# Lectures on the Syzygies and Geometry of Algebraic Varieties

Preliminary Draft

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# Preface

The object of these lectures is to survey a body of work centered around the syzygies and geometry of algebraic varieties.

Classically the equations defining projective varieties attracted a certain amount of attention, e.g. Petri's theorem on canonical curves or in Mumford's papers on abelian varieties from the 1960s. However two developments in the early 1980s revitalized and redirected such questions. First, as computer-assisted computations began to be practical, Castelnuovo–Mumford regularity came into focus as a measure of algebraic complexity. More importantly, Mark Green [88] realized that classical results on defining equations were just the beginning of a much more general picture involving higher syzygies. The past forty years have seen a great deal of activity on these matters, touching on a wide range of topics in algebraic geometry. The time seemed ripe for a summary of some of this research, and this is what we have tried to produce here.

The material in these lectures sits at the interface of commutative algebra and algebraic geometry. While we hope that these notes might have something to offer an algebraically inclined reader, our outlook is primarily geometric. So for example we largely ignore the vast commutative algebra literature dealing with ideals and modules of combinatorial origin. On the other hand, we do include an introduction to Boij–Söderberg theory: this is a celebrated recent development in commutative algebra that is not yet widely known in the algebro-geometric community. The specific topics and questions we take up are previewed in the introductory lecture.

Our aim has been to give an invitation to the area rather than a comprehensive overview, and this has guided a number of design decisions. We provide reasonably complete proofs of the most central results, but for many specialized or more advanced topics we often make simplifying assumptions or omit demonstrations altogether. In the same spirit we relegate to brief notes, or skip over entirely, many topics that would belong in a more exhaustive account. We also work exclusively with varieties and rings defined over the complex numbers: this is necessary for some of the material, and it seemed preferable to maintain uniform hypotheses throughout. We trust that in the more algebraic discussions – for example Lecture 1 on Hilbert's syzygy theorem – it will be clear that this assumption is extraneous. We have tried to pitch the presentation at roughly the level of an intermediate or advanced graduate class. A first course in algebraic geometry should provide sufficient background for much of the material, although we do assume facility with coherent cohomology.

Several related book-length surveys have previously appeared, most notably Eisenbud's text [60] and the notes [7] of Aprodu and Nagel. Compared to [60] the present lectures have a more geometric focus. Our perspective is closer to [7], but we cover substantially more ground. Peeva's volume [157] contains an account of the algebraic theory.

Concerning matters of organization, each lecture consists of several sections, many of which are further divided into subsections. Statements and equations are numbered consecutively within each section. So for example Mumford's theorem on regularity, Theorem 3.1.2, is the second enumerated statement in Section 3.1 of Lecture 3; it happens to appear in Section 3.1.A. The finest structural unit is the non-numbered `TeX` "paragraph" (such as the paragraph of Acknowledgements below). Each lecture ends with a brief section of Notes giving references and sources that did not appear in the body of the text.[1] A summary of Notations and Conventions appears at the end of the Introduction.

**Acknowledgements.** First and foremost, we wish to recognize our debt to Mark Green. A good deal of the material discussed here originates with him, and the second author in particular has been greatly influenced by a close collaboration with Green in the 1980s. We have also profited from many discussions over the years with David Eisenbud, Daniel Erman, Gabi Farkas and Frank Schreyer.

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**Concerning this draft.** This is a preliminary draft of this volume. A few sections, including the Notes to several chapters, have yet to be written: they are indicated by a diamond $\diamond$. We also intend to flesh out some other material. The authors welcome corrections, comments, suggestions and complaints.

[1] Some of these have yet to be written.

# Contents

# Introduction

In this introductory lecture, we preview informally some of the results and questions to which these notes will be devoted.

We start by recalling Hilbert's theorem on syzygies. Consider the polynomial ring

$$S \;=\; \mathbf{C}[z_0, \ldots, z_r]$$

in $r+1$ variables over the complex numbers, and let $E$ be a finitely generated graded $S$-module. Choose homogeneous generators

$$m_1\,,\,\ldots\,,\,m_b \;\in\; E,$$

with $m_j$ of degree $a_{0,j}$, i.e. $m_j \in E_{a_{0,j}}$. These determine a surjective mapping

$$\oplus\, S(-a_{0,j}) \xrightarrow{\;\varepsilon\;} E \longrightarrow 0 \quad , \quad (f_1, \ldots, f_b) \mapsto \Sigma\, f_i \cdot m_i.$$

Next, choose generators for $\ker(\varepsilon)$, and continue step by step to build a *graded free resolution* $P_\bullet$ of E:

$$\begin{array}{ccccccccc} \ldots \longrightarrow & \oplus\, S(-a_{2,j}) & \xrightarrow{\delta_2} & \oplus\, S(-a_{1,j}) & \xrightarrow{\delta_1} & \oplus\, S(-a_{0,j}) & \xrightarrow{\varepsilon} & E \longrightarrow 0 & \\ & \| & & \| & & \| & & & (*) \\ & P_2 & & P_1 & & P_0 & & & \end{array}$$

If at each stage we choose the generators minimally in a suitable sense, we can arrange that the matrices defining the $\delta_i$ do not have non-zero constant entries. In this case it is elementary that $P_\bullet$ is unique up to isomorphism.

The story begins with a remarkable result discovered by Hilbert:

**Theorem 1.** *The process just described terminates after at most $r+1$ steps. In other words, any finitely generated $S$-module $E$ has a minimal graded free resolution $(*)$ of length at most $r+1$, unique up to isomorphism.*

Lecture 1 is devoted to the proof of Hilbert's theorem, as well as a discussion of some of the invariants of $E$ – such as the length of the resolution – that one reads off of (*).

**Example 2** (**Resolution of the twisted quartic curve**)**.** Here is a concrete example of the construction of a free resolution. The computations can be hard to carry out by hand even in simple cases, and the calculations that follow were made with the computer program `Macaulay2`. Let $C \subseteq \mathbf{P}^3$ be the twisted quartic curve arising as the image of the embedding

$$\mathbf{P}^1 \hookrightarrow \mathbf{P}^3 \quad , \quad [\, s\, ,\, t\,] \mapsto [\, s^4\, ,\, s^3t\, ,\, st^3\, ,\, t^4\,].$$

(Note that the monomial $s^2t^2$ is missing!) We take $E$ to be the ideal $I_C \subseteq S = \mathbf{C}[x, y, z, w]$ of $C$. This ideal is generated by a quadric and three cubics:

$$Q \;=\; yz - xw \;,\; F_1 \;=\; z^3 - yw^2 \;,\; F_2 \;=\; xz^2 - y^2w \;,\; F_3 \;=\; y^3 - x^2z.$$

The module of syzygies among these generators is spanned by the four relations:

$$\begin{array}{ccccccccc} z^2Q & - & yF_1 & + & wF_2 & & & = & 0 \\ ywQ & - & xF_1 & + & zF_2 & & & = & 0 \\ xzQ & & & - & yF_2 & - & wF_3 & = & 0 \\ y^2Q & & & - & xF_2 & - & zF_3 & = & 0 \end{array}$$

Up to scalars there is a single relation, with degree one coefficients, among the rows of the coefficient matrix, namely:

$$x \cdot (z^2, -y, w, 0) \;-\; y \cdot (yw, -x, z, 0) \;-\; z \cdot (xz, 0, -y, -w) \;+\; w \cdot (y^2, 0, -x, -z) \;=\; 0.$$

Thus the resoluton of $E = I_C$ takes the form

$$0 \longleftarrow I_C \longleftarrow S(-2) \oplus S(-3)^3 \longleftarrow S(-4)^4 \longleftarrow S(-5) \longleftarrow 0.$$

It is interesting to note that in "nice" cases – such as with Example 10 below – the resolution of the ideal of a curve in $\mathbf{P}^r$ has length $r - 2$. As we shall see, the failure of the twisted quartic to be embedded by a complete linear series precludes this from happening here. □

Theorem 1 raises a number of questions. To begin with:

**Question 3.** Is there any structure to the totality of all resolutions of some large class of $S$-modules?

Until quite recently, one wouldn't have imagined that there is anything sensible to be said in this direction. However a remarkable conjecture of Boij and Söderberg – subsequently proven by Eisenbud and Schreyer following contributions from Fløystad and Weyman – shows that Question 3 actually has a very nice answer. We present an introduction to this theory in Lecture 2.

There are very few modules $E$ for which one can write down the resolution $P_\bullet$ explicitly. Instead, much of our attention will be focused on questions relating to the grading of $P_\bullet$. This involves two sorts of invariants. To begin with, the *Betti numbers* $b_{i,j} = b_{i,j}(E)$ are defined by writing

$$P_i \ = \ \oplus_j \, S(-j)^{b_{i,j}}.$$

In other words, $b_{i,j}(E)$ counts the number of generators in degree $j$ of the $i^{\text{th}}$ module of syzygies of $E$. So determining the grading of $P_\bullet$ amounts to answering:

**Question 4.** Which Betti numbers $b_{i,j}(E)$ are non-zero?

The *Castelnuovo–Mumford regularity* of $E$ is a weaker – and hence more accessible – invariant of the grading of $P_\bullet$. It is defined as:

$$\operatorname{reg}(E) \ =_{\text{def}} \ \max \big\{ \, j - i \ \big| \ b_{i,j}(E) \neq 0 \, \big\}.$$

Thus

$$\operatorname{reg}(E) \ \leq m \quad \Longleftrightarrow \quad \begin{array}{l} E \text{ is generated in degrees } \leq m \text{ and for every } i \text{ the } i^{\text{th}} \\ \text{module of syzygies of } E \text{ is generated in degrees } \leq i+m. \end{array}$$

**Example 5.** Let $C \subseteq \mathbf{P}^3$ be the twisted quartic curve from Example 2. Then $\operatorname{reg}(I_C) = 3$. On the other hand, suppose that $X \subseteq \mathbf{P}^3$ is the curve of degree $k^2$ arising as the complete intersection of two surfaces $F_1, F_2$ of degree $k$. Then the ideal $I_X$ is generated by $F_1$ and $F_2$, and the syzyies among them are spanned by the Koszul relation

$$F_2 \cdot F_1 \ - \ F_1 \cdot F_2 \ = \ 0.$$

Thus the resolution of $I_X$ has the shape $0 \longrightarrow S(-2k) \longrightarrow S(-k)^2 \longrightarrow I_X \longrightarrow 0$, and one sees that $\operatorname{reg}(I_X) = 2k - 1$. □

Lecture 3 is devoted to an overview of regularity and some of its applications. One thinks of regularity as measuring the algebraic complexity of $E$: algorithms for explicitly computing syzygies proceed degree by degree, so $\operatorname{reg}(E)$ controls their overall running time. Hence a guiding problem here is:

**Question 6.** What sort of upper bounds can one give on the Casteluovo–Mumford regularity $\operatorname{reg}(E)$?

Having introduced these invariants, for what modules $E$ should we study them? In commutative algebra, there is a huge literature devoted to the syzygetic properties of ideals of combinatorial origin. For example a simplicial complex $\Delta$ determines a *Stanley–Reisner* monomial ideal $I_\Delta$ whose syzygies are computed via the homology of $\Delta$ and its subcomplexes. Or again, there are various sorts of ideals that one can associate to a graph $G$, and it is interesting to relate homological properties of these ideals to the combinatorics of $G$. We will give a small sampling of results along these lines, but they are not our main concern.

Rather we will focus on modules arising geometrically, most notably the ideals defining "nice" varieties in projective space. Turning first to regularity, consider a complex projective variety

$$X \subseteq \mathbf{P}^r \quad \text{with} \quad \dim X = n,$$

and denote by $I = I_X \subseteq S$ its homogeneous ideal. We set $\operatorname{reg}(X) = \operatorname{reg}(I_X)$, so $\operatorname{reg}(X)$ measures the "algebraic complexity" of $X$. A result of Mumford interprets this invariant in terms of classical geometric quantities. Specifically, the regularity of $X$ satisfies $\operatorname{reg}(X) \leq m$ if and only if:

$(i)$. Hypersurfaces of degree $m-1$ trace out a complete linear series on $X$, i.e. the restriction

$$\rho_{m-1} : H^0\big(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(m-1)\big) \longrightarrow H^0\big(X, \mathcal{O}_X(m-1)\big)$$

is surjective; and

$(ii)$. $H^i\big(X, \mathcal{O}_X(m-i-1)\big) \;=\; 0 \quad \text{for} \quad i > 0.$

The question then is to bound $\operatorname{reg}(X)$ in terms of accessible geometric or algebraic invariants of $X$.

Lecture 4 is devoted to results in this direction. Leaving the precise statements for later, we can summarize the picture as

**Meta-Theorem 7.** *If $X$ is non-singular, then the regularity $\operatorname{reg}(X)$ of $X$ is bounded linearly in its invariants.*

More explicitly, best possible results are known in terms of the degrees of the defining equations of $X$. There are also linear bounds involving the degree of $X$ itself, although in this case the (presumably!) optimal statements have only been established in small dimensions.

