MAT513 Homework 8
Due Wednesday, April 6

Do *any four* of the problems below. If you also do a fifth problem, it will count as extra credit.

1. For a function $f \colon A \to \mathbb{R}$ with $c \in \bar{A}$, recall that $\lim\limits_{x \to c} = L$ means that for every $\varepsilon > 0$ there exists $\delta > 0$ so that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$ and $x \in A$.

   Let $\lceil x \rceil$ denote the smallest integer greater than or equal to x (for example, $\lceil 0.5 \rceil = 1 = \lceil 1 \rceil$).

   (a) Suppose we take $\varepsilon = 1$. What is the largest value of δ we can use in the definition of $\lim\limits_{x \to \pi} \lceil x/2 \rceil$?

   (b) Suppose we take $\varepsilon = .01$. What is the largest value of δ we can use in the definition of $\lim\limits_{x \to \pi} \lceil x/2 \rceil$?

   (c) Write a proof, using the definition of limit above, that $\lim\limits_{x \to \pi} \lceil x/2 \rceil = 2$.

   (d) Consider $g(x) = \dfrac{1}{\lceil x \rceil}$. A student makes the (false) claim that $\lim\limits_{x \to 4} g(x) = \dfrac{1}{4}$. Give an explanation of why this cannot be true by exhibiting the largest ε for which there is no δ that satisfies the definition.

2. Calculate each of the limits below or show they do not exist, fully justifying each of your answers. You can use the definition or other facts we know from the definition.

   (a) $\lim\limits_{x \to 2} x^3$

   (b) $\lim\limits_{x \to 2} \dfrac{|x - 2|}{x - 2}$

   (c) $\lim\limits_{x \to 0} f(x)$ where $f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

3. Introductory calculus courses typically refer to the limit of a function *from the right* (or the left) as the limit obtained by "letting x approach a from the right (or left) side".

   Adapt the definition of $\lim\limits_{x \to a} f(x)$ to give a proper definition for the statements

   $$\lim\limits_{x \to a^+} f(x) = L \qquad \text{and} \qquad \lim\limits_{x \to a^-} f(x) = M.$$

   Then prove that

   $$\lim\limits_{x \to a} f(x) = L \qquad \text{if and only if} \qquad \lim\limits_{x \to a^+} f(x) = L = \lim\limits_{x \to a^-} f(x).$$

4. Thomae's function is defined as follows:

   $$T(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{q} & \text{if } x = \frac{p}{q}, \text{ with } p \in \mathbb{Z} \smallsetminus \{0\}, q \in \mathbb{N} \text{ and } \gcd(p, q) = 1 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

   Show that $\lim\limits_{x \to c} T(x) = 0$ where c is any irrational number, but $\lim\limits_{x \to r} T(x)$ does not exist if $r \in \mathbb{Q}$.

5. Recall that a function $f : A \to \mathbb{R}$ is continuous on A if for every $c \in A$ we have $\lim_{x \to c} = f(c)$.

A student says that there is no reason we need to define continutity in terms of limits, and that we can just say that *a function is continuous (on an interval) if we can draw the graph from start to finish without ever picking up our pencil (or crayon).*

Write a paragraph or more responding to the student's claim. Write your answer in such a way that it can be understood by a high-school student.