MAT513 Homework 9
Due Wednesday, April 13

1. Here are several invented definitions which are variations on the definition of continuity. In each case, if you give an example you must justify that it meets the stated criteria.

(a) A function $f\colon \mathbb{R} \to \mathbb{R}$ is **onetinuous** at c if for every $\varepsilon > 0$, we have $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < 1$.
Give an example of a function g that is onetinuous on all of $\mathbb{R}$, and another function h that is continuous at every $c \in \mathbb{R}$, onetinuous at $c = 0$, but not onetinuous at $c = 2$, or explain why no such function can exist.

(b) A function $f\colon \mathbb{R} \to \mathbb{R}$ is **equaltinuous** at c if for every $\varepsilon > 0$, whenever $|x - c| < \varepsilon$ we also have $|f(x) - f(c)| < \varepsilon$.
Give an example of a function f which is not onetinuous at any $c \in \mathbb{R}$, but is equaltinuous at every $c \in \mathbb{R}$, or explain why no such function can exist.

(c) A function $f\colon \mathbb{R} \to \mathbb{R}$ is **lesstinuous** at $c \in \mathbb{R}$ if for every $\varepsilon > 0$, there is a δ with $0 < \delta < \varepsilon$ so that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$.
Find a function f which is lesstinuous on all of $\mathbb{R}$ but is nowhwere equaltinuous, or explain why no such function can exist.

(d) Is every lesstinuous function continuous? Is every continuous function lesstinuous? Explain.

2. Let A and B be subsets of $\mathbb{R}$, with $f\colon A \to B$ and $g\colon B \to \mathbb{R}$. Prove that if f is continuous at $c \in A$ and g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that the set $K = \{\, x \mid f(x) = 0 \,\}$ is a closed set.

4. Suppose that $f\colon [a,b] \to [a,b]$ is continuous. Prove that f has a **fixed point**; that is, that there is a $c \in [a,b]$ so that $f(c) = c$.

5. Observe that if a and b are real numbers, then we can define $\max(a,b) = \dfrac{(a+b)+|a-b|}{2}$; this can readily be extended to a finite set of numbers $\{\, a_1, a_2, \ldots, a_n \,\}$ via

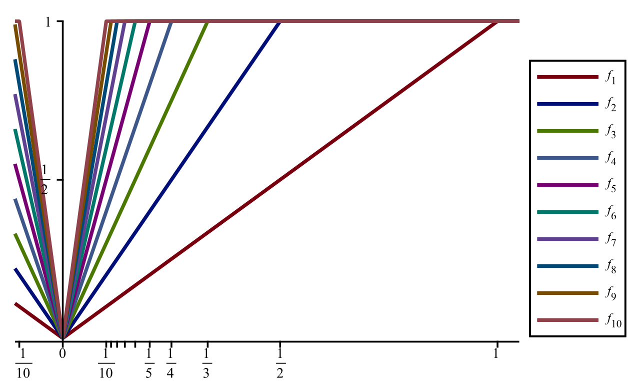
$$\max\{\, a_1, a_2, \ldots, a_n \,\} = \max(a_1, \max\{\, a_2, a_3, \ldots, a_n \,\}).$$

(a) Show that if $f_1, f_2, \ldots, f_n$ are continuous, then $g(x) = \max(f_1(x), f_2(x), \ldots, f_n(x))$ is also continuous.

(b) For each positive integer n, define

$$f_n(x) = \begin{cases} 1 & \text{if } |x| \geq 1/n \\ n|x| & \text{if } |x| < 1/n \end{cases}$$

For each n, $f_n\colon \mathbb{R} \to \mathbb{R}$ is continuous.
Write an explicit formula for the function $h(x) = \sup\{\, f_1(x), f_2(x), f_3(x), \ldots \,\}$.
Is $h(x)$ continuous?



6. Assume that the temperature $T(x)$ of a point x on the equator of the Earth is a continuous function. As a corollary to the Intermediate Value Theorem, at every moment there is a point x on the equator with the property that its antipodal point (the point $-x$ which is immediately opposite it on a line through the center of the Earth) has exactly the same tempertature, that is $T(x) = T(-x)$.

Write a paragraph or two explaining this in a way that it can be understood by a high school student.