By contrast, using examples of Mayr and Mayer from complexity theory, Bayer and Stillman observed in the early 1980s that the regularity of arbitrary homogeneous ideals can exhibit doubly exponential regularity growth. However the schemes witnessing this behavior were highly non-reduced. For a long time, it was unclear what to expect for reduced and irreducible, but possibly singular, varieties. This question was recently settled by McCullough and Peeva, who showed how to construct prime ideals encoding the pathological behavior of the Mayr–Mayer–Bayer–Stillman (or any other) examples. Lecture 4 also contains a sketch of these constructions.

Let us turn now to the finer invariants involving syzygies. Here the natural questions involve embeddings arising from complete linear series. So consider a smooth complex projective variety $X$ of dimension $n$, and let $L$ be a very ample line bundle on $X$ defining an embedding

$$\phi_L : X \subseteq \mathbf{P}H^0(X, L) = \mathbf{P}^{r(L)}.$$

In other words, we take a basis $s_0, \dots, s_r \in \Gamma\big(X, L\big)$, and set $\phi_L(x) = [s_0(x), \dots, s_r(x)]$. Thus $r(L) = h^0(L) - 1$, and we are working over the polynomial ring

$$S \ = \ \mathrm{Sym}\, H^0\big(X, L\big).$$

(In practice we will usually ask that $L$ satisfy a suitable positivity condition.)

Now consider

$$E \ = \ E_L \ = \ \oplus\, H^0\big(X, L^{\otimes m}\big),$$

viewed as an $S$-module. Observe that by construction $E_1 = S_1$. It will often happen that $L$ is *normally generated*, meaning that the natural maps

$$\mathrm{Sym}^k H^0\big(X, L\big) \longrightarrow H^0\big(X, L^{\otimes k}\big)$$

are surjective for every $k$. In this case $E$ is generated in degree 1, and hence $E = S/I_X$, where $I_X \subseteq S$ is the homogeneous ideal of $X$ in $\mathbf{P}H^0(L)$. Knowing the syzygies of $E$ is then equivalent to having the resolution of $I_X$.

The prototypical results involve curves, so we'll start there. Suppose that $X$ is a smooth complex projective curve of genus $g$, and let $L$ be a very ample non-special line bundle on $X$ of degree $d$. Thus $r = d - g$ and $L$ defines an embedding $X \subseteq \mathbf{P}^{d-g}$. The classical facts are summarized in:

**Proposition 8.** *If $d \geq 2g + 1$ then $L$ is normally generated, and if $d \geq 2g + 2$ then the homogeneous ideal $I_{X/\mathbf{P}^{d-g}}$ is generated by quadrics.*

But until the early 1980s, this was the end of the story: nobody thought to ask what happens for example when $d \geq 2g + 3$.

It was Mark Green who realized that the classical picture is a special case of a more general result involving higher syzygies:

**Theorem 9.** *Assume that*

$$d \ = \ \deg(L) \ \geq \ 2g + 1 + k.$$

*Then for $k \geq 1$ the first $k$ modules of syzygies of $E = S/I_X$ are generated entirely in degree $k + 1$.*

(A slightly more technical statement also handles the case $k = 0$.) When $k = 1$ this recovers the fact that $I_X$ is generated by quadrics. When $k = 2$, the assertion is that if one chooses degree two generators $q_\alpha \in I_X$, then the module of syzygies among the $q_\alpha$ is spanned by relations of the form

$$\Sigma\, \ell_\alpha \cdot q_\alpha \ = \ 0 \ \text{ where the } \ell_\alpha \text{ are polynomials of degree 1.}$$

Several proofs of the Theorem appear in Lecture 5, which is devoted to Koszul cohomology: this is the fundamental tool used to study graded pieces of syzygy modules.

**Example 10.** It is instructive to see concretely how Theorem 9 works in the two simplest cases.

(a). Take $X = \mathbf{P}^1$ and consider the embedding of $X \subseteq \mathbf{P}^3$ by the complete linear series of degree 3, i.e.

$$X = \mathbf{P}^1 \hookrightarrow \mathbf{P}^3 \text{ via } [s,t] \mapsto [s^3, s^2t, st^2, t^3].$$

Thus we are in the case $k = 2$ of Theorem 9. It is classical (and elementary) that one can describe $X$ as the locus in $\mathbf{P}^3$ where the $2 \times 3$ matrix of linear forms

$$\begin{bmatrix} x & y & z \\ y & z & w \end{bmatrix}$$

has rank $\leq 1$. Thus $I_X$ is generated by the three quadrics

$$Q_1 = yw - z^2 \ , \ Q_2 = xw - yz \ , \ Q_3 = xz - y^2.$$

By repeating each of the rows of the matrix and expanding out the resulting determinant, one finds that these satisfy two relations with linear coefficients:

$$\begin{aligned} x \cdot Q_1 \ - \ y \cdot Q_2 \ + \ z \cdot Q_3 \ &= \ 0 \\ y \cdot Q_1 \ - \ z \cdot Q_2 \ + \ w \cdot Q_3 \ &= \ 0 \end{aligned}$$

So the resolution of $S/I_X$ has the form

$$0 \longleftarrow S/I_X \longleftarrow S \longleftarrow S(-2)^3 \longleftarrow S(-3)^2 \longleftarrow 0,$$

as predicted by Green's statement. □

(b). Now let $X$ be a curve of genus 1 and $L$ a line bundle of degree $4 = 2g + 1 + 1$, defining an embedding $X \subset \mathbf{P}^3$. In this case $X$ is the complete intersection of two quadrics $Q_1$ and $Q_2$, so $S/I_X$ is resolved by the Koszul complex:

$$0 \longleftarrow S/I_X \longleftarrow S \longleftarrow S(-2)^2 \longleftarrow S(-4) \longleftarrow 0.$$

So we see that the first syzygies of $S/I_X$ are generated in degree 2, as Green predicts, but the second syzygies are generated in degree 4. □

Green's theorem for curves raises a number of questions. To begin with:

**Question 11.** What is the analogue of Theorem 9 for embeddings of other varieties $X$?

There was a great deal of work in this direction in the 1980s and 1990s, involving for instance Veronese embeddings of $\mathbf{P}^n$, abelian varieties, and adjoint-type bundles on any smooth variety. These are discussed in Lecture 6. The picture that emerges is that – just as in the case of curves – the first $k$ modules of syzygies of such an embedding are generated in the lowest possible degree $k+1$ for a value $k$ depending linearly on the positivity of the embedding line bundle.

As we shall see shortly, Green's theorem determines the grading of all but a fixed number of terms in the resolution of a curve of large degree. However in higher dimension the situation is quite different. Namely, suppose that $X$ has dimension $n$, and consider for $d \gg 0$ the line bundle $L_d = \mathcal{O}_X(dA)$ where $A$ is an ample divisor on $X$. Then $L_d$ defines an embedding

$$X \hookrightarrow \mathbf{P}^{r_d} \quad \text{where } r_d \sim \mathrm{C} \cdot d^n$$

for some constant $C$ depending on $X$ and $A$, and the resolution of the resulting module $S/I_X$ has length $\approx r_d$. Thus when $n \geq 2$, the linearity statements established in connection with Question 11 ignore most of the syzygy modules that occur. This raises:

**Question 12.** What is the asymptotic behavior of the syzygies of $L_d$ as $d \to \infty$?

Question 12 is the subject of Lecture 8. It turns out that for most values of $i$, the $i^{\text{th}}$ module of syzygies of the ideal of $X$ has generators in $n$ different degrees. In fact, write $b_{i,j}(L_d)$ for the Betti numbers of the module $S/I_X$ under the embedding defined by $L_d$.

**Theorem 13** ([52], [155])**.** *Fix an integer $q$ with $1 \leq q \leq n$. There exist positive constants $C_1, C_2$ having the property that for $d \gg 0$:*

$$b_{i,i+q}(L_d) \neq 0$$

*for every value of $i$ in the range*

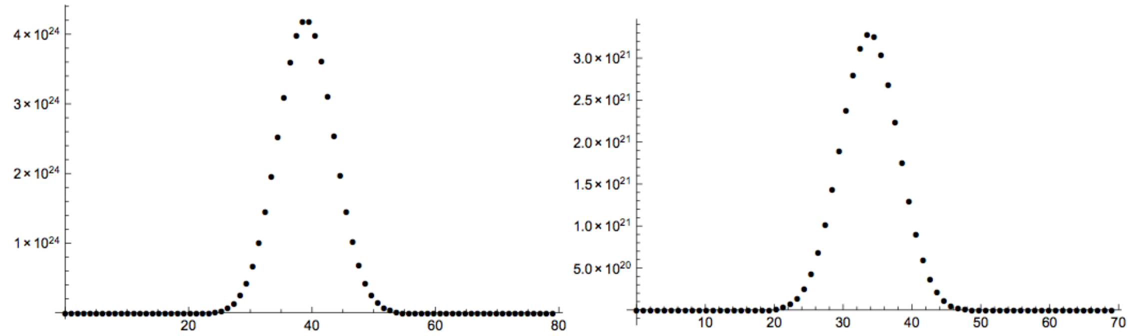
$$C_1 \cdot d^{q-1} \;\leq\; i \;\leq r_d \;-\; C_2 \cdot d^{n-1}. \tag{*}$$

*Moreover the lower bound in $(*)$ is sharp: there is a constant $C_3 > 0$ such that*

$$b_{i,i+q}(L_d) \;=\; 0 \quad \textit{for} \quad i \;\leq\; C_3 \cdot d^{q-1}$$

It is also natural to wonder about the Betti numbers themselves. For degree $d \gg 0$ embeddings of curves, one finds that the $b_{i,i+1}$ are approximated by the binomial coefficients $\binom{d-g}{i}$. Hence when suitably normalized they approach a Gaussian distribution as $d \to \infty$: see Figure 1, which plots the Betti numbers of degree 80 embeddings of curves of genus 0 and 10. One conjectures that a similar picture holds for the non-vanishing Betti numbers $b_{i,i+q}$ in all dimensions, but this is already unknown for the Veronese embeddings of $\mathbf{P}^2$.

In Lecture 7 we return to curves and discuss some results and conjectures of a more delicate nature. Consider again the embedding $X \subseteq \mathbf{P}^r$ of a curve of genus $g \geq 2$ determined by a line bundle $L = L_d$ of degree $d \gg 0$, so that $r = d - g$. The statements discussed so far hold uniformly for all curves of given genus. By contrast, the most interesting questions bring the instrinsic geometry of $X$ into play. Specifically, the resolution $P_\bullet$ of $S/I_X$ has length




$g = 0$ $g = 10$

Figure 1: Betti numbers of curves of degree 80

$r-1$, while Theorem 9 determines the grading of the first $d-2g-1 = r-1-g$ terms of $P_\bullet$. In other words, if $P_\bullet$ is the resolution of the embedding of $X$ by a line bundle $L_d$ of degree $d \gg 0$, then only the modules

$$P_{r-g}\,,\, P_{r-g+1}\,,\,\ldots\,,\, P_{r-1}$$

might have generators in more than one degree. It turns out that what actually happens is controlled by the *gonality* $\operatorname{gon}(X)$ of $X$, i.e. the least degree of a branched covering $X \longrightarrow \mathbf{P}^1$.

**Theorem 14** ([53]). *Assume that $d = \deg(L) \gg 0$. Then $P_{r-i}$ is generated in a single degree if and only if $i \leq \operatorname{gon}(X)$. In particular, one can read off the gonality of a curve from the grading of the resolution associated to any one line bundle bundle of sufficiently large degree.*

The proof, which is surprisingly quick, involves the geometry of vector bundles on the symmetric products of $X$.

Finally, we consider canonical curves. Assuming still that $X$ is a curve of genus $g \geq 2$, recall that the canonical bundle $\omega_X = \Omega^1_X$ is globally generated and defines the *canonical mapping*

$$\phi : X \longrightarrow \mathbf{P}^{g-1}$$

of $X$. This is an embedding unless $X$ is hyperelliptic, and classical results of Noether and Petri assert:

(a). (Noether). A non-hyperelliptic canonical curve $X \subseteq \mathbf{P}^{g-1}$ is projectively normal; and

(b). (Petri). The homogeneous ideal $I_{X/\mathbf{P}^{g-1}}$ of a non-hyperelliptic canonical curve fails to be generated by quadrics if and only if $X$ is trigonal – ie $\operatorname{gon}(X) = 3$ – or else is a smooth plane quintic.

