513, 2/2/22

1

I REALIZED WE KEEP USING THIS, BUT ONLY HALF OF YOU SAID YOU COULD PROVE IT.

$\boxed{\underline{\text{THM}} \quad \sqrt{2} \text{ IS IRRATIONAL}}$

Pf/ SPOZE $\sqrt{2}$ IS A RATIONAL NUMBER. THEN THERE ARE $p, q \in \mathbb{Z}$ WITH $q > 0$ AND $\gcd(p,q) = 1$, SO THAT

$$\sqrt{2} = p/q.$$

BUT THEN $2 = p^2/q^2$, THAT IS $2q^2 = p^2$.

SO p^2 IS EVEN, THAT IS, $p = 2r$ FOR SOME $r \in \mathbb{Z}$

SINCE p^2 IS EVEN, p IS ALSO EVEN, SINCE THE SQUARE OF AN ODD NUMBER IS ALWAYS ODD

SO

$$2q^2 = (2r)^2 = 4r^2$$

IE

$$q^2 = 2r^2$$

SINCE $r \in \mathbb{Z}$, $r^2 \in \mathbb{Z}$, SO q IS EVEN, IE $q = 2s$.

HENCE BOTH p AND q ARE DIVISIBLE BY 2, A CONTRADICTION. ☒

- REMINDER ABOUT DECIMAL REPRESENTATION (BASE 10)

<u>FIRST</u> $\mathbb{N}$.

- 5162 MEANS $5 \times 10^3 + 1 \cdot 10^2 + 6 \cdot 10 + 2 \cdot 1$.

WE CAN USE OTHER BASES. EG, IN BASE 2

$$\begin{array}{cccccccccccccc} & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & \\ 5162_{10} = & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0_2 \end{array}$$

$$= 2^{12} + 2^{10} + 2^5 + 2^3 + 2$$

$$= 4096 + 1024 + 32 + 8 + 2$$

IN BASE 16, USING ABCDEF FOR #'S ABOVE 10

- $5162 = 142A_{16} = 1 \cdot 16^3 + 4 \cdot 16^2 + 2 \cdot 16 + 10$

OTHER BASES AREN'T JUST CURIOSITIES.

- DIGITAL ELECTRONICS USE BINARY (BASE 2) OR POWERS OF 2.
- BASE 3 WILL BE USEFUL SOON WHEN WE DISCUSS CANTOR SETS
- SOMERIANS & BABYLONIANS USED BASE 60 (WITH SOME BASE 10 MIXED IN)
- MAYANS USED BASE 20

WHAT ABOUT FRACTIONS? (STICK TO BASE 10 FOR NOW)

"WE ALL KNOW"

$.5 = \frac{1}{2}$ $\quad .25 = 1/4$ $\quad$ | $\quad .33333\ldots = 1/3$

$.125 = \frac{1}{8}$

FOR A FINITE DECIMAL, WE JUST MEAN THE SAME THING AS THE WHOLE NUMBER REP, THEN DIVIDE BY A POWER OF 10^d, WHERE THERE ARE d DIGITS AFTER THE DECIMAL; ~~EG~~ EG.

$$123.7826 = \frac{1237826}{10^4}$$

BUT OF COURSE, NOT ALL DECIMALS ARE FINITE, AS $\frac{1}{3}$ SHOWS [WHICH ARE FINITE IN BASE 10? WHY?]

WHEN WE WRITE AN DECIMAL # w/ n DIGITS AFTER POINT WE MEAN

$$w.d_1 d_2 d_3 \cdots d_n = w + \sum_{j=1}^{n} \frac{d_j}{10^j} \qquad \Big|\; d_j \in \{0,1,2,\dots,8,9\},\; w \in \mathbb{N}$$

SO, WHAT IF THIS IS AN INFINITE LIST,

eg $0.3333\ldots$

OBVIOUSLY IT MUST MEAN

$$w + \sum_{j=1}^{\infty} \frac{d_j}{10^j}$$

WHERE $$\sum_{j=1}^{\infty} \frac{d_j}{10^j} = \lim_{n \to \infty} \sum_{j=1}^{n} \frac{d_j}{10^j}.$$

BUT OF COURSE, WE HAVEN'T DISCUSSED OR DEFINED WHAT A LIMIT IS.

FIRST, DOES THIS LIMIT CONVERGE NO MATTER WHAT d_j IS?

CAN DO BETTER OF COURSE

FOR EACH n, WE HAVE

$$w + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n} < w + \frac{10}{10} + \frac{10}{10^2} + \cdots + \frac{10}{10^n}$$

$$= w + \left(1 + \frac{1}{10} + \cdots + \frac{1}{10^{n-1}}\right)$$

$$= w + \left(\frac{1 - 1/10^n}{1 - 1/10}\right) < w + \frac{1}{9/10}$$

$$= w + 19$$

$$< w + 2$$

SO BOUNDED ABOVE, FOR ANY n.

SO BY A_0C, THIS IS A REAL NUMBER NO MATTER WHAT d_j ARE.

IE

THEOREM EVERY DECIMAL (FINITE OR INFINITE) IS A REAL NUMBER

NOTE THAT WE <u>DIDN'T</u> SHOW THAT IT IS A UNIQUE REAL NUMBER (IE WE COULD HAVE TWO THE SAME) NOR WHAT IT ACTUALLY IS.

[NOR DID WE DEFINE LIMIT LETS' DO THAT NOW]

FOR SIMPLICITY, AT FIRST LETS JUST LOOK AT <u>INCREASING</u> <u>SEQUENCES</u>.

THAT IS, WE HAVE A SEQUENCE (OR ∞-LIST INDEXED BY $\mathbb{N}$)

$a_0, a_1, a_2, a_3, a_4, \dots$ (IN THE CASE OF DECIMALS, $a_n = a_{n-1} + \frac{d_n}{10^n}$, A SPECIAL CASE)

FOR NOW, LET'S ASSUME

$$a_0 < a_1 < a_2 < \cdots$$

CONSIDER THE SET $S = \{a_0, a_1, a_2, \dots, a_n, \dots\}$

S IS A SUBSET OF $\mathbb{R}$, SO $\sup S$ EXISTS. BY A_0C

SO WHAT SHOULD WE SAY THE LIMIT OF THIS INCREASING SEQUENCE $a_0, a_1, a_2, a_3 \ldots$ SHOULD BE?

OBVIOUSLY $\sup S$

THAT DOESN'T TELL US WHAT $\sup S$ IS.

SPECIAL CASE OF LIMIT.

HOW CAN WE GET OUR HANDS ON $\sup S$?

RECALL THAT ~~IF L IS AN UPPER~~ $L = \sup S \iff$ FOR EVERY $\varepsilon > 0$, THERE IS SOME $a_n \in S$ WITH $L - \varepsilon < a_n$ (AND L IS AN UPPER BOUND FOR S)

IE $L > a_n + \varepsilon$

~~ALSO~~

SO THE FOLLOWING IS A REASONABLE DEFINITION:

DEF: LET $\{a_n\}$ BE AN INCREASING SEQUENCE OF NUMBERS

$a_0 < a_1 < a_2 < \cdots < a_n < a_{n+1} < \cdots$

THE LIMIT OF $\{a_n\}$ IS L IF

- L IS AN UPPER BOUND FOR $\{a_n\}$, IE $L > a_n$ FOR ALL n
- FOR EVERY $\varepsilon > 0$, THERE IS SOME N SO THAT FOR ALL $n \geq N$, $L - a_n < \varepsilon$