MAT513 Homework 6
Due Wednesday, March 23

1. Let $I = \{ x \mid 0 < x < 1 \}$ be the open unit interval $(0,1)$, and let S be the open unit square, that is, $S = \{ (x,y) \mid 0 < x < 1 \text{ and } 0 < y < 1 \} = (0,1) \times (0,1)$.

(a) Find an injective function (that is, a one-to-one function) $f : I \to S$. This should be *very easy*: f does not need to be surjective (onto).

(b) Use the fact that every real number x has a decimal expansion to produce an injective function $g : S \to I$. Is your function g a surjection (onto)?
It might be helpful to remember that every real number which has a "terminating" decimal expansion (such as 0.25) can also be written as an infinite decimal (e.g., $0.2499\overline{9}\cdots$ or $0.2500\overline{0}\cdots$).

As a consequence of the Schröder-Bernstein Theorem (which says that if there are injective functions $f : A \to B$ and $g : B \to A$, then there is a bijective function $h : A \to B$), this shows that the unit interval and the unit square have the same cardinality.

2. A real number $x \in \mathbb{R}$ is called **algebraic** if there are integers $a_0, a_1, a_2, \ldots, a_n$ so that

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0,$$

that is, $x \in \mathbb{R}$ is algebraic if it is a root of a polynomial with integer coefficients (note that rational numbers are algebraic, since each is the root of a degree 1 polynomial). Real numbers which are not algebraic are called **transcendental** numbers.

(a) Show that $\sqrt{2}$ and $\sqrt{3} + \sqrt{2}$ are algebraic.

(b) Fix $n \in \mathbb{N}$ and let A_n be set of algebraic numbers which are roots of polynomials of degree n. Show that each A_n is a countable set. (Hint: the Fundamental Theorem of Algebra is relevant here; you may assume it.)

(c) Prove that the set of algebraic numbers is a countable set.

(d) What is the cardinality of the set of transcendental numbers?

3. In both parts below, justify your answer fully by establishing an bijection between the set in question and a set of known cardinality. (The goal is to establish cardinality, so a bijection is not strictly necessary if your argument is complete.)

(a) Let $\mathcal{F}$ be the set consisting of all functions from $\{ 0, 1 \}$ to $\mathbb{N}$. What is the cardinality of $\mathcal{F}$?

(b) Let $\mathcal{G}$ be the set consisting of all functions from $\mathbb{N}$ to $\{ 0, 1 \}$. What is the cardinality of $\mathcal{G}$?

4. Let $\mathcal{C}$ denote the middle-thirds Cantor set. Prove that

$$\mathcal{C} + \mathcal{C} = \{ x + y \mid x, y \in \mathcal{C} \} = [0, 2] .$$

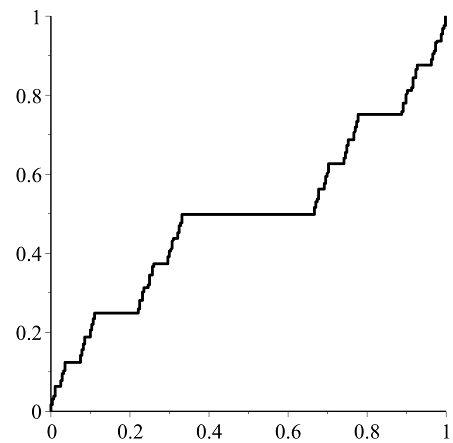
That is, any real number z with $0 \le z \le 2$ can be written as $z = x + y$, where x and y are in $\mathcal{C}$. (The other direction is obvious: since $\mathcal{C} \subset [0,1]$, certainly $0 \le x + y \le 2$.)
This result is illustrated on this web page.

Below is a suggested outline of a proof (you still need to fill in the details).

- Let $\mathcal{C}_n$ be the "level-n Cantor set", that is
$$\mathcal{C}_n = \bigcup_{i=0}^{2^n-1} \left[\frac{2i}{3^n}, \frac{2i+1}{3^n}\right] .$$
- Given any $z \in [0,2]$, observe that there are $x_1 \in \mathcal{C}_1$ and $y_1 \in \mathcal{C}_1$ with $x_1 + y_1 = z$. Show that for any $n \in \mathbb{N}$, there are numbers $x_n \in \mathcal{C}_n$ and $y_n \in \mathcal{C}_n$ with $x_n + y_n = z$.
- The sequences $\{x_n\}$ and $\{y_n\}$ may not converge, but they can be used to construct points $x \in \mathcal{C}$ and $y \in \mathcal{C}$ so that $x + y = z$. Specifically, the sequences $\{x_n\}$ and $\{y_n\}$ must have convergent subsequences (why is this?).
(You can assume that if a sequence of points $c_n \in \mathcal{C}$ converges, then the limit is also in $\mathcal{C}$; this follows from the compactness of $\mathcal{C}$. We haven't yet covered compact sets, so just take it as true for now.)
- This gives the desired result.

5. We discussed how the cardinality of the Cantor set $\mathcal{C}$ can be shown to be the same as $\mathbb{R}$ by constructing a surjective function $f : \mathcal{C} \to [0,1]$. This function f can be extended to a function $F : [0,1] \to [0,1]$ as follows:

- Express x in base 3.
- If the representation contains a 1, replace all digits after the first 1 by a 0.
- Replace any remaining 2s with 1s.
- Interpret the result in base 2. This is $F(x)$.



The resulting function F is called the Cantor Function; an approximation of its graph is shown at right. The graph of the Cantor Function is also the best-known example of a "Devil's Staircase" (in fact, it is often called *the* Devil's Staircase).

In the late 1980s, the composer György Ligeti was was inspired† by the Cantor Function and wrote *L'escalier du diable (The Devil's Staircase)* as the 13th piece in his Études. This work incorporated self-similarity and rhythmic structure echoing the 2/3 patterns in the Cantor Set and the Devil's Staircase.

Listen to a performance of Ligeti's Étude No.13 on YouTube, Spotify, bandcamp, or elsewhere. Do you perceive any relations between the music and the Cantor Set? Does it help your appreciation/understanding of it in any way?

† See pp.51-59 of Lauren Halsey: "An Examination of Rhythmic Practices and Influences in Keyboard Works of György Ligeti". Masters thesis, University of North Carolina Greensboro (2012).