In thinking about how the classical statements might generalize to higher syzygies, Green realized that the picture should be governed by the so-called *Clifford index* $\operatorname{Cliff}(X)$ of $X$ in much the same way that the gonality gon$(X)$ controls the the syzygies of large degree embeddings. (In fact, the statement of Theorem 14 was motivated by Conjecture 15.) Without giving the precise definition here, suffice it to say that

$$\begin{aligned} \mathrm{Cliff}(X) &= 0 \iff X \text{ is hyperelliptic;} \\ \mathrm{Cliff}(X) &= 1 \iff X \text{ is trigonal or a smooth plane quintic.} \end{aligned}$$

Allowing ourselves a somewhat informal statement, one arrives at arguably the most famous open question in the subject:

**Conjecture 15** (**Green's conjecture**). *One can read off* Cliff$(X)$ *from the grading of the resolution $P_\bullet$ of the homogeneous ideal $I_{X/\mathbf{P}^{g-1}}$ of $X$ in its canonical embedding.*

(More precisely, the conjecture predicts that for a non-hyperelliptic curve $X$, Cliff$(X)$ is equal to the least integer $i$ such that $P_{i-1}$ has generators in more than one degree.) It is relatively elementary to show that curves of low Clifford index have special sysygies; as always, the difficulty is to show that algebraic peculiarities can be accounted for geometrically.

While Conjecture 15 remains open as of this writing, Voisin [190, 192] some years ago established the most important special case:

**Theorem 16** (**Voisin**). *Green's conjecture holds for general curves, i.e. for all curves parameterized by a Zariski-dense open set of the moduli space $\mathfrak{M}_g$.*

Voisin's argument involved an interesting new way of seeing syzygies on the Hilbert scheme of points of a variety, combined with some very involved calculations on the Hilbert schemes of $K3$ surfaces. Recently, Kemeny found a very simple proof (in the even genus case) working directly with vector bundles on a K3 surface. Lecture 7 is devoted to a discussion of these results and conjectures for curves.

Depending on the reader's interests, various paths are possible through these notes. Lectures 1, 3.1 and 5 are essential, but most of the other material can be sampled, non-sequentially, according to taste. The later sections of Lecture 6 and Lecture 7 assume a little more background in algebraic geometry than required elsewhere. Lecture 2, on Boij–Söderberg theory, is not used in other chapters except for a brief appearance in Section 8.4. We do occasionally draw on Kodaira-type vanishing theorems, but we include statements and references for convenience.

# Notation and Conventions

In conclusion, we fix notation and conventions.

1. As indicated in the Preface, we work exclusively over the complex numbers. A variety is a reduced (and usually irreducble) **C**-scheme. A scheme is a possibly non-reduced algebraic **C**-scheme.

2. Given a vector space or vector bundle $V$, $\mathbf{P}(V)$ denotes the projective space or bundle of one-dimensional *quotients* of $V$. The $k^{\text{th}}$ symmetric product of $V$ is written $\mathrm{Sym}^k(V)$ or sometimes just $S^kV$.

3. By a "sheaf" on a variety or scheme $X$, we understand a coherent algebraic sheaf on $X$.

4. If $X$ and $Y$ are varieties or schemes, we often write without further comment

$$\mathrm{pr}_1 : X \times Y \longrightarrow X \quad , \quad \mathrm{pr}_2 : X \times Y \longrightarrow Y$$

for the two projections. When $\mathcal{F}$ and $\mathcal{G}$ are sheaves on $X$ and $Y$ respectively, we denote by

$$\mathcal{F} \boxtimes \mathcal{G} \ =_{\text{def}} \ \mathrm{pr}_1^*\mathcal{F} \otimes \mathrm{pr}_2^*\mathcal{G}$$

their exterior product on $X \times Y$. On occasion we use $D \dotplus E$ for the corresponding exterior sum of divisors.

5. When $X \subseteq \mathbf{P}^r$ is a subvariety or subscheme of projective space, we write $\mathcal{O}_X(1) = \mathcal{O}_{\mathbf{P}^r}(1)|X$ for the restriction to $X$ of the hyperplane line bundle on $\mathbf{P}^r$.

6. Given a smooth projective variety $X$, we denote by $K_X$ a canonical divisor of $X$, and by $\omega_X$ its canonical bundle.

7. If $V$ is a complex vector space and $X$ is a variety, we denote by $V_X = V \otimes_{\mathbf{C}} \mathcal{O}_X$ the trivial vector bundle on $X$ with fibre $V$.

8. Given an $\mathbf{R}$-valued function $f(d)$ of a natural number $d \in \mathbf{N}$ we say that

$$f \ \in \ \Theta(d^q)$$

if there exist positive real numbers $C_1, C_2 > 0$ such that

$$C_1 \cdot d^q \ \leq \ f(d) \ \leq C_2 \cdot d^q$$

for all sufficiently large $d$. We write $f(d) \sim C \cdot d^q$ if

$$\lim_{d \to \infty} \frac{f(d)}{d^q} \ = \ C.$$

# Lecture 1

# Hilbert's Theorem on Syzygies

To set the stage, we devote this first lecture to a discussion of Hilbert's theorem on syzygies and related matters. Hilbert's result asserts the existence of free resolutions of a module, the most interesting point being the finiteness of these resolutions in the non-singular situation. We begin by outlining the proof in the local setting, where the algebraic ideas are particularly transparent. In the second section, we recall (without proof) some basic definitions and facts about the homological invariants associated to a module over a local ring. Finally, we extend the discussion to graded resolutions over the polynomial ring and to sheaves on projective space.

## 1.1 The local setting

The present section is devoted to proving that a finitely generated module over a regular local ring has an essentially unique finite minimal free resolution. The analogous statement holds for graded modules over a polynomial ring, and that setting will eventually become our main focus. While the essential ideas are the same in both situations, the graded case requires an additional layer of book-keeping. Therefore it seems preferable to start locally. This also allows for easy examples of infinite resolutions over some non-regular rings.

### 1.1.A Set-up and statement.

Fix a Noetherian local ring $(A, \mathfrak{m})$ of dimension $n$. In keeping with our convention of working geometrically over the complex numbers, we will assume for concreteness that we are in one of the following situations:

- $A = \mathcal{O}_x X$ is the local ring of an $n$-dimensional complex algebraic variety at a point $x \in X$;

- $A = \widehat{\mathcal{O}_x X}$ is the completed local ring of $X$ at $x$; or
- $A = \mathbf{C}\{z_1, \ldots, z_n\}$ is the ring of convergent power series with complex coefficients.

Thus the residue field $\mathbf{k} = A/\mathfrak{m}$ is a copy of $\mathbf{C}$, but we write $\mathbf{k}$ to emphasize its structure as an $A$-module. The reader should also keep in mind that everything we say in this lecture actually holds in much greater generality, often with no change in the proofs.

Now consider a finitely generated $A$-module $E$. A classical idea is to describe $E$ by means of generators and relations, which amounts to writing it as the cokernel of a map of free $A$-modules:

$$A^{b_1} \xrightarrow{\delta_1} A^{b_0} \longrightarrow E \longrightarrow 0.$$

However it is hard to read off invariants of $E$ from such a presentation. Hilbert realized that it is much better to next choose generators for the kernel of $\delta_1$, and then continue step by step to build a *free resolution* of $E$, i.e. a long exact sequence

$$\ldots \longrightarrow A^{b_2} \xrightarrow{\delta_2} A^{b_1} \xrightarrow{\delta_1} A^{b_0} \xrightarrow{\varepsilon} E \longrightarrow 0 \tag{1.1.1}$$

of free $A$-modules of finite rank.

There are then two natural questions to ask. First, does the process terminate at some point, i.e. is $\ker(\delta_\ell)$ is already free for some $\ell$? Second, to what extent is such a resolution unique?

Let us start with the latter issue. Without further hypotheses, nothing guarantees any uniqueness in (1.1.1). For example one could introduce an extraneous generator and immediately kill it with a relation. This would lead to a unit appearing as an entry in the matrix describing the relevant map in the resolution. To rule this sort of thing out, we make:

**Definition 1.1.1.** The free resolution (1.1.1) is called *minimal* if all of the entries in the matrix describing each $\delta_i$ lie in the maximal ideal $\mathfrak{m}$. In other words, one asks that

$$\mathrm{im}(\delta_i) \subseteq \mathfrak{m} \cdot A^{b_{i-1}} \tag{1.1.2}$$

for every index $i > 0$. □

We remark that this is the essential place where the hypothesis that $A$ be local enters the picture: there is no natural notion of of minimality for a morphism of modules over a non-local (and non-graded) ring.

We outline in the next subsection the proof of the elementary:

**Proposition 1.1.2 (Existence and uniqueness of minimal resolutions).** *Any finitely generated $A$-module $E$ admits a (possibly infinite) minimal free resolution*

$$P_\bullet : \qquad \ldots \longrightarrow A^{b_2} \xrightarrow{\delta_2} A^{b_1} \xrightarrow{\delta_1} A^{b_0} \xrightarrow{\varepsilon} E \longrightarrow 0, \tag{1.1.3}$$

*which is unique up to isomorphism.*

In particular, the *Betti numbers* $b_i = b_i(E)$ are therefore invariants of $E$: we will discuss these below.

The more subtle point is that in the non-singular setting these resolutions are necessarily finite. Specifically, recall that $A$ is *regular* if the maximal ideal $\mathfrak{m}$ can be generated by $n = \dim A$ elements. In our situation this means that $A$ is either the local ring $A = \mathcal{O}_x X$ at a *smooth* point $x \in X$, its completion the formal power series ring $\mathbf{C}[[z_1, \ldots, z_n]]$, or the ring of convergent power series. For these rings, the facts we require from the general theory will be elementary or evident.

Hilbert's remarkable discovery was that when one works over a *regular* local ring, the step-by-step construction of resolutions automatically terminates: for some $\ell \leq \dim(A) - 1$, $\ker \delta_\ell$ is already free.

**Theorem 1.1.3 (Hilbert's Syzygy Theorem).** *Assuming that $A$ is regular of dimension $n$, $E$ admits a minimal free resolution of length at most $n$. In other words, there exists a long exact sequence*

$$0 \longrightarrow A^{b_n} \xrightarrow{\delta_n} \ldots \xrightarrow{\delta_3} A^{b_2} \xrightarrow{\delta_2} A^{b_1} \xrightarrow{\delta_1} A^{b_0} \longrightarrow E \longrightarrow 0 \tag{1.1.4}$$

*of maps satisfying the minimality condition* (1.1.2).

(We allow here the possibility that some of the $b_i = 0$, so that the length of the resolution might be $< n$.) The proof of Theorem 1.1.3 appears in §1.1.C.

**Example 1.1.4.** Let $A = \mathbf{C}[[x, y]]$, fix an integer $a \geq 1$, and consider

$$E \;=\; \mathfrak{m}^a \;=\; (x^a, x^{a-1}y, \ldots, xy^{a-1}, y^a).$$

Writing $e_0, \ldots, e_a$ for the indicated generators of $\mathfrak{m}^a$, the syzygies among the $e_i$ are spanned by the relations

$$y \cdot e_i - x \cdot e_{i+1} \;=\; 0 \quad (0 \leq i \leq a-1).$$

Thus the miminal resolution of $E$ has the form:

$$0 \longrightarrow A^a \xrightarrow{\begin{pmatrix} y & 0 & \ldots & 0 \\ -x & y & \ldots & 0 \\ & & \vdots & \\ 0 & 0 & \ldots & y \\ 0 & 0 & \ldots & -x \end{pmatrix}} A^{a+1} \longrightarrow \mathfrak{m}^a \longrightarrow 0. \tag{1.1.5}$$

Note that this resolution has length one even though $A$ has dimension 2. On the other hand, by splicing the exact sequence $0 \longrightarrow \mathfrak{m}^a \longrightarrow A \longrightarrow A/\mathfrak{m}^a \longrightarrow 0$ onto the right of (1.1.5) one arrives at a length two resolution for the $A$-module $A/\mathfrak{m}^a$. These observations are explained by a theorem of Auslander and Buchsbaum (Corollary 1.2.4). □

### 1.1.B Existence and uniqueness.

We start by sketching (somewhat informally) the existence and uniqueness assertions of Proposition 1.1.2.

