MAT513 Homework 5

Due Wednesday, March 2

1. For each of the following, either give an example of such a sequence, or give an arguement why its existence is impossible.

(a) A sequence that does contains no terms equal to 0 or 1, but has subsequences which converge to each of these.

(b) A monotone sequence that diverges, but has a convergent subsequence.

(c) A sequence that contains subsequences which converge to each point in the infinite set

$$\left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \right\} .$$

(d) An unbounded sequence with a convergent subsequence.

(e) A sequence which is bounded but contains no subsequence that converges.

2. Another made-up definition: Call a sequence $\{a_n\}$ **pseudo-Cauchy** if, for every $\varepsilon > 0$, there exists a $K_\varepsilon \in \mathbb{N}$ such that $|a_{n+1} - a_n| < \varepsilon$ for all $n > K_\varepsilon$.

Either prove that all pseudo-Cauchy sequences are also Cauchy (and hence converge), or give an example of a divergent sequence which is pseudo-Cauchy.

3. Call a sequence $\{x_n\}$ **contractive** if there is some number C with $0 < C < 1$ so that

$$|x_{n+1} - x_n| < C|x_n - x_{n-1}| \quad \text{for all } n \in \mathbb{N} .$$

Show that every contractive sequence is Cauchy.
Hint: it might be useful to use $1 + C + C^2 + \cdots + C^n = (1 - C^{n+1})/(1 - C)$.

4. Prove the Alternating Series Test.
Hint: either show that the sequence of partial sums is Cauchy, use the Nested Intervals Property, or use the Monotone Convergence Theorem on the subsequences $\{s_{2n}\}$ and $\{s_{2n+1}\}$. All three methods work, but give somewhat different proofs.

5. An invented defintion: A series **subverges** if the sequence of partial sums $\{s_n\}$ has a convergent subsequence. For each of the statements below, decide if it is true or false; justify your answer with either a sketch of a proof or a counter example.

(a) If $\{a_n\}$ is bounded, then $\sum a_n$ subverges.

(b) Every convergent series is subvergent.

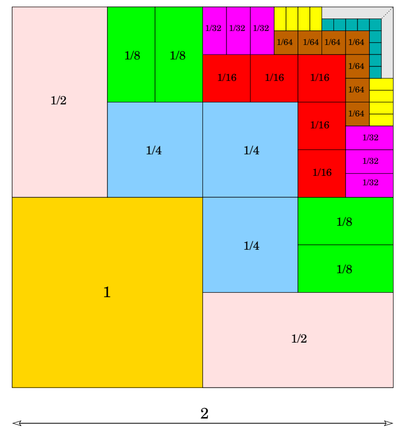
(c) If $\sum |a_n|$ subverges, then $\sum a_n$ also subverges.

(d) If $\sum a_n$ subverges, then $\{a_n\}$ has a convergent subsequence.

6. Consider the picture at right below, a "proof without words" of something. What is being proven?

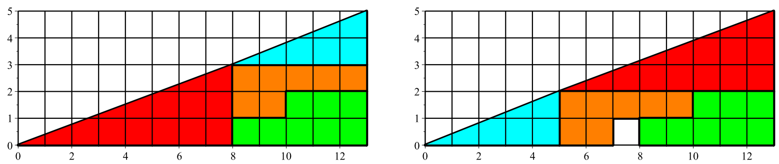
First, write an explanation of what is being demonstrated by this image in a way that can be understood by a student who knows something (not a lot) about infinite series.

Then, discuss whether you think this constitutes a convincing proof. Even if not, is this image helpful? Explain.



Roger B. Nelsen,
Mathematics Magazine **62**
(Dec. 1989), pp.332–333

You might want to consider the image below, a "standard proof that $65/2 = 63/2$", as part of your discussion.



(See also Wikipedia: "Missing square puzzle").