LAST TIME:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

- LET'S ASSUME WE KNOW $\mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; p, q \in \mathbb{Z} \text{ WITH } q > 0, \gcd(p,q) = 1 \right\}$.

- GOAL OF COURSE IS TO UNDERSTAND $\mathbb{R}$ BETTER, AS WELL AS $f: \mathbb{R} \to \mathbb{R}$.

(*) $\mathbb{Q}$ IS A FIELD.

THAT IS, IT IS A SET $\mathbb{F}$ WITH BINARY OPERATIONS $+, \cdot$ SO THAT

- (COMMUTATIVITY) $\forall x, y \in \mathbb{F}$, $x + y = y + x$, $x \cdot y = y \cdot x$.
- (ASSOCIATIVITY) $\forall x, y, z \in \mathbb{F}$ $(x+y)+z = x+(y+z)$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
- (IDENTITY ~~INVERSES~~). $\exists!\, 0 \in \mathbb{F}$, $\exists!\, 1 \in \mathbb{F}$ WITH $0 \neq 1$, AND $\forall x \in \mathbb{F}\ x + 0 = x$, $\forall x \in \mathbb{F}\ x \cdot 1 = x$.

(INVERSES)

- (DISTRIBUTIVITY) $\forall x, y, z \in \mathbb{F}$ $x \cdot (y+z) = x \cdot y + x \cdot z$.

$\forall x, \exists (-x)$ SO THAT $x + (-x) = 0$

$\forall x \neq 0, \exists x^{-1}$ SO THAT $x \cdot x^{-1} = 1$

* $\mathbb{Q}$ is ORDERED. THAT IS, $\exists$ RELATION $\leq$ SO

$\forall x, y, z \in \mathbb{F}$

- AT LEAST ONE OF $x \leq y$ OR $y \leq x$ HOLDS.
- IF $x \leq y$ AND $y \leq x$, THEN $x = y$
- IF $x \leq y$ AND $y \leq z$, THEN $x \leq z$.

WE WRITE $x < y$ IF $x \leq y$ AND $x \neq y$.

AN ORDERED FIELD SATISFIES

$\forall x, y, z$

- $y \leq z \Rightarrow x + y \leq x + z$
- $x \geq 0$ AND $y \geq 0 \Rightarrow x \cdot y \geq 0$.

$\mathbb{R}$ IS A COMPLETE ORDERED FIELD. CONTAINING $\mathbb{Q}$ AS A SUBFIELD.

WHAT DOES THIS MEAN?

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$\mathbb{R}$ is $\mathbb{Q}$ "WITH THE GAPS FILLED IN",

WHATEVER THAT MEANS.

HERE'S ONE WAY:

$\boxed{\text{AXIOM OF COMPLETENESS: EVERY BOUNDED NONEMPTY SET OF REAL NUMBERS HAS A LEAST UPPER BOUND.}}$

SEVERAL ~~MANY~~ APPROACHES. ONE SUCH IS VIA DEDEKIND CUTS

WHICH SEEMS VERY ABSTRACT:

DEF $A \subset \mathbb{Q}$ IS A CUT IF

(C1) $A \neq \emptyset$, $A \neq \mathbb{Q}$

(C2) IF $r \in A$, THEN EVERY $q < r$ IS ALSO IN A

(C3) A HAS NO MAXIMUM ELEMENT. THAT IS,
IF $r \in A$, THEN THERE IS $s \in A$ WITH $r < s$.

~~THEN $\mathbb{R}$~~

DEF: $\mathbb{R}$ IS THE SET OF ALL CUTS IN $\mathbb{Q}$

ANOTHER WAY (DUE TO CANTOR)

IN ESSENCE, WE TAKE CONVERGENT SEQUENCES OF RATIONALS, (CAUCHY SEQUENCES) AND DEFINE $\mathbb{R}$ TO BE THE SET OF LIMITS OF SUCH SEQUENCES.

TROUBLE: ~~CERTAINLY TWO SUCH~~

- WHAT DOES THIS MEAN? USUALLY WE DEFINE CONVERGENCE IN TERMS OF THE LIMIT. BUT IF THE LIMIT ISN'T A THING ~~-~~ WE KNOW??

→ USE CAUCHY SEQUENCES OF RATIONALS..

- STILL, MANY SUCH CAUCHY SEQUENCES CAN HAVE THE SAME LIMIT, SO ACTUALLY EQUIVALENCE CLASSES OF CAUCHY SEQS. ~~IE~~ IE

  ~~CONVER $\{x_n\}$~~, CONVERGENT $\{x_n\}$, $\{y_n\}$ SUCH THAT $(x_n - y_n) \to 0$ AS $n \to \infty$

- ALSO, THINKING OF #'S AS SEQUENCES IS HARD. OR IS IT?

SORT OF ANOTHER WAY

RESTRICT TO SPECIAL SEQUENCES, THAT IS, (INFINITE) DECIMALS.

HERE WE HAVE THE ISSUE THAT $0.99\overline{9} = 1.00\overline{0}$

BUT OK.

IS THIS OK? WHY OR WHY NOT?

WHEN WE SAY "DECIMALS" IT SEEMS WE ARE RESTRICTING TO BASE 10.

SOMETIMES OTHER BASES ARE USEFUL.

LETS MESS WITH THOSE A BIT.