The first point to observe is that it is elementary to build a (possibly infinite) minimal resolution $P_\bullet$ of $E$. Specifically, choose to begin with elements $e_1, \ldots, e_{b_0} \in E$ whose residues form a basis of the $\mathbf{k}$-vector space $E/\mathfrak{m}E$. These are called *minimal generators* of $E$, and it follows from Nakayama's Lemma that they do in fact generate $E$. We use the $e_j$ to define a surjective map $\varepsilon : A^{b_0} \longrightarrow E$. Next, pick minimal generators of $\ker(\varepsilon)$ to produce a presentation

$$A^{b_1} \xrightarrow{\delta_1} A^{b_0} \xrightarrow{\varepsilon} E \longrightarrow 0.$$

By construction, $\delta_1$ determines the zero map $\mathbf{k}^{b_1} \xrightarrow{0} \mathbf{k}^{b_0}$ after tensoring by $\mathbf{k} = A/\mathfrak{m}$, and it follows that $\delta_1$ satisfies the condition of Definition 1.1.1. Continue step by step in the same manner to construct the required minimal resolution. Note that in general there is no reason to suppose that the process terminates: see Example 1.1.7

We next outline why a minimal resolution (1.1.1) is unique up to isomorphism. In fact, suppose given a second minimal resolution

$$P'_\bullet : \quad \ldots \longrightarrow A^{b'_2} \xrightarrow{\delta'_2} A^{b'_1} \xrightarrow{\delta'_1} A^{b'_0} \xrightarrow{\varepsilon'} E \longrightarrow 0 \tag{1.1.6}$$

Then in the first place $b_0 = \dim_{\mathbf{k}} E/\mathfrak{m}E = b'_0$. Moreover one constructs in the evident way a diagram

$$\begin{array}{ccc} A^{b'_0} & \xrightarrow{\varepsilon'} & E \\ {\scriptstyle u_0}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ A^{b_0} & \xrightarrow[\varepsilon]{} & E, \end{array}$$

where $u_0$ is an isomorphism (mod $\mathfrak{m}$). By Nakayama this implies that $u_0$ itself is an isomorphism, and again one continues in this fashion to build step by step an isomorphism $P_\bullet \cong P'_\bullet$.

This discussion shows that Proposition 1.1.2 is essentially formal. The finiteness of minimal resolutions in the case that $A$ is regular is more interesting. A number of approaches are known, including constructive ones in the (not quite local) graded case [44, Chapter 6.2]. Arguably the quickest is homological in nature. This is the path we will follow.

For this, recall that if $R$ is a commutative ring, and if $M, N$ are any $R$-modules, then then one can form the modules $\mathrm{Tor}_i^R(M, N)$. These may be defined by starting with a projective resolution $P_\bullet \xrightarrow{\varepsilon} M$ of $M$, and taking

$$\mathrm{Tor}_i^R(M, N) \ = \ H_i\big(P_\bullet \otimes_R N\big).$$

Up to isomorphism these are independent of the choice of resolution. Thus $\mathrm{Tor}_0(M, N) = M \otimes N$, and the higher $\mathrm{Tor}_i$ are the derived functors of $\otimes$. It follows from the definition that if $N$ is annihilated by an ideal $\mathfrak{a} \subseteq R$ then so are the Tor's. The most important fact for our purposes is the *symmetry of Tor*, namely the isomorphism

$$\operatorname{Tor}_i^R(M,N) \;\cong\; \operatorname{Tor}_i^R(N,M).$$

Very concretely, what this means is that we can alternatively compute $\operatorname{Tor}_i^R(M,N)$ by starting with a projective resolution $Q_\bullet \longrightarrow N$, and then $\operatorname{Tor}_i^R(M,N) = H_i\big(M \otimes Q_\bullet\big)$.

Now return to our finitely generated module over a local ring $(A,\mathfrak{m})$. Consider the modules $\operatorname{Tor}_i^A(E,\mathbf{k})$, where as always $\mathbf{k} = A/\mathfrak{m}$. These are finite dimensional vector spaces over $\mathbf{k}$. We compute them by tensoring the (conceivably infinite) minimal resolution (1.1.3) by $\mathbf{k} = A/\mathfrak{m}$ and taking the cohomology of the resulting complex. But thanks to minimality, the differentials in $P_\bullet \otimes \mathbf{k}$ are the zero maps. Therefore

$$b_i \;=\; \dim_{\mathbf{k}} \operatorname{Tor}_i^A(E,\mathbf{k}). \tag{1.1.7}$$

Theorem 1.1.3 is hence a consequence of

**Theorem 1.1.5.** *Let $(A,\mathfrak{m})$ be a Noetherian regular local ring of dimension $n$, and let $E$ be any finitely generated $A$-module. Then*

$$\operatorname{Tor}_i^A(E,\mathbf{k}) \;=\; 0 \quad \textit{for } i > n.$$

The proof of Theorem 1.1.5 is outlined in the next subsection. The key point will be to compute the Tor's in question starting from a free resolution of $\mathbf{k}$.

We conclude this subsection with a few examples.

**Example 1.1.6.** Assuming that $A$ is regular of dimension $n$, suppose given a possibly non-minimal or infinite resolution (1.1.1) of the module $E$. Then $\ker(\delta_j)$ is a free $A$-module for every $j \geq n-1$. (In fact, chasing through (1.1.1) shows that

$$\operatorname{Tor}_i^A\big(\ker(\delta_j),\mathbf{k}\big) \;=\; \operatorname{Tor}_{i+j+1}^A(E,\mathbf{k}).$$

so the assertion follows from 1.1.5.) □

**Example 1.1.7 (Infinite resolutions).** If $A$ is not regular, then it need not be – and in fact never is – the case that every finitely generated $A$-module $E$ has a finite free resolution (compare Remark 1.2.5). The simplest illustration is obtained by resolving the residue field over the ring of dual numbers $\mathbf{C}[[x]]/(x^2)$. For a more interesting example, consider the formal local ring

$$A \;=\; \mathbf{C}[[x,y]]/(y^2 - x^3)$$

of a cuspidal curve. Writing $\bar{x}, \bar{y}$ for the images of $x, y$ in $A$, it is an amusing exercise to check that the residue field $\mathbf{k} = A/\mathfrak{m}$ has the infinite periodic resolution

$$\cdots \xrightarrow{\cdot B} A^2 \xrightarrow{\cdot B} A^2 \xrightarrow{\cdot B} A^2 \xrightarrow{(\bar{x},\bar{y})} A \longrightarrow \mathbf{k} \longrightarrow 0,$$

where $B$ is the $2 \times 2$ matrix

$$B = \begin{pmatrix} \bar{y} & \bar{x}^2 \\ -\bar{x} & -\bar{y} \end{pmatrix}.$$

This is an illustration of a general result of Eisenbud [57] concerning resolutions over hypersurface rings: see Example 1.2.15 below. □

**Example 1.1.8 (Betti numbers).** The ranks of the terms in a minimal resolution of $M$ are called the *Betti numbers* $b_i(M)$ of $M$. One has

$$b_i(E) = \dim_{\mathbf{k}} \mathrm{Tor}_i^A(E, \mathbf{k}) = \dim_{\mathbf{k}} \mathrm{Ext}_A^i(E, \mathbf{k}). \qquad (*)$$

So for example, if $A$ is the formal local ring of a cusp as in the previous example, then $b_0(\mathbf{k}) = 1$ while $b_i(\mathbf{k}) = 2$ for all $i \geq 1$. (For the second equality in (*), note that the differentials in the complex $\mathrm{Hom}(P_\bullet, \mathbf{k})$ also vanish thanks to minimality.)

## 1.1.C The Koszul complex and vanishing of Tor.

The key to proving Theorem 1.1.5 is to construct explicity the minimal resolution of the residue field $\mathbf{k}$. This is given by the Koszul complex associated to generators of $\mathfrak{m}$. We start with some remarks concerning these complexes and regular sequences of elements of $A$. .

Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $n$, and suppose given any elements

$$x_1, \ldots, x_\ell \in \mathfrak{m}.$$

The *Koszul complex* $K_\bullet(x) = K_\bullet(x_1, \ldots, x_\ell)$ associated to these elements is a free minimal complex of length $\ell$. Concretely, start with a free $A$-module $L$ of rank $\ell$, with basis $e_1, \ldots, e_\ell$. Then put $K_p(x) = \Lambda^p L$, with the differential $\delta : \Lambda^p L \longrightarrow \Lambda^{p-1} L$ determined by the rule

$$\delta\big(e_{i_1} \wedge \ldots \wedge e_{i_p}\big) = \sum_{k=1}^{p} (-1)^{k+1} x_{i_k} \cdot e_{i_1} \wedge \ldots \wedge \widehat{e_{i_k}} \wedge \ldots \wedge e_{i_p}.$$

More usefully, $K_\bullet(x)$ may be identified with the $\ell$-fold tensor product

$$(A \xrightarrow{\cdot x_1} A) \otimes \ldots \otimes (A \xrightarrow{\cdot x_\ell} A) \qquad (1.1.8)$$

of the two-term complexes arising from multiplication by the $x_i$. From this last description one sees that $H_0(K_\bullet) = A/(x_1, \ldots, x_\ell)$. Example 1.1.11 indicates some further properties of these complexes.

In general there is no reason that $K_\bullet(x)$ should be acyclic: for instance, we have not assumed that the $x_i$ are distinct. However recall that $x_1, \ldots, x_\ell$ are said to form a *regular sequence* if multiplication by $x_i$ determines an injective mapping

$$A/\big(x_1, \ldots, x_{i-1}\big) \xrightarrow{\cdot x_i} A/\big(x_1, \ldots, x_{i-1}\big)$$

for every $1 \leq i \leq \ell$. A rather elementary induction proves:

**Lemma 1.1.9.** *If $x_1, \ldots, x_\ell \in \mathfrak{m}$ form a regular sequence, then the Koszul complex $K_\bullet(x)$ is acyclic.*

We refer to [59, Chapter 17], [32, Chapter 1.6] or [175, Chapter IV.A] for details. Example 1.1.11 presents a sketch.

Now suppose that $A$ is regular of dimension $n$, and choose elements

$$z_1, \ldots, z_n \in A$$

that generate $\mathfrak{m}$. Then:

**Proposition 1.1.10.** *The elements $z_1, \ldots, z_n$ form a regular sequence.* □

The assertion is clear in the main cases of interest enumerated above. See [59, Chapter 10.3], [32, Chapter 2.2] for the proof for arbitrary regular local rings.

Theorem 1.1.5 – and with it Theorem 1.1.3 – now follows at once:

*Proof of Theorem 1.1.5.* Assuming that $(A, \mathfrak{m})$ is regular of dimension $n$, consider the Koszul complex $K_\bullet(z)$ associated to generators $z_1, \ldots, z_n \in \mathfrak{m}$. Lemma 1.1.9 and Proposition 1.1.10 imply that $K_\bullet(z_1, \ldots, z_n)$ gives a length $n$ minimal resolution of $\mathbf{k} = A/\mathfrak{m}$ having the shape:

$$0 \longrightarrow A \longrightarrow A^n \longrightarrow \ldots \longrightarrow A^{\binom{n}{2}} \longrightarrow A^n \longrightarrow A \longrightarrow \mathbf{k} \longrightarrow 0. \tag{1.1.9}$$

On the other hand, by the symmetry of Tor:

$$\mathrm{Tor}_i^A\big(E, \mathbf{k}\big) \ = \ H_i\big(E \otimes K_\bullet(z_1, \ldots, z_n)\big),$$

and the group on the right evidently vanishes for $i > n$ since the Koszul complex has length $n$. □

**Example 1.1.11** (**Koszul Complexes**)**.** Let $(A, \mathfrak{m})$ be a Noetherian local ring, and $E$ a finitely generated $A$-module. Given a sequence $x$ of elements $x_1, \ldots, x_\ell \in \mathfrak{m}$, set $K_\bullet(x, E) = K_\bullet(x) \otimes E$. The description (1.1.8) of Koszul complexes as tensor products shows that the $K_\bullet(x, E)$ behave very simply with respect to taking subsequences of $x$. Specifically, denote by $x'$ the truncated sequence $x_1, \ldots, x_{\ell-1} \in \mathfrak{m}$. Then:

(i). There is an exact sequence of complexes

$$0 \longrightarrow K_\bullet(x', E) \longrightarrow K_\bullet(x, E) \longrightarrow K_\bullet(x', M)[-1] \longrightarrow 0.$$

(ii). In the resulting long exact sequence of homology groups

$$\ldots \xrightarrow{\pm x_\ell} H_p(K_\bullet(x', E)) \longrightarrow H_p(K_\bullet(x, E)) \longrightarrow H_{p-1}\big(K_\bullet(x', E)\big) \xrightarrow{\pm x_\ell} H_{p-1}\big(K_\bullet(x', E)\big) \longrightarrow \ldots$$

the connecting homomorphisms $H_{p-1}\big(K_\bullet(x', E)\big) \to H_{p-1}\big(K_\bullet((x', E)\big)$ are given by multiplication by $\pm x_\ell$.

One can deduce the acyclicity Lemma 1.1.9 from (ii). See [32] for further information. □

## 1.2 Some Local Invariants

In this section we recall, largely without proof, some classical results about homological invariants attached to a finitely generated module over a local ring. In particular, the theorem of Auslander and Buchsbaum (Theorem 1.2.3) computes the length of a resolution in cases when it is finite.

