MAT513 Homework 4

Due Wednesday, February 23

1. (**Cesàro Means**) Given a sequence $\{x_n\}$, define a new sequence whose terms are the arithmetic mean of the first n terms. That is, let

$$y_n = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}.$$

(a) Show that if $\{x_n\}$ converges to a limit L, then the sequence of averages $\{y_n\}$ also converges to the same limit.

(b) Give an example where the sequence of averages $\{y_n\}$ converges but the original sequence $\{x_n\}$ does not.

2. (**Calculating square roots**) Consider the sequence defined by $x_1 = 2,\ x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$.

(a) Show that x_n^2 is always greater than 2, and then use this to conclude that $\{x_n\}$ is a decreasing sequence. Then show that $\lim x_n = \sqrt{2}$.

(b) Modify the sequence in part (a) to obtain a sequence that converges to $\sqrt{c}$.

3. An invented definition:
A sequence $\{a_n\}$ is **quasi-increasing** if for all $\varepsilon > 0$, there exists an $K \in \mathbb{N}$ so that whenever $n > m \geq K$ it follows that $a_n > a_m - \varepsilon$.

(a) Give an example of a sequence which is quasi-increasing but is not monotone or eventually monotone.

(b) Give an example of a quasi-increasing sequence that is divergent and not monotone.

(c) Is there an analogue of the Monotone Convergence Theorem for quasi-increasing sequences? That is, suppose $\{a_n\}$ is bounded and quasi-increasing. Must it also converge? If so prove it; if not, give a counter-example.

4. Prove the **Comparison Test for Series:** Let $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$.

- If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.
- If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.

Use the Monotone Convergence Theorem (a bounded monotone sequence has a limit).

5. Use the Cauchy Condensation Test† to show that the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$.
Hint: consider the cases $p \leq 0$ and $p > 0$ separately.

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†**Theorem:** Suppose $\{a_n\}$ satisfies $0 \leq a_{n+1} \leq a_n$ for all n. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.

6. Read the excerpt consisting of pp. 239–249 from chapter 8 of Outliers by Malcom Gladwell at https://www.math.stonybrook.edu/~scott/mat513.spr22/HW/gladwell.pdf

Then write a paragraph or two in response. Is this relevant to teaching mathematics? How so (or why not)?