MAT513 Homework 11
Due Monday, May 2

Do *any six* of the problems below. If you also do a seventh problem, it will count as extra credit.

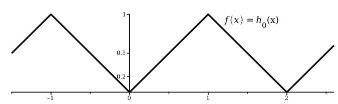
1. Compute the limits below. You can, of course, use your knowledge of standard derivatives from elementary calculus.

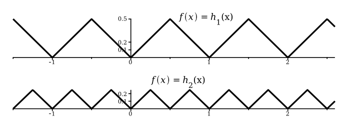
(a) $\lim_{x \to 0} x \ln(1+x)$

(b) $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{\sin x}{x^3} \right)$

2. As discussed in class, for $x \in \mathbb{R}$, define the function

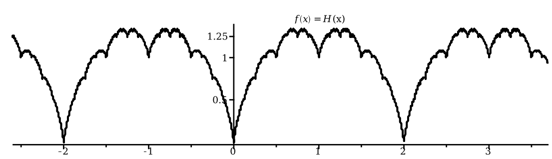
$$h_0(x) = \begin{cases} |x| & \text{if } -1 \le x \le 1 \\ h_0(x-2) & \text{if } x > 1 \\ h_0(x+2) & \text{if } x < -1 \end{cases}$$





and for $n \in \mathbb{N}$, let $h_n(x) = h_0(2^n x)/2^n$.

Then let $H(x) = \sum_{n=0}^{\infty} h_n(x)$.



For each $x \in \mathbb{R}$, $H(x)$ converges absolutely, since each term in the series is nonnegative and $H(x) \le \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$.

Show that $H(x)$ is continuous on $\mathbb{R}$ but is not differentiable at any $x \in \mathbb{R}$.

(Hint: for continuity, observe that h_k is continuous for all k, so any finite sum these is also continuous – you still need to account for the tail of the series, however. To see non-differentiablilty everywhere, show that for any x, there are cusps arbitrarily close to x, and hence there are points of non-differentiablilty in any neighborhood of x.)

3. Let $f(x) = x^2$ on $[\frac{1}{2}, 3]$.

(a) Let $\mathcal{P}$ be the partition $\left\{ \frac{1}{2}, 1, 2, 3 \right\}$ and compute $L(f, \mathcal{P})$ and $U(f, \mathcal{P})$, where L and U are the lower and upper sums with respect to the partition.

(b) Let $\mathcal{Q} = \left\{ \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3 \right\}$ and compute $L(f, \mathcal{Q})$ and $U(f, \mathcal{Q})$.

4. Let f be continuous on $[a,b]$ and suppose that $f(x) \ge 0$ for all $x \in [a,b]$. Prove that if $L(f) = 0$, then $f(x) = 0$ for all $x \in [a,b]$.

5. Prove the **Mean Value Theorem for Integrals**: If f is continuous on $[a,b]$, then there exists $c \in (a,b)$ for which

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

(This value $f(c)$ is called the **average value of *f*** on the interval $[a,b]$.)

6. Assume that functions $u(x)$ and $v(x)$ have continuous derivatives on $[a,b]$. Derive the formula for **integration by parts**:

$$\int_a^b u(t)v'(t)\,dt = \Big(u(b)v(b) - u(a)v(a)\Big) - \int_a^b v(t)u'(t)\,dt.$$

7. Write an explanation of the intuitive heuristic behind the second part of the Fundamental Theorem of Calculus: Let $G(x)$ be the function that measures the area under the graph of g from a to x. Then the derivative of G at x is the height $y = g(x)$, since the approximate change from x to $x+h$ is essentially the area of the rectangle with base h and height y.

You might want to include a relevant picture, and relate this to the explicit statement of the relevant part of the FTC. Give your explanation so that it can be understood by a beginning calculus student.