### 1.2.A Depth and the theorem of Auslander and Buchsbaum

As in the previous section, let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $n$, which we do not for the time being require to be regular. Recall that the *depth* $\operatorname{depth}(A)$ of $A$ is the maxmal length of a regular sequence in $\mathfrak{m}$. More generally, given a non-zero finitely generated $A$-module $E$, the depth of $E$ is the length of any maximal $\mathfrak{m}$-sequence in $E$. In other words, $\operatorname{depth}(E)$ is the largest integer $d$ for which there exist elements $x_1, \ldots, x_d \in \mathfrak{m}$ having the property that multiplication by $x_i$ defines an injective map

$$E/(x_1, \ldots, x_{i-1})E \xrightarrow{\cdot x_i} E/(x_1, \ldots, x_{i-1})E$$

for every $1 \leq i \leq d$. One has the bounds

$$\operatorname{depth}(A) \leq \dim(A) \quad \text{and} \quad \operatorname{depth}(E) \leq \dim(E), \tag{1.2.1}$$

where $\dim(E)$ denotes the dimension of the ring $A/\operatorname{Ann}(E)$.

**Example 1.2.1.** Let $A = \mathbf{C}[[x_1, \ldots, x_n]]$, and let $E = (x_1, \ldots, x_\ell)$ be the ideal generated by the first $\ell$ variables. Then $\operatorname{depth}(E) = (n+1) - \ell$. (The elements $x_\ell, x_{\ell+1}, \ldots, x_n$ form a regular sequence for $E$.) □

**Remark 1.2.2 (Homological interpretations of depth).** There are several equivalent computations of the depth of a non-zero finitely generated $A$-module $E$. First,

$$\operatorname{depth}(E) = \min\left\{ i \mid \operatorname{Ext}^i_A(\mathbf{k}, E) \neq 0 \right\}. \tag{1.2.2}$$

In the same vein,

$$\operatorname{depth}(E) = \min\left\{ i \mid H^i_{\mathfrak{m}}(E) \neq 0 \right\}, \tag{1.2.3}$$

where $H^i_{\mathfrak{m}}(E)$ denotes the local cohomology of $E$ supported at the maximal ideal. Finally, choose generators $x_1, \ldots, x_t \in \mathfrak{m}$. Then:

$$\operatorname{depth}(E) = t - \max\left\{ i \mid H_i\big(K_\bullet(x) \otimes E\big) \neq 0 \right\}. \tag{1.2.4}$$

The first characterization (1.2.2) is a result of Rees: see [32, Theorem 1.2.8]. For the second and third, we refer to [32, Theorem 3.5.7] and [32, Theorem 1.6.17]. □

Consider now a minimal free resolution (1.1.3) of $E$. The *projective dimension* $\mathrm{pd}(E)$ of $E$ is defined to be the length of the resolution (with $\mathrm{pd}(E) = \infty$ if the resolution is infinite):

$$\mathrm{pd}(E) \ = \ \max\{ i \mid b_i(E) \neq 0 \}\,.$$

Equivalently, $\mathrm{pd}(E)$ (if finite) is the least integer $p$ such that $\mathrm{Tor}_i^A(E, E') = 0$ for all $i > p$ and all $A$-modules $E'$.

A fundamental result of Auslander and Buchsbaum computes the projective dimension of $E$ in terms of its depth:

**Theorem 1.2.3** (**Auslander–Buchsbaum**). *Let $E$ be a finitely generated $A$-module of finite projective dimension. Then*

$$\mathrm{pd}(E) + \mathrm{depth}(E) \ = \ \mathrm{depth}(A).$$

The theorem is proved by an induction on the invariants in question: see [32, Theorem 1.3.3] or [59, Chapter 19.3].

Our main focus is on resolutions over a regular local ring. In this case $\mathrm{depth}(A) = \dim(A)$, and every finitely generated module has finite projective dimension. Hence:

**Corollary 1.2.4.** *Assume that $A$ is regular of dimension $n$. Then*

$$\mathrm{pd}(E) \ = \ n - \mathrm{depth}(E)$$

*for every finitely generated $A$-module $E$*

The Corollary also follows directly from (1.2.4).

**Remark 1.2.5** (**The Auslander–Buschbaum–Serre characterization of regular local rings**). Auslander–Buchsbaum and Serre proved that if $A$ is a Noetherian local ring with the property that $\mathrm{pd}(E) < \infty$ for every finitely generated $A$-module $E$, then in fact $A$ must be regular. See for example [59, Chapter 19.3] for a detailed account. □

**Example 1.2.6.** Let $A = \mathbf{C}[[x, y, z]]$, and denote by $E_1$ and $E_2$ the cokernel and kernel respectively of the mapping $A^3 \longrightarrow A^3$ given by the matrix

$$\begin{pmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{pmatrix}.$$

Then $\mathrm{depth}(E_1) = 1$, and $\mathrm{depth}(E_2) = 2$. More generally, assume that $A$ is regular of dimension $n$, and suppose that $E$ is a $k^{\text{th}}$ *syzygy module*, meaning that $E$ sits in an exact sequence

$$0 \longrightarrow E \longrightarrow A^{p_1} \longrightarrow A^{p_2} \longrightarrow \ldots \longrightarrow A^{p_k}. \tag{1.2.5}$$

Then $\mathrm{depth}(E) \geq \min\{\, k\,,\, n\,\}$.

## 1.2.B The Cohen-Macaulay condition.

The depth of a ring or module is a rather subtle invariant. The theorem of Auslander and Buchsbaum is particularly powerful in situations where one can work with dimensions instead. This leads to the notion of Cohen-Macaulay rings and modules.

**Definition 1.2.7 (Cohen-Macaulay).** Let $A$ be a local Noetherian ring of dimension $n$, and let $E$ be a finitely generated $A$-module. One says that $E$ is a *Cohen-Macaulay module* if

$$\operatorname{depth}(E) \ = \ \dim(E).$$

The ring $A$ is Cohen-Macaulay if it Cohen-Macaulay as a module over itself, i.e. if it contains a regular sequence of length $n$.

**Example 1.2.8 (Regular local rings are Cohen-Macaulay).** Proposition 1.1.10 asserts that regular local rings satisfy the requirement of Definition 1.2.7.

The next Proposition illustrates the usefulness of the Cohen-Macaulay condition.

**Proposition 1.2.9.** *Keeping notation as above, suppose that $A$ and $E$ are Cohen-Macaulay of dimensions $n$ and $m$ respectively. Suppose that $x_1, \ldots, x_\ell \in \mathfrak{m}$ is a collection of elements having the property that*

$$\overline{A} = A/(x_1, \ldots x_\ell) \quad \textit{and} \quad \overline{E} = E/(x_1, \ldots x_\ell)E$$

*have dimensions $n - \ell$ and $m - \ell$. Then $x_1 \ldots, x_\ell$ form a regular sequence for both $A$ and $E$, and both quotients are again Cohen-Macaulay.*

We refer to [32, Theorems 2.1.2, 2.1.3] or [175, Proposition II.13] for the proof.

**Remark 1.2.10 (Complete intersections).** When $A$ is regular, a quotient $A/(x_1, \ldots, x_\ell)$ satisfying the condition of the Proposition is called a *complete intersection*. The Koszul complex $K(x_1, \ldots, x_\ell)$ gives the minimal resolution of $A/(x_1, \ldots, x_\ell)$ as an $A$-module. □

**Remark 1.2.11 (Unmixedness theorem).** Suppose that $A$ is Cohen-Macaulay, and that $I \subseteq A$ is the ideal generated by a sequence $x_1, \ldots, x_\ell \in \mathfrak{m}$ of elements satisfying the conditions of Proposition 1.2.9. Then $I$ is *unmixed*, i.e. all of its associated primes have codimension $= \ell$. (See [32, Theorem 2.1.6] or [59, Corollary 18.14].) This was established by Macaulay (for polynomial rings) and by Cohen in general. □

**Example 1.2.12.** Let $R = \mathbf{C}[[s^3, s^2t, st^2, t^3]]$ denote the formal cone over a twisted cubic curve. Then $R$ is a Cohen-Macaulay ring of dimension two, but not a complete intersection. (The elements $s^3, t^3 \in R$ form a regular sequence.) If one uses the indicated generators to write $R$ as a quotient of $A = \mathbf{C}[[x, y, z, w]]$, then $R$ has dimension 2 and depth 2 considered as an $A$-module, and it admits a resolution of the shape

$$0 \longrightarrow A^2 \longrightarrow A^3 \longrightarrow A \longrightarrow R \longrightarrow 0. \qquad \square$$

Restrictions behave particularly well in the Cohen-Maculay setting, in which case the hypotheses of the following statement are tested dimensionally.

**Proposition 1.2.13 (Restrictions).** *Let $E$ be a finitely generated module over a local ring $A$, with minimal resolution*

$$P_\bullet \ : \quad \ldots \longrightarrow A^{b_2} \xrightarrow{\delta_2} A^{b_1} \xrightarrow{\delta_1} A^{b_0} \longrightarrow E \longrightarrow 0.$$

*Suppose that $x_1, \cdots x_\ell \in \mathfrak{m}$ are a collection of elements forming a regular sequence for both $A$ and $E$, and put*

$$\overline{A} = A/(x_1, \ldots x_\ell) \quad , \quad \overline{E} = E/(x_1, \ldots x_\ell)E.$$

*Then the reduction $\overline{P}_\bullet = P_\bullet \otimes_A \overline{A}$ of $P_\bullet$ $(\mathrm{mod}\ (x_1, \ldots, x_\ell))$ is the minimal resolution of $\overline{E}$ as an $\overline{A}$-module.*

*Proof.* By induction it suffices to prove this when $\ell = 1$, in which case multiplication by $x_1$ determines a short exact sequence

$$0 \longrightarrow P_\bullet \xrightarrow{\cdot x_1} P_\bullet \longrightarrow \overline{P}_\bullet \longrightarrow 0$$

of complexes. It then follows from the long exact sequence of cohomology that $\overline{P}_\bullet$ is the minimal resolution of $\overline{E}$. □

**Remark 1.2.14 (Hilbert–Burch theorem).** Let $A$ be a regular local ring of dimension $n$, let $B$ be an $(r+1) \times r$ matrix of elements of $\mathfrak{m}$, and consider the ideal $I = I_r(B) \subseteq A$ generated by the $r \times r$ minors of $B$. For sufficiently generic $B$ one expects $\dim A/I = n - 2$. When this happens $A/I$ is Cohen-Macaulay, and $I$ admits the minimal

$$0 \longrightarrow A^r \xrightarrow{\cdot B} A^{r+1} \xrightarrow{\delta} A \longrightarrow A/I \longrightarrow 0, \tag{1.2.6}$$

where $\delta$ arises by attaching suitable signs to the minors of $B$. Conversely, if $I \subseteq A$ is an ideal having projective dimension 1, then $I = a \cdot I_r(B)$ for some $(r+1) \times r$ matrix and non-zero element $a \in A$. Example 1.1.4 illustrates this result. We refer to [32, Theorem 1.4.17] or [59, Chapter 20.4] for the proof (as well as the statement of the theorem in its natural generality). The exact sequence (1.2.6) is a special case of the Eagon-Northcott resolution of determinantal ideals: see Section 1.3.D. □

**Example 1.2.15 (Modules on a hypersurface ring).** Eisenbud [57] proved some celebrated results about modules over a hypersurface ring. Let $A = \mathbf{C}[[z_1, \ldots, z_n]]$ and fix $0 \neq f \in A$. The quotient $\overline{A} = A/(f)$ is called a hypersurface ring: it is a Cohen-Macaulay ring of dimension $n-1$. Assuming for simplicity that $f$ is irreducible, so that $\overline{A}$ is a domain, let $E$ be a Cohen-Macaulay module over $\overline{A}$ of maximal dimension $n-1$. Then $E$ also has depth $n-1$ considered as an $A$-module, hence admits a length one resolution over $A$, say

$$0 \longrightarrow A^b \xrightarrow{u} A^b \longrightarrow M \longrightarrow 0.$$

Then there exists a mapping $v : A^b \longrightarrow A^b$ such that $u \circ v = v \circ u = f \cdot \mathrm{Id}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^b & \xrightarrow{u} & A^b & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f\cdot\mathrm{Id}} & \swarrow{\scriptstyle v} & \downarrow{\scriptstyle f\cdot\mathrm{Id}} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & A^b & \xrightarrow{u} & A^b & \longrightarrow & e & \longrightarrow & 0. \end{array} \tag{1.2.7}$$

The pair $(u, v)$ is called a matrix factorization of $f$. Eisenbud shows that as a module over $\overline{A}$, $E$ admits the infinite periodic resolution

$$\ldots \xrightarrow{\bar{v}} \overline{A}^b \xrightarrow{\bar{u}} \overline{A}^b \xrightarrow{\bar{v}} \overline{A}^b \xrightarrow{\bar{u}} \overline{A}^b \longrightarrow E \longrightarrow 0, \tag{*}$$

where $\bar{u}, \bar{v}$ are the maps obtained by reducing $u, v$ modulo $f$. See Example 1.1.7 above for a concrete instance of this construction where $E$ is the maximal ideal in $k[[x, y]]/(y^2 - x^3)$. (The existence of $v$ such that $u \circ v = f \cdot \mathrm{Id}$ encodes the vanishing of $f \cdot \mathrm{Id}_M$, and the fact that $v \circ u = f \cdot \mathrm{Id}$ then follows from the relation $u \circ v \circ u - u \circ f \cdot \mathrm{Id} = 0$ together with the injectivity of $u$. As for the exactness of (*), suppose for instance that $\bar{u}(\bar{x}) = 0$ for some $x \in A^b$. Then $u(x) - f \cdot y = 0$ for $y \in A^b$, and hence $f \cdot \big(x - v(y)\big) = 0$. This implies that $x = v(y)$.) Such matrix factorizations have lately proved of importance in several areas of mathematics and theoretical physics (eg [161]). □

## 1.3 Graded syzygies

In this section we turn to graded modules over a polynomial ring, and in particular to the syzygies associated to coherent sheaves on projective space. A general principle holds that the theory here is very close to that for modules over a local ring, and Hilbert's theorem is an excellent case in point. Therefore we content ourselves with a rather brief indication of how the local picture goes over to the graded setting. We then introduce the formalism of Betti tables, which are used to summarize numerics of the resolutions. In the final subsection we focus on the algebraic properties of modules arising from coherent sheaves on projective space.

