MAT513 Homework 3
Due Wednesday, February 16

1. Are each of the following possible? If so, give an example. If not, give a proof showing why it is impossible.

(a) The sequence $\{a_n\}$ is divergent and $\{b_n\}$ is convergent, while $\{a_n b_n\}$ converges.

(b) The sequence $\{a_n\}$ is unbounded and $\{b_n\}$ is convergent, but $\{a_n - b_n\}$ is bounded.

2. What happens if we reverse the order of the quantifiers in the definition of convergence? That is, consider the following invented definition:

A sequence $\{x_n\}$ **verconges** *to* L *if there exists* $\varepsilon > 0$ *such that for all* $M \in \mathbb{N}$*, we have* $|x_n - L| < \varepsilon$ *for all* $n \in \mathbb{N}$ *with* $n \geq M$.

Give an example of a sequence which is vercongent. Is there an example of a vercongent sequence which diverges (in the ordinary sense)? Can a sequence verconge to L and also to K with $L \neq K$?

Explain in ordinary English what this definition is describing.

3. Using the definition of convergence of a sequence, prove that each of the following sequences converges to the given limit.

(a) $\left\{\dfrac{2n+1}{5n+4}\right\}_{n=0}^{\infty} \longrightarrow \dfrac{2}{5}$.

(b) $\left\{\dfrac{n^2}{2n^3-1}\right\}_{n=0}^{\infty} \longrightarrow 0$.

4. Prove or disprove (by giving a counterexample) each of the following:

(a) If $\{s_n\}$ converges to S, then $\{|s_n|\} \to |S|$.

(b) If $\{|s_n|\}$ converges, then $\{s_n\}$ is convergent.

(c) $\lim s_n = 0$ if and only if $\lim |s_n| = 0$.

5. Let $P(x)$ be some property that x has.

- A sequence $\{a_n\}$ is said to **eventually** have property P if there exists $M \in \mathbb{N}$ such that $P(a_n)$ holds for all $n \geq M$.
- A sequence $\{a_n\}$ is said to **frequently** have property P if for every $M \in \mathbb{N}$, there exists $n \geq M$ for which $P(a_n)$ holds.

(a) Does the sequence $\{(-1)^n\}$ eventually take on the value 1? Does it frequently take on the value 1?

(b) Which is stronger? That is, does eventually always imply frequently? Does frequently always imply eventually? Give a proof or counterexample.

(c) Rephrase the usual definition of convergence of a sequence using either eventually or frequently (one works, the other doesn't. Which is it?)

6. Write a paragraph or so explaining how the Nested Intervals Property tells us that every (possibly infinite) decimal determines a unique real number (although some real numbers have more than one decimal expansion).