MAT513 Homework 1

Due Wednesday, February 2

1. Use induction to prove that $5^{2n}-1$ is divisible by 8 for all $n \in \mathbb{N}$.
(Hint: observe that $5^{2k+2}-1 = 5^{2k+2}-5^2+5^2-1$.)

2. We saw in class that the set $\{0,1\}$ is a field when $+$ and $\cdot$ are computed modulo† 2, and we noted that something analogous is true for a set $\{0,1,\ldots,p-1\}$ with any prime p (i.e., $p = 2,3,5,7,11,\ldots$). Show that the set $\{0,1,2,3\}$ is *not* a field when arithmetic is done modulo 4.

3. Since $\mathbb{Q}$ is a field, we know that the sum and product of two rational numbers is also rational. Prove that if t is irrational and $a \in \mathbb{Q}$ with $a \neq 0$, then $a+t$ and at are both also irrational.

4. Let a and b be two real numbers a and b with $a < b$.

- **(a)** Show that there is an irrational number t for which $a < t < b$.
  Hint: consider the numbers $a-\sqrt{2}$ and $b-\sqrt{2}$ and use the previous exercise.‡
- **(b)** Now show that there is also a rational number r with $a < r < b$.

5. Let $\mathbb{F}$ be the set of all rational functions, that is, the set

$$\mathbb{F} = \left\{ f(x) \mid f(x) = \frac{p(x)}{q(x)} \text{ where } p(x) \text{ and } q(x) \text{ are polynomials with coefficients in } \mathbb{R} \right\}$$

Using the usual rules for addition and multiplication of rational functions, $\mathbb{F}$ can easily be shown to be a field.

Given a rational function $f \in \mathbb{F}$, we can define f to be *positive* whenever the leading coefficients (the coefficient of the highest power) of the numerator and denominator of f have the same sign. For example, the functions $\dfrac{3x^2+4x-1}{7x^5+5}$ and $\dfrac{2-5x^5}{17-2x^2}$ are both positive, but $\dfrac{1-2x^2}{x^4-15}$ is not.

We can define an ordering on $\mathbb{F}$ by saying that $f > g$ exactly when $f-g$ is positive in the above sense. This means $\mathbb{F}$ is an ordered field. Observe that we can view $\mathbb{N}$ as a subset of F by viewing $n \in \mathbb{N}$ as the constant function $f(x) = n$.

- **(a)** Show by giving explicit counterexamples, that the field $\mathbb{F}$ does not have the Archimedian property§ . That is, find $f \in \mathbb{F}$ so that $f > n$ for all $n \in \mathbb{N}$, and also, find $g \in \mathbb{F}$ for which $0 < g < 1/n$ for all $n \in \mathbb{N}$.
- **(b)** Show that $\mathbb{F}$ doesn't satisfy the completeness axiom by finding a subset $B \subset \mathbb{F}$ which is bounded above, but has no least upper bound. Justify your answer fully.

6. Write a paragraph or two responding to the following statement: "Because of the density of $\mathbb{Q}$ in $\mathbb{R}$, every measurement corresponding to a real number can be approximated by a rational number to within the precision of any device we can use measure it with. Thus, in science or engineering, it suffices to work only with real numbers that are fractions (or finite decimals, if you prefer)."

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†That is, we take the result to be the remainder after we divide it 2.

‡(You might find it easier to do part (b) first, but this is not necessary.)

§ The Archimedian property says that (1) if x and y are real numbers, then there is $n \in \mathbb{N}$ such that $nx > y$, and (2) if $y > 0$ is real, there is $n \in \mathbb{N}$ such that $1/n < y$. We will discuss this in class on Jan 31.