### 1.3.A Hilbert's theorem for graded S-modules.

We start by fixing notation. Let $V$ be a vector space of dimension $n + 1$ over $\mathbf{C}$. We denote by $S = \mathrm{Sym}\, V$ the symmetric algebra on $V$, viewed as a graded $\mathbf{C}$-algebra. Upon choosing a basis $z_0, \ldots, z_n \in V$, $S$ is identified with the polynomial ring

$$S = \mathbf{C}[z_0, \ldots, z_n],$$

graded by assigning each $z_i$ degree one. Let $S_d \subseteq S$ be the degree $d$ component of $S$, consisting of homogeneous polynomials of degree $d$, so that $S_d = \mathrm{Sym}^d(V)$. Write

$$S_+ \;=\; \oplus_{d>0} S_d \;\subseteq\; S$$

for the irrelevant maximal ideal consisting of all polynomials of positive degree; to emphasize the analogy with the local setting, we sometimes denote this by $\mathfrak{m} \subseteq S$. Note that $S_1 = V$ and that $S/\mathfrak{m} = \mathbf{k}$ (concentrated in degree 0), where as above we write $\mathbf{k}$ for the copy of $\mathbf{C}$ arising as the residue field.

In the graded setting, the theorem on syzygies asserts the existence of graded minimal free resolutions:

**Theorem 1.3.1.** *Let $E$ be a finitely generated graded $S$-module. Then $E$ admits a minimal graded free resolution*

$$0 \longrightarrow P_{n+1} \xrightarrow{\delta_{n+1}} P_n \xrightarrow{\delta_n} \ldots \longrightarrow P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\varepsilon} E \longrightarrow 0 \tag{1.3.1}$$

*of length at most $n+1$, where*

$$P_i \ = \ \oplus_j \ S(-j)^{b_{i,j}}$$

*is a free graded $S$-module having $b_{i,j}$ generators in degree $j$. Moreover this resolution is unique up to isomorphism.*

As in the local setting, the minimality condition means that the matrices defining the maps contain no non-zero constant entries. In this case, the Betti numbers $b_{i,j}(E)$ arise as the dimensions of the graded pieces of the graded vector space $\operatorname{Tor}_i^S(E, \mathbf{k})$:

$$b_{i,j}(E) \ = \ \dim_{\mathbf{C}} \ \operatorname{Tor}_i^S(E, \mathbf{k})_j.$$

The Koszul resolution (1.1.9) in the local setting is here replaced by the graded Koszul complex:

$$0 \longrightarrow \Lambda^{n+1}V \otimes_{\mathbf{C}} S(-n-1) \longrightarrow \Lambda^n V \otimes_{\mathbf{C}} S(-n) \longrightarrow \ldots \longrightarrow V \otimes_{\mathbf{C}} S(-1) \longrightarrow S \longrightarrow \mathbf{C} \longrightarrow 0. \tag{1.3.2}$$

The finiteness of the resolution (1.3.1) follows, as in the local case, from the fact that the resolution (1.3.2) has length $n+1$. Observe that in the present setting infinite resolutions do not arise.

**Remark 1.3.2 (Koszul cohomology groups).** This discussion shows that $\operatorname{Tor}_i^S(E, \mathbf{k})_j$ is computed as the cohomology of a complex of vector spaces:

$$\Lambda^{i+1}V \otimes E_{j-i-1} \longrightarrow \Lambda^i V \otimes E_{j-i} \longrightarrow \Lambda^{i+1}V \otimes E_{j-i+1}.$$

In particular, $b_{i,j}(E)$ is the dimension of this cohomology group. This description will become central starting in Lecture 5, but for now we don't dwell on it. □

**Example 1.3.3 (A monomial ideal).** Take $S = \mathbf{C}[x, y]$ and consider the homogeneous ideal $I$ generated by

$$f_1 \ = \ xy \ \ , \ \ f_2 \ = \ x^3 \ \ , \ \ f_3 \ = \ y^5.$$

One checks by hand that $S/I$ has the resolution

$$0 \longrightarrow S(-4) \oplus S(-6) \xrightarrow{\cdot B} S(-2) \oplus S(-3) \oplus S(-5) \longrightarrow S \longrightarrow S/I \longrightarrow 0,$$

where $A$ is the matrix

$$B = \begin{pmatrix} x^2 & y^4 \\ -y & 0 \\ 0 & -x \end{pmatrix}.$$

Note that (up to sign) the generators of $I$ are the $2 \times 2$ minors of $B$: this is an illustration of the Hilbert-Burch theorem (Remark 1.2.6) in the homogeneous setting. □

**Example 1.3.4 (A monomial complete intersection).** Consider the ideal

$$I = (x^2, y^2, z^3) \subseteq \mathbf{C}[x, y, z] = S.$$

The three generators form a regular sequence, so $S/I$ is resolved by the graded Koszul complex:

$$0 \longrightarrow S(-7) \longrightarrow S(-4) \oplus S^2(-5) \longrightarrow S^2(-2) \oplus S(-3) \longrightarrow S \longrightarrow S/I \longrightarrow 0. \quad \square$$

**Example 1.3.5 (Linear resolution of $\mathfrak{m}^a$).** The computations of Example 1.1.4 hold in the present homogeneous setting. Specifically, let $S = \mathbf{C}[x, y]$, and let $I = (x, y)^a$. Then $I$ has the graded resolution

$$0 \longrightarrow S(-a-1)^a \xrightarrow{\cdot B} S(-a)^{a+1} \longrightarrow I \longrightarrow 0,$$

where $B$ is the matrix appearing in equation (1.1.5). Note that the entries of $A$ all have degree one: one says in this case that $I$ has a *linear resolution*. □

**Example 1.3.6 (Four points in the plane).** Let $X \subseteq \mathbf{P}^2$ consist of four (distinct) points, denote by $I \subseteq S = \mathbf{C}[x, y, z]$ the homogeneous ideal of $X$, and set $E = S/I$. There are three different possibilities for the resolution of $E$.

(a). No three points of $X$ are collinear. In this case $X$ is the complete intersection of two conics, and $E$ is resolved by a Koszul complex:

$$0 \longrightarrow S(-4) \longrightarrow S(-2)^2 \longrightarrow S \longrightarrow E \longrightarrow 0.$$

(b). Three but not all four of the points of $X$ lie on the line cut out by the linear form $\ell$. In this case it is still true that $X$ imposes independent conditions on curves of degree $\geq 2$, and therefore we can apply the following useful

> Fact: If $X \subseteq \mathbf{P}^n$ is a finite set imposing independent conditions on hypersurfaces of degree $m-1$, then the homogeneous ideal $I_X$ of $X$ is generated in degrees $\leq m$, and the $i^{\text{th}}$ module of syzygies of $I_X$ is generated in degrees $\leq m + i$.

(This is a very special case of the theory of Castelnuovo–Mumford regularity, although in the case at hand one could argue directly.) Returning to our four points $X \subseteq \mathbf{P}^2$, with three collinear, choose linear forms $\ell_1, \ell_2$ passing through the fourth point $P \in X$.

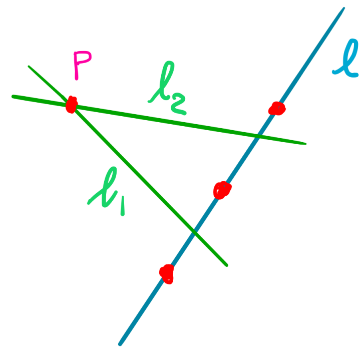



Figure 1.1: Four points in the plane

(See Figure 1.1.) Then $I$ contains the two conics $q_1 = \ell \cdot \ell_1$ and $q_2 = \ell \cdot \ell_2$ but these don't generate $I$, and they satisfy the weight one syzygy

$$\ell_2 \cdot q_1 \;-\; \ell_1 \cdot q_2 \;=\; 0. \tag{*}$$

Observing that $\dim I_3 = 10-4 = 6$, it follows from (*) that that $S_1 {\cdot} q_1$ and $S_1 {\cdot} q_2$ together span a codimension one subspace of $I_3$. Therefore $I$ requires a minimal generator $f$ of degree 3, and since $I$ is in any event generated in degrees $\leq 3$ (by the Fact), it follows that $I = (q_1, q_2, f)$. Keeping in mind that the syzygies among $q_1, q_2$ and $f$ appear in degrees $\leq 4$, the reader will enjoy checking and that the minimal resolution of $S/I$ takes the form

$$0 \longrightarrow S(-3) \oplus S(-4) \longrightarrow S(-2)^2 \oplus S(-3) \longrightarrow S \longrightarrow S/I \longrightarrow 0. \tag{1.3.3}$$

(c). All four points of $X$ are collinear. Then $X$ is the complete intersection of a quartic and a line, and $E$ is again resolved by a graded Koszul complex:

$$0 \longrightarrow S(-5) \longrightarrow S(-1) \oplus S(-4) \longrightarrow S \longrightarrow E \longrightarrow 0.$$

A generalization of this discussion appears in Example 1.3.21. □

The homological invariants attached to a module over a local ring have natural graded analogues. For instance, the depth of a finitely generated graded $S$-module $E$ is the maximal length of a regular sequence for $E$ consisting of elements from $S_+$. One can moreover take these elements to be homogeneous, and since $\mathbf{k} = \mathbf{C}$ is infinite one can focus if one likes on linear forms. (See [32, Propositions 1.5.11, 1.5.12].) As in the local situation, the projective dimension of a graded $S$-module is the length of its minimal resolution. The Auslander-Buchsbaum theorem 1.2.4 continues to hold, namely

**Theorem 1.3.7 (Graded Auslander–Buchsbaum).** *For any finitely generated graded $S$-module $E$, one has*

$$\mathrm{pd}(E) \;=\; (n+1) - \mathrm{depth}(E). \qquad \square$$

In the same spirit, the Cohen-Macaulay condition $\operatorname{depth}(E) = \dim(E)$ is tested with homogeneous elements, and the graded analogues of Propositions 1.2.9 and 1.2.13 remain valid. In particular, in the Cohen–Macaulay setting, resolutions restrict well to linear subspaces:

**Proposition 1.3.8 (Restrictions of resolutions).** *Let $\overline{V}$ be a quotient of $V$, with $\dim \overline{V} = m+1$ and $\overline{S} = \operatorname{Sym}(\overline{V})$ the corresponding quotient of $S$, so that $\overline{S}$ is isomorphic to a polynomial ring in $m+1$ variables. Let $E$ be a Cohen–Macaulay $S$-module of dimension $d$, and suppose that*

$$\overline{E} \ =_{def} \ E \otimes_S \overline{S}$$

*has dimension $d-(n-m)$. Then $\overline{E}$ is Cohen–Macaulay as an $\overline{S}$-module. Moreover, if $F_\bullet$ is the minimal graded free resolution of $E$, then $\overline{F}_\bullet = F_\bullet \otimes_S \overline{S}$ is the minimal resolution of $\overline{E}$.* □

**Example 1.3.9 (Saturated ideals).** Recall that a homogeneous ideal $I \subseteq S$ is said to be *saturated* if given $f \in S_d$ with the property that $\mathfrak{m} \cdot f \in I$, then $f \in I$. These are the ideals that arise most naturally from geometric constructions, and if $I$ is saturated then $\operatorname{depth}(S/I) \geq 1$. So such an ideal has a resolution of length $\leq n-1$. □

## 1.3.B Betti tables.

It is often convenient to display the Betti numbers $b_{i,j}(E)$ of a graded $S$-module in tabular form. The *Betti table* of $E$ is the array in which $b_{i,i+j}(E)$ appears in the $i^{\text{th}}$ column and $j^{\text{th}}$ row. Schematically:

| | $\dots$ | $i$ | $i+1$ |
|---|---|---|---|
| $\dots$ | $\dots$ | $\dots$ | $\dots$ |
| $j$ | $\dots$ | $b_{i,i+j}$ | $b_{i+1,i+j+1}$ |
| $j+1$ | $\dots$ | $b_{i,i+j+1}$ | $b_{i+1,i+j+2}$ |
| $\dots$ | $\dots$ | $\dots$ | $\dots$ |

Thus the $i^{\text{th}}$ column of the diagram gives the number of generators in each degree of the free $S$-module $P_i$ in (1.3.1). Note that the grading conventions are such that two adjacent entries on the same row of the table correspond to a map in the resolution given by a matrix of linear forms. One reason for this convention is that the smallest and largest integers $j$ such that $b_{i,j} \neq 0$ typically grow with $i$, and displaying $b_{i,j}$ in the $j^{\text{th}}$ row and $i^{\text{th}}$ column would lead to large diagrams whose non-zero entries were concentrated near the diagonal. This convention originated with early versions of the computer program `Macaulay`, so we speak of a `Macaulay`-style display.

