MAT513 Homework 2

Due Wednesday, February 9

1. As we noted in class, some fractions (such as $\frac{1}{2}$ or $\frac{3}{16}$) have finite decimal representations, while others (e.g. $\frac{1}{7}$) must be represented as infinite decimals. Show that any rational p/q has a finite decimal representation (with k digits after the decimal point) if and only if the denominator q is a divisor of 10^k for some $k \in \mathbb{N} \cup \{0\}$.

2. In order to write a repeating decimal that repeats immediately after the decimal point (as in, say $2/7 = 0.285714\overline{285714}\cdots$) as a rational number, the following procedure works.

- Let k be the length of the repeating block, set x to be the repeating decimal, and let b be the k-digit integer in the repeating block. (In the example of $x = 2/7$, we have $k = 6$, $b = 285714$.)
- Observe that $10^k x = b + x$, and consequently $x = b/(10^k - 1)$. (In our example, we obtain $x = 285714/999999$ which simplifies to $x = 2/7$ since both numerator and denominator are divisible by 142857.

Explain how to modify this procedure to work for a repeating decimal that doesn't repeat immediately after the decimal point (as in $0.98647\overline{47}\cdots$) and use it to write $0.98647\overline{47}\cdots$ as a rational number in the form p/q, where p and q have no common divisors.

3. Let $y_1 = 1$ and for each $n \in \mathbb{N}$, let $y_{n+1} = \dfrac{3y_n + 4}{4}$.

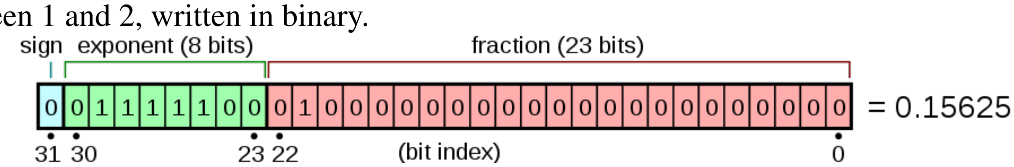
(a) Show that $y_n < 4$ for all $n \in \mathbb{N}$. (Hint: use induction.)

(b) Now show that $y_1, y_2, y_3, \ldots$ forms an increasing sequence.

(c) Conclude that the sequence converges. What is the limit?

4. Most computers represent numbers internally in binary (base 2). Each number is represented in a *word*, which is a block of binary digits (bits) of a given length (word sizes of 32 or 64 bits are most common, but other sizes are also used). Let's assume the wordsize is 32 bits. One bit is reserved for a sign (it is a 1 for a negative). This means we can represent $2^{31} - 1$ positive integers, zero will be the number with all zero bits (and the sign also zero), and then there are 2^{31} possibilities where the sign bit is on. Consequently, integers between -2^{31} and $2^{31} - 1$ can be represented[†].

To represent non-integer numbers, a different "floating point" format saves another 8 bits for an exponent, and the remaining 23 bits are used to represent the fractional part of a number between 1 and 2, written in binary.



The above example represents the binary number $0.00101_2 = 1.01_2 \times 2^{-3} = 0.15625_{10}$. Exponents have 127 added to them, so in the example above, the exponent 1111100_2 corresponds to -3.

Which rational numbers can be represented EXACTLY using the "single precision" 32-bit floating point numbers described above, with a floating-point exponent of zero?[‡] Explain your

[†]The representation of negative numbers is typically different from positive ones, using "twos-complement" notation. There is no -0, so there is an "extra" negative number.

[‡]Note that unlike for 32-bit integers, the only difference between a positive and negative number is changing the sign bit.

answer with a clear justification. (Note that the number $1/3$ cannot be represented exactly in floating point, nor can $1/10$.)

5. There is no problem number 5. And the cake was a lie.

6. Consider Zeno's paradox of Achilles and the Tortoise; a version[§] is given below.

The Tortoise challenged Achilles to a race, claiming that he would win as long as Achilles gave him a small head start. Achilles laughed at this, for of course he was a mighty warrior and swift of foot, whereas the Tortoise was heavy and slow.

"How big a head start do you need?" he asked the Tortoise with a smile.

"Ten meters," the latter replied.

Achilles laughed louder than ever. "You will surely lose, my friend, in that case," he told the Tortoise, "but let us race, if you wish it."

"On the contrary," said the Tortoise, "I will win, and I can prove it to you by a simple argument."

"Go on then," Achilles replied, with less confidence than he felt before. He knew he was the superior athlete, but he also knew the Tortoise had the sharper wits, and he had lost many a bewildering argument with him before this.

"Suppose," began the Tortoise, "that you give me a 10-meter head start. Would you say that you could cover that 10 meters between us very quickly?"

"Very quickly," Achilles affirmed.

"And in that time, how far should I have gone, do you think?"

"Perhaps a meter— no more," said Achilles after a moment's thought.

"Very well," replied the Tortoise, "so now there is a meter between us. And you would catch up that distance very quickly?"

"Very quickly indeed!"

"And yet, in that time I shall have gone a little way farther, so that now you must catch that distance up, yes?"

"Ye-es," said Achilles slowly.

"And while you are doing so, I shall have gone a little way farther, so that you must then catch up the new distance," the Tortoise continued smoothly.

Achilles said nothing.

"And so you see, in each moment you must be catching up the distance between us, and yet I — at the same time — will be adding a new distance, however small, for you to catch up again."

"Indeed, it must be so," said Achilles wearily.

[§] from Smith, B. Sidney: "Zeno's Paradox of the Tortoise and Achilles", Platonic Realms Interactive Mathematics Encyclopedia is given below. (Or watch a video version on YouTube.)

"And so you can never catch up," the Tortoise concluded sympathetically.

"You are right, as always," said Achilles sadly — and conceded the race.

Write a paragraph or so explaining how the idea of a convergent sequence resolves the apparent paradox. Suppose $A(t)$ represents the position of Achilles at time t, and $T(t)$ represents that of the Tortoise. From the above story, Achilles moves ten times as fast as the Tortoise. What can you say about the sequence of times $\{t_n\}$ given by $A(t_n) = T(t_{n-1})$, where $t_0 = 0$, $A(0) = 0$, $T(0) = 10$?

Not to hand in: Consider the paradox called Thomson's Lamp, devised in 1954 by James F. Thomson.

There is a lamp which has a toggle switch: Hitting the switch once turns the lamp on and hitting it again turns the lamp off.

Suppose you do the following: Turn the lamp on, then wait 1 minute. Turn the lamp off, wait 30 seconds. Then turn the lamp on and wait 15 seconds. Continue in this way: after a time period exactly half of the previous one, hit the switch.

At the end of two minutes, is the lamp on or off?

Compare this to the Ross-Littlewood paradox, where an infinite number of balls are added and removed from a vase (and I won't describe here, but you can look it up). Infinity is tricksy, even tricksier than hobbitses!