MAT513 Homework 7

Due Wednesday, March 30

Do *any five* of the problems below. If you also do a sixth problem, it will count as extra credit.

1. In class on 3/23 we showed that if $K \subseteq \mathbb{R}$ is compact, then it must be closed and bounded. Prove the converse: If $K \subseteq \mathbb{R}$ is is closed and bounded, then it is compact.

2. We had the following theorem:

For any $A \subseteq \mathbb{R}$, the closure $\overline{A}$ is a closed set, and is the smallest closed set which contains A.

Provide a proof[†] of this theorem.

3. A notion dual to the closure of a set is the interior of a set. Specifically, given a set E, the **interior of E** is denoted by $\mathring{E}$ and is defined as

$$\mathring{E} = \{ x \in E \mid \text{there exists a neighborhood } V_\varepsilon(x) \subseteq E \}.$$

There is a symmetry between many properties regarding closure and the interior of sets.

(a) We know that a set E is closed if and only if $E = \overline{E}$. Show that a set U is open if and only if $U = \mathring{U}$.

(b) Recall that E^c denotes the complement of E, that is, $E^c = \mathbb{R} \setminus E$.
Show that $\overline{E}^c$ is the interior of E^c. Also show that $(\mathring{E})^c = \overline{E^c}$.

4. Let K and L be compact subsets of $\mathbb{R}$. We can define a distance between the sets K and L as

$$d(K,L) = \inf_{x \in K,\ y \in L} \{ |x - y| \}.$$

(a) Show that if K and L are disjoint compact sets, then $d(K,L) > 0$.

(b) Give an example of disjoint closed sets A and B for which $d(A,B) = 0$.

5. A set $P \subseteq \mathbb{R}$ is called **perfect** if it is closed and contains no isolated points (so every interval $[a,b]$ is a perfect set). A set is $D \subseteq \mathbb{R}$ is **totally disconnected** if, for every x and y in the set D with $x < y$, there is a point $c \notin D$ so that $x < c < y$.

Show that the Cantor set is a perfect set which is totally disconnected.

6. Below are several statements about compact sets; some are true and some are not. Prove the true ones, and give a counterexample for the false ones.

(a) Let K_n be a compact set for each $n \in \mathbb{N}$. Then $K = \bigcup_{n=0}^{\infty} K_n$ must be compact.

(b) Let F_λ be a compact set for each $\lambda \in \Lambda$. Then $F = \bigcap_{\lambda \in \Lambda} F_\lambda$ must be compact.

(c) Let A be an arbitrary subset of $\mathbb{R}$, and let K be compact. Then $A \cap K$ is compact.

(d) Let K be compact and F be closed. Then $K \setminus F = \{ x \in K \mid x \notin F \}$ is open.

[†]Hint for problem 2: Let $L = \{ x \mid x \text{ is a limit point of } A \}$. Recall that $\overline{A} = A \cup L$. It is immediate that $\overline{A}$ contains all its limit points. To show that this is the *smallest* closed set containing A, you need to show that if $x \in \overline{A}$, $A \setminus \{x\}$ is not closed or it does not contain all of A. The crux of the proof is to show that L is closed, that is if x is a limit point of L, then $x \in L$.