As a first example, consider the resolution of the module $S/I$ presented in Example 1.3.3. To begin with, it is helpful to rewrite the complex with maps going from right to left:

$$0 \longleftarrow S/I \longleftarrow S \longleftarrow S(-2) \oplus S(-3) \oplus S(-5) \longleftarrow S(-4) \oplus S(-6) \longleftarrow 0.$$

The three columns of the table refer respectively to the free modules

$$S \quad , \quad S(-2) \oplus S(-3) \oplus S(-5) \quad , \quad S(-4) \oplus S(-6),$$

and the Betti table is:

| | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 1 | – | – |
| 1 | – | 1 | – |
| 2 | – | 1 | 1 |
| 3 | – | – | – |
| 4 | – | 1 | 1 |

Note that one often uses a dash to indicate a zero entry.

**Example 1.3.10.** The Koszul complex

$$0 \longleftarrow S/I \longleftarrow S \longleftarrow S^2(-2) \oplus S(-3) \longleftarrow S(-4) \oplus S^2(-5) \longleftarrow S(-7) \longleftarrow 0$$

from Example 1.3.4 gives rise to the table

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1 | – | – | – |
| 1 | – | 2 | – | – |
| 2 | – | 1 | 1 | – |
| 3 | – | – | 2 | – |
| 4 | – | – | – | 1 |

**Example 1.3.11.** The resolution (1.3.3) of the homogeneous coordinate ring of four points in the plane with three collinear is summarized by the table

| | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 1 | – | – |
| 1 | – | 2 | 1 |
| 2 | – | 1 | 1 |

## 1.3.C Sheaves on projective space.

From a geometric point of view, the most natural graded modules are those arising from coherent sheaves on projective space.

As above, let $V$ be a vector space of dimension $n+1$ over the field $\mathbf{C}$. We denote by $\mathbf{P} = \mathbf{P}(V)$ the $n$-dimensional projective space of one-dimensional quotients of $V$. Thus

$$V \;=\; H^0\big(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(1)\big),$$

and $S$ is the homogeneous coordinate ring of $\mathbf{P}$.

Now let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}$. We wish to associate a finitely generated graded $S$-module

$$E = E_{\mathcal{F}},$$

to $\mathcal{F}$. There are two cases to consider. Suppose to begin with that $\mathcal{F}$ has no zero-dimensional associated primes: in other words, assume that $\mathcal{F}$ does not contain a non-zero subsheaf $\mathcal{F}_0 \subseteq \mathcal{F}$ that is supported on a finite subset of $\mathbf{P}$. (This hypothesis will be satisfied in most cases of interest to us.) Then $H^0(\mathcal{F}(m)) = 0$ for $m \ll 0$ and therefore, as shown by Serre in FAC,

$$\Gamma_*(\mathcal{F}) =_{\text{def}} \oplus_{m\in\mathbf{Z}} H^0\big(\mathbf{P}, \mathcal{F}(m)\big)$$

is finitely generated. In this case we take $E_{\mathcal{F}} = \Gamma_*(\mathcal{F})$.

If $\mathcal{F}$ does contain a non-zero subsheaf supported on points, then $H^0(\mathcal{F}(m)) \neq 0$ for every $m \in \mathbf{Z}$, and hence $\Gamma_*(\mathcal{F})$ is not finitely generated. Therefore we truncate. Specifically, there exists in any event a very negative integer $m_0$ with the property that $\dim H^0\big(\mathbf{P}, \mathcal{F}(m)\big)$ becomes constant for $m \leq m_0$. In this case we take $E_{\mathcal{F}} = \oplus_{m\geq m_0} H^0\big(\mathbf{P}, \mathcal{F}(m)\big)$ for some such $m_0$. This of course depends on the choice of $m_0$, but we allow this to remain implicit when possible. We will use one of the abbreviations

$$E_{\mathcal{F}} = \oplus_{m\gg-\infty} H^0\big(\mathbf{P}, \mathcal{F}(m)\big) = \Gamma_{\gg-\infty}\big(\mathcal{F}\big) \tag{1.3.4}$$

to cover both cases. Note that the dimension of $E_{\mathcal{F}}$ as an $S$-module is one greater than the dimension of the support of $\mathcal{F}$, considered as a subscheme of projective space.

Conversely, a finitely generated graded $S$-module $E$ determines a coherent sheaf $\widetilde{E} = \mathcal{F}_E$. If $E$ has a presentation $\oplus S(-b_j) \xrightarrow{u} \oplus S(-a_i) \longrightarrow E \longrightarrow 0$, then $\mathcal{F}_E$ may for example be realized as the cokernel of the map of vector bundles defined by the same matrix $u$:

$$\mathcal{F}_E = \operatorname{coker}\big( \oplus \mathcal{O}_{\mathbf{P}}(-b_j) \xrightarrow{u} \oplus \mathcal{O}_{\mathbf{P}}(-a_i)\big).$$

There is a canonical homomorphism

$$E \longrightarrow \Gamma_*\big(\mathcal{F}_E\big)$$

which is an isomorphism in sufficiently large degrees. Starting with a coherent sheaf $\mathcal{F}$, one recovers $\mathcal{F}$ as the sheaf associated to its module:

$$\mathcal{F} = \widetilde{E_{\mathcal{F}}}.$$

Recall also that the cohomology of $\mathcal{F}_E$ is computed by the local cohomology of $E$ or $\Gamma_*(\mathcal{F})$. Specifically, given $i > 0$, denote by $H^i_*\big(\mathbf{P}, \mathcal{F}\big)$ the graded $S$-module

$$H^i_*\big(\mathbf{P}, \mathcal{F}\big) =_{\text{def}} \oplus_{m\in\mathbf{Z}} H^i\big(\mathbf{P}, \mathcal{F}(m)\big). \tag{1.3.5}$$

Then for $i > 0$:

$$H^i_*\big(\mathbf{P}, \mathcal{F}\big) = H^{i+1}_{\mathfrak{m}}\big(\Gamma_*(\mathcal{F})\big), \tag{1.3.6}$$

where as above $\mathfrak{m}$ denotes the irrelevant maximal ideal. We refer to [106, Chapters 2, 3] or [59, Appendix 4] for a detailed discussion of the correspondence between graded $S$-modules and coherent sheaves on projective space.

The depth (and hence the projective dimension) of $E = \Gamma_*(\mathcal{F})$ is determined by the vanishing of cohomology:

**Proposition 1.3.12.** *Let $\mathcal{F}$ be a coherent sheaf that does not have any zero-dimensional associated primes, and let $E = E_{\mathcal{F}}$ be the corresponding $S$-module. Then $E$ has depth $\geq 2$, and* $\mathrm{depth}(E)$ *is the largest integer $p \geq 2$ having the property that*

$$H^j_*\big(\mathbf{P}, \mathcal{F}\big) \ = \ 0 \quad \textit{for all} \ \ 0 \ < \ j \ < \ p-1.$$

In fact, $\mathrm{depth}(E) \geq p$ if and only if $H^i_{\mathfrak{m}}(E) = 0$ for $i < p$ (Remark 1.2.2), and so the assertion follows from (1.3.6). An alternative approach is indicated in Example 1.3.22.

**Example 1.3.13.** If $\mathcal{F}$ has a non-trivial subsheaf supported at points, then $E_{\mathcal{F}} = \Gamma_{\gg -\infty}(\mathcal{F})$ has depth $= 1$, and hence projective dimension $n$. □

**Example 1.3.14.** Suppose that $X \subseteq \mathbf{P}$ is an irreducible subvariety of dimension $d \geq 1$, and let

$$R_X \ = \ \oplus\, H^0\big(\mathbf{P}, \mathcal{O}_X(m)\big) \ = \ \Gamma_*(\mathbf{P}, \mathcal{O}_X)$$

be the graded ring of $X$.[1] Then $R_X$ is a Cohen-Macaulay $S$-module if and only if

$$H^j_*\big(X, \mathcal{O}_X\big) \ = \ 0 \ \ \text{for every} \ \ 0 < j < d.$$

This condition is satisfied for instance by the twisted quartic curve $C \subseteq \mathbf{P}^3$ discussed in the Introduction, and one finds that $R_C$ has a resolutiom of the shape

$$0 \longrightarrow S(-3)^3 \longrightarrow S(-2)^5 \longrightarrow S \oplus S(-1) \longrightarrow R_C \longrightarrow 0. \qquad \square$$

**Example 1.3.15 (Homogeneous ideal of a subscheme).** Let $X \subseteq \mathbf{P} = \mathbf{P}^n$ be a non-empty subscheme with ideal sheaf $\mathcal{I} = \mathcal{I}_X \subseteq \mathcal{O}_{\mathbf{P}}$. Then the homogeneous ideal

$$I_X \ = \ \Gamma_*(\mathcal{I}_X) \ \subset \ S$$

of $X$ is saturated (Example 1.3.9), and so $S/I_X$ has projective dimension $\leq n$. Moreover, $\mathrm{pd}(S/I_X) < n$ if and only if there exists an integer $q$ such that

$$H^j_*\big(\mathbf{P}, \mathcal{I}_X\big) \ = \ 0 \ \ \text{for} \ \ 1 \leq j \leq q,$$

and then

$$\mathrm{pd}\big(S/I_X\big) \ = \ n-q,$$

for the largest such $q$. □

[1] Note that as an $S$-module, $R_X$ may have generators in degrees $> 0$, and hence might not be a quotient of $S$.

**Example 1.3.16** (**The twisted quartic curve, revisited**)**.** Let $C \subseteq \mathbf{P}^3$ be the twisted quartic curve considered in Example 2 from the Introduction. Then $H^1\big(\mathbf{P}^3, \mathcal{I}_C(1)\big) \neq 0$, so the previous Example 1.3.15 explains the fact that $\mathrm{pd}(S/I_C) = 3$. More generally, if $X \subseteq \mathbf{P}^n$ is embedded by an incomplete linear series, then $\mathrm{pd}(S/I_X) = n$. See also Example 1.3.22 (iv). □

**Example 1.3.17** (**Projectively Cohen–Macaulay subvarieties**)**.** Observe that if $X \subseteq \mathbf{P}$ is a variety of dimension $d$, with ideal sheaf $\mathcal{I}_X \subseteq \mathcal{O}_\mathbf{P}$ and homogeneous ideal $I_X \subseteq S$, then the graded $S$-module $S/I_X$ has dimension $d+1$. One says that $X \subseteq \mathbf{P}$ is *projectively* or *arithmetically Cohen–Macaulay* if $S/I_X$ is Cohen–Macaulay as an $S$-module, i.e. if $\mathrm{depth}(S/I_X) = d+1$. In view of Example 1.3.15 this is equivalent to the vanishing

$$H^j\big(\mathbf{P}, \mathcal{I}_X(m)\big) \ = \ 0 \quad \text{for all} \ \ 1 \leq j \leq d \ \ \text{and all} \ \ m \in \mathbf{Z}.$$

These vanishings are in turn equivalent to the two conditions

$$\begin{aligned} H^1\big(\mathbf{P}, \mathcal{I}_X(m)\big) \ &= \ 0 \ \ \text{for all} \ \ m \in \mathbf{Z} \\ H^j\big(X, \mathcal{O}_X(m)\big) \ &= \ 0 \ \ \text{for all} \ \ 0 < j < d \ \ \text{and all} \ \ m \in \mathbf{Z}. \end{aligned}$$

The vanishing of the $H^1$ is automatic if one re-embeds $X$ by a sufficiently high Veronese, but if $H^j\big(X, \mathcal{O}_X\big) \neq 0$ for some $0 < j < d$, then $X$ does not admit any aritmetically Cohen–Macaulay embeddings. □

