N|, |L|+1 \}$.

ALGEBRA OF LIMITS

THM: SPOZE $\lim a_n = A$ $\lim b_n = B$

- $\lim(c\, a_n) = cA$ FOR ANY $c \in \mathbb{R}$.
- $\lim(a_n + b_n) = A + B$
- $\lim(a_n b_n) = A \cdot B$
- $\lim(a_n / b_n) = A/B$ PROVIDED $B \neq 0$.

- COMPARISON

(i) IF $a_n \geq 0$ FOR ALL n, THEN $A \geq 0$.

(ii) IF $a_n \leq b_n$ FOR ALL n, THEN $A \leq B$.

(iii) IF THERE IS $C \in \mathbb{R}$ WITH

$C \leq b_n$ FOR ALL n, THEN $C \leq B$

$a_n \leq C$ .... $A \leq C$

LIMITS ONLY DEPEND ON THE TAIL OF THE SEQ,

SO "FOR ALL n" SHOULD BE

"FOR ALL n SUFF. LARGE".

LETS PROVE ONE (YOU DO THE REST)

- $\lim (a_n + b_n) = A + B$

MUST SHOW THAT FOR ANY $\varepsilon > 0$, $\exists N$ SOTHAT IF $n > N$,

$$|(a_n + b_n) - (A+B)| < \varepsilon.$$

HOW TO DO? SPLIT IN HALF.

PICK $\varepsilon_a = \varepsilon/2$. THEN FOR n BIG, (ie $> N_a$)

HAVE $|a_n - A| < \varepsilon_a = \varepsilon/2$.

$\varepsilon_b = \varepsilon/2$. FOR $(n > N_b)$

HAVE $|b_n - B| < \varepsilon_b = \varepsilon/2$.

TAKE $N = \max(N_a, N_b)$.

THEN

TRIANGLE ≠

$$|(a_n + b_n) - (A+B)| \le |a_n - A| + |b_n + B|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

THE SQUEEZE THEOREM

SPOZE $x_n \le y_n \le z_n$ FOR n LARGE.

IF $\lim x_n = \lim z_n = L$,

THEN $\lim y_n = L$.

Pf/ DISCUSS.