LAST TIME, FOR AN INCREASING BOUNDED SEQUENCE $\{a_n\}_{n=0}^{\infty} = a_0, a_1, a_2 \ldots$

$$\lim_{n\to\infty} a_n = \sup\{a_n \mid n \in \mathbb{N}\}$$

(AND IF b_n IS DECREASING AND BOUNDED BELOW,

$$\lim_{n\to\infty} b_n = \inf\{b_n \mid n \in \mathbb{N}\}.$$)

WANT TO GENERALIZE TO ARBITRARY SEQUENCES.

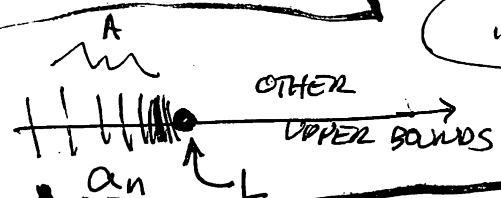
RECALL $\sup A = L \iff$

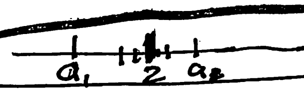
- L IS AN UPPER BOUND FOR A.
- FOR EVERY $\varepsilon > 0$, THERE IS $a_n \in A$ WITH $a_n > L - \varepsilon$

(IE, THERE IS NO SMALLER UPPER BOUND)

CAPTURES IDEA OF "a_n IS ARBITRARILY CLOSE TO L"

INSERT
EXAMPLE OF $\{2 + \frac{(-1)^n}{n}\}_{n=1}^{\infty} \to 2.$





<u>DEF</u> FOR ANY SEQUENCE $\{a_n\}_{n=0}^{\infty}$, WE SAY

$\lim_{n\to\infty} a_n = L$ IF AND ONLY IF

FOR EVERY $\varepsilon > 0$, THERE IS $N(\varepsilon) = N \in \mathbb{N}$ SO THAT

FOR ALL $n \geq N$, $|a_n - L| < \varepsilon$.

IN OTHER WORDS, L IS THE LIMIT IF ~~FEE~~ NO MATTER HOW CLOSE YOU WANT IT, (ε) a_n ~~GE~~ EVENTUALLY GETS THAT CLOSE TO L AND STAYS ~~THE~~ AT LEAST THAT CLOSE.

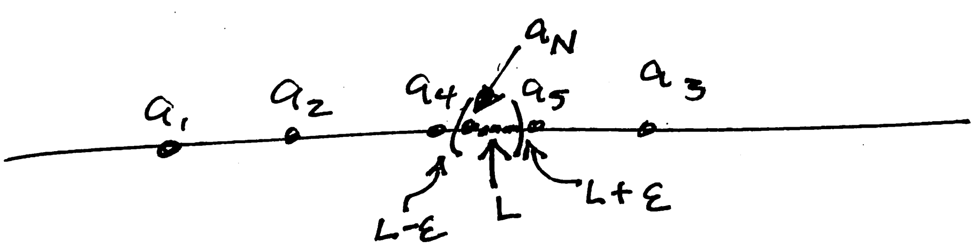
IE $|a_n - L| < \varepsilon$ FOR ALL $n > N$

~~ALTERNATIVE (SET EQUIVALENT)~~:

DEF: FOR $a \in \mathbb{R}$, THE SET $V_\varepsilon = \{ x \in \mathbb{R} \mid |x-a| < \varepsilon \}$
$= (a-\varepsilon, a+\varepsilon)$
IS AN <u>ε-NEIGHBORHOOD OF a</u>

ALTERNATIVE (TOPOLOGICAL) DEF OF LIMIT.

DEF: THE SEQUENCE $\{a_n\}$ CONVERGES TO L IF, GIVEN ANY ε-NEIGHBORHOOD $V_\varepsilon(L)$ OF L, THERE IS A POINT OF THE SEQUENCE AFTER WHICH ALL REMAING TERMS LIE IN $V_\varepsilon(L)$



THEY ARE JUST TWO DIFFERENT WAYS OF SAYING THE SAME THING:

"AFTER SOME POINT IN THE SEQUENCE, THE REST OF THE TERMS ARE AS CLOSE TO L AS YOU LIKE."

(THAT POINT DEPENDS ON HOW CLOSE YOU WANT)

LETS DO SOME EXAMPLES.

① $\left\{2 + \frac{(-1)^n}{n}\right\} \to 2.$ $\quad$ [$1,\ 2\tfrac{1}{2},\ 1\tfrac{2}{3},\ 2\tfrac{1}{4},\ 1\tfrac{4}{5},\ \ldots$

PROOF: FIX $\varepsilon > 0$.
TAKE $N > 1/\varepsilon$. THEN WE HAVE FOR ALL

$n > N$, $\left|\left(2 + \frac{(-1)^n}{n}\right) - 2\right| = \left|\frac{(-1)^n}{n}\right| < \frac{1}{N} < \frac{1}{1/\varepsilon} = \varepsilon.$ $\boxed{\varepsilon}$

② $\lim_{n\to\infty} \left\{\frac{1}{\sqrt{n}}\right\}_{n=1}^{\infty} = 0.$

WANT $\frac{1}{\sqrt{n}} < \varepsilon$,
SO $n > 1/\varepsilon^2$.

THIS WORKS...

③ $\lim_{n\to\infty} \{n^2\}$ DNE. [IE DIVERGES]

PROP: THE LIMIT OF A SEQUENCE IS UNIQUE, IF IT EXISTS

Pf: SPOZE $\lim a_n = L$ AND $\lim a_n = K$
WITH $L \neq K$.

IF $L \neq K$, $|L - K| > \varepsilon$ FOR SOME ε ... TROUBLE