**Example 1.3.18** (**Locally Cohen–Macaulay varieties**)**.** Even if $X \subseteq \mathbf{P}$ fails to be projectively Cohen–Macaulay, it may happen that the local ring $\mathcal{O}_x X$ is a Cohen-Macaulay module over $\mathcal{O}_x \mathbf{P}^n$ for every point $x \in X$. In this case one says that $X$ is locally Cohen–Macaulay. For example, if $X$ is smooth or more generally a local complete intersection, then it is locally Cohen–Macaulay. Example 1.3.22 gives a further perspective on the difference between the local and arithmetic Cohen–Macaulay properties. □

Recall that sheafification is an exact functor. Therefore, starting with a finitely generated $S$-module $E$, the sheafification of its minimal graded resolution (1.3.1) is a locally free resolution of $\mathcal{F} = \mathcal{F}_E$ whose terms are direct sums of line bundles:

$$0 \longrightarrow \mathcal{P}_{n+1} \longrightarrow \ldots \longrightarrow \mathcal{P}_1 \longrightarrow \mathcal{P}_0 \longrightarrow \mathcal{F} \longrightarrow 0, \tag{1.3.7}$$

where

$$\mathcal{P}_i \ = \ \oplus_j \ \mathcal{O}_\mathbf{P}(-j)^{b_{i,j}}.$$

For example, let $E = (x,y)^a \subseteq S = \mathbf{C}[x,y]$. As we saw in Example 1.3.5, this has the resolution

$$0 \longrightarrow S(-a-1)^a \longrightarrow S(-a)^{a+1} \longrightarrow E \longrightarrow 0 \tag{1.3.8}$$

Now $\widetilde{E} = \mathcal{O}_{\mathbf{P}^1}$, and (*) sheafifies to a resolution on $\mathbf{P}^1$ of $\mathcal{O}_{\mathbf{P}^1}$:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^1}(-a-1)^a \longrightarrow \mathcal{O}_{\mathbf{P}^1}(-a)^{a+1} \longrightarrow \mathcal{O}_{\mathbf{P}^1} \longrightarrow 0. \tag{1.3.9}$$

Note that $E$ is a proper submodule of $S = \Gamma_*(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1})$, so we do not recover (1.3.8) by applying $\Gamma_*$ termwise to (1.3.9) : this is a reflection of the non-exactness of $\Gamma_*$. Observe in the same spirit that a locally free resolution as in (1.3.7) need not be the sheafification of a minimal resolution of graded $S$-modules, as the maps on global sections of the $\mathcal{P}_i$ might not yield an acyclic complex.

We conclude this subsection with a number of Examples and Remarks.

**Example 1.3.19** (**Modules of finite length**)**.** Let $E$ be a finitely generated graded $S$-module with the property that $E_m = 0$ for $m \gg 0$: this is equivalent to assuming that $\dim_{\mathbf{C}}(E) < \infty$. Then $\operatorname{depth}(E) = 0$, so the minimal graded free resolution $P_\bullet$ of $E$ has length $n+1$:

$$0 \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow E \longrightarrow 0. \tag{*}$$

The hypothesis on $E$ implies that $\mathcal{F}_E = \widetilde{E} = 0$, so (*) sheafifies to a long exact sequence

$$0 \longrightarrow \mathcal{P}_{n+1} \longrightarrow \mathcal{P}_n \longrightarrow \ldots \longrightarrow \mathcal{P}_1 \longrightarrow \mathcal{P}_0 \longrightarrow 0 \tag{**}$$

of vector bundles on $\mathbf{P}$ each of whose terms is a direct sum of line bundles. For example, when $S = \mathbf{C}[x, y]$ the resolution

$$0 \longrightarrow S(-a-1)^a \longrightarrow S(-a)^{a+1} \longrightarrow S \longrightarrow S/(x, y)^a \longrightarrow 0$$

of $S/(x, y)^a$ considered above gives rise to the exact sequence (1.3.9). Conversely, a length $n+1$ long exact sequence of split vector bundles (**) arises from a resolution of a module of finite length, namely

$$E \ = \ \operatorname{coker}\big(\Gamma_*(\mathcal{P}_1) \longrightarrow \Gamma_*(\mathcal{P}_0)\big).$$

This picture will play a central role in our discussion of Boij–Söderberg theory in Lecture 2. □

**Remark 1.3.20** (**Hilbert–Burch theorem in the graded case**)**.** Let $I \subseteq S$ be an ideal such that $R = S/I$ is Cohen-Macaulay of dimension $n-1$. Then $I$ is generated by the maximal minors of an $r \times (r+1)$ matrix $B$ of homogeneous polynomials, and after attaching suitable signs to these generators, $S/I$ admits the resolution

$$0 \longrightarrow \bigoplus^{r} S(-c_j) \xrightarrow{\cdot B} \bigoplus^{r+1} S(-d_i) \longrightarrow S \longrightarrow S/I \longrightarrow 0.$$

(See [60, Chapter 3A].) A concrete illustration appears in the next example. □

**Example 1.3.21** (**Finite sets in $\mathbf{P}^2$**)**.** Let $X \subseteq \mathbf{P}^2$ be a (reduced) finite set, with ideal sheaf $\mathcal{I}_X \subseteq \mathcal{O}_{\mathbf{P}^2}$, and denote by

$$I_X \ = \ H^0_*\big(\mathbf{P}^2, \mathcal{I}_X\big) \ \subseteq \ S$$

the saturated homogeneous ideal of $X$. Then $S/I_X$ has depth 1, and hence $I_X$ is generated by the minors of an $(r+1) \times r$ matrix of homogeneous polynomials thanks to Remark 1.3.20.

For example, suppose that $X \subseteq \mathbf{P}^2$ consists of five general points. Then $I_X$ is generated by the $2 \times 2$ minors of a $3 \times 2$ matrix $B$ of the the form

$$\begin{pmatrix} q_1 & q_2 \\ \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{pmatrix},$$

where the $q_i$ have degree 2 and the $\ell_{ij}$ have degree 1. The determinant of the bottom two rows is the equation of the unique conic containing the five points. We refer to Chapter 3 of Eisenbud's text [60] for a detailed discussion of the syzygies of finite subsets of $\mathbf{P}^2$. $\square$

**Example 1.3.22** (**Local versus global projective dimension**)**.** The difference between the projective dimension of a graded module and the pointwise projective dimensions of (the stalks of) the corresponding sheaf can be understood in terms of vector bundles on projective space. We illustrate how this goes in the first non-trivial case, leaving it to the reader to formulate the general statement.

Suppose then that $C \subseteq \mathbf{P}^3$ is a smooth curve. Let $\mathcal{I}_C \subseteq \mathcal{O}_{\mathbf{P}^3}$ be the ideal sheaf of $C$, and write $I_C \subseteq S$ for the saturated homogeneous ideal of $C$, so that

$$I_C \ = \ \Gamma_*\big(\mathbf{P}^3, \mathcal{I}_C\big).$$

(i). The ideal $I_C$ has projective dimension 1 if and only if $H^1_*\big(\mathbf{P}^3, \mathcal{I}_C\big) = 0$, in which case it admits a minimal resolution (1.3.1) having the shape

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow I_C \longrightarrow 0.$$

Otherwise $\operatorname{pd}(I_C) = 2$, and the resolution takes the form

$$0 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow I_C \longrightarrow 0.$$

(ii). On the other hand, sheafifying the surjection $P_0 \longrightarrow I_C$ gives rise to an exact sequence of sheaves

$$0 \longrightarrow Q \longrightarrow \mathcal{P}_0 \longrightarrow \mathcal{I}_C \longrightarrow 0. \qquad (*)$$

Since $C$ is locally Cohen-Macaulay, the stalks of $\mathcal{I}_C$ have projecive dimension 0 or 1 at every point of $\mathbf{P}^3$. Therefore $Q$ is locally free, and (*) is a length one resolution of $\mathcal{I}_C$ by vector bundles on $\mathbf{P}^3$. By construction $H^0_*\big(\mathbf{P}^3, \mathcal{P}_0\big) \longrightarrow H^0_*\big(\mathbf{P}^3, \mathcal{I}_C\big)$ is surjective, and hence $H^1_*\big(\mathbf{P}^3, Q\big) = 0$. Moreover

$$H^2_*\big(\mathbf{P}^3, Q\big) \ = \ H^1_*\big(\mathbf{P}^3, \mathcal{I}_C\big).$$

(iii). Now recall the useful:

FACT: A vector bundle $U$ on $\mathbf{P}^n$ is a direct sum of line bundles if and only if

$$H^i_*\big(\mathbf{P}^n, U\big) \ = \ 0 \ \text{ for all } \ 0 < i < n. \qquad (**)$$

(Proof by induction on $n$ starting with Grothendieck's theorem on decomposability of vector bundles on $\mathbf{P}^1$, or see [147, Theorem 2.3.1].) Thus one recovers the first statement in (i). Moreover if one uses generators for $H^0_*\big(\mathbf{P}^3, Q\big)$ to construct a surjective map $u : \mathcal{P}_1 \longrightarrow Q \longrightarrow 0$, one sees that

$$\mathcal{P}_2 \ =_{\text{def}} \ \ker(u)$$

satisfies (**), and is therefore a direct sum of line bundles.

(iv). As a concrete illustration of this discussion, consider again the twisted quartic curve $C \subseteq \mathbf{P}^3$ from Example 2 in the Introduction. The resolution of $I_C$ constructed there sheafifies to an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^3}(-5) \xrightarrow{\delta} \mathcal{O}_{\mathbf{P}^3}(-4)^4 \longrightarrow \mathcal{O}_{\mathbf{P}^3}(-3)^3 \oplus \mathcal{O}_{\mathbf{P}^3}(-2) \longrightarrow \mathcal{I}_C \longrightarrow 0,$$

where $\delta$ is given by the matrix $(x, -y, -z, w)$. It follows from the Euler sequence that

$$\text{coker}(\delta) \ = \ T_{\mathbf{P}^3}(-5) \ = \ \Omega^2_{\mathbf{P}^3}(-1),$$

and so one arrives at the exact sequence

$$0 \longrightarrow \Omega^2_{\mathbf{P}^3}(-1) \longrightarrow \mathcal{O}_{\mathbf{P}^3}(-3)^3 \oplus \mathcal{O}_{\mathbf{P}^3}(-2) \longrightarrow \mathcal{I}_C \longrightarrow 0.$$

The group $H^2\big(\mathbf{P}^3, \Omega^2_{\mathbf{P}^3}\big)$ receives the non-zero class in $H^1\big(\mathbf{P}^3, \mathcal{I}_C(1)\big)$ arising from the failure of $C$ to be linearly normal. □

## 1.3.D Some explicit resolutions

We list briefly a few resolutions that arise in several contexts. One can work either in the setting of rings – as in [59, Appendix A.2.6] – or with vector bundles, as in [128, Appendix B.2]. We work in the latter setting, and refer to [128] for details and proofs.

Suppose then that $X$ is a smooth variety of dimension $n$. Let $E$ be a vector bundle of rank $e$ on $X$, and suppose given a section $s \in \Gamma\big(X, E\big)$. Denote by $Z = \text{Zeroes}(s) \subseteq X$ the zero-scheme of $s$. The *Koszul complex* determined by $s$ is:

$$0 \longrightarrow \det E^* \longrightarrow \Lambda^{e-1}E^* \longrightarrow \ldots \longrightarrow \Lambda^2 E^* \longrightarrow E^* \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0. \tag{1.3.10}$$

This is exact provided that $Z$ has codimension $e$ in $X$ (or if $Z = \varnothing$). In any event, it is exact off $Z$.

There are also various *Eagon–Northcott complexes* associated to a mapping $u : E \longrightarrow F$ between vector bundles of ranks $e \geq f$. We consider the degeneracy locus

$$Z \ = \ \big\{x \in X \mid \text{rank} u(x) \ \leq \ f\big\}.$$

Provided that $Z$ has the expected codimension $e - f + 1$ – or else $Z = \varnothing$ – the two Eagon–Northcott complexes

$$0 \to \begin{matrix} \Lambda^e E \\ \otimes \\ (S^{e-f}F)^* \otimes \Lambda^f F^* \end{matrix} \to \cdots \to \begin{matrix} \Lambda^{f+1} E \\ \otimes \\ F^* \otimes \Lambda^f F^* \end{matrix} \to \begin{matrix} \Lambda^f E \\ \otimes \\ \Lambda^f F^* \end{matrix} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0 \tag{1.3.11}$$

$$0 \to \begin{matrix} \Lambda^e E \\ \otimes \\ (S^{e-f-1}F)^* \otimes \Lambda^f F^* \end{matrix} \to \cdots \to \begin{matrix} \Lambda^{f+2} E \\ \otimes \\ F^* \otimes \Lambda^f F^* \end{matrix} \to \begin{matrix} \Lambda^{f+1} E \\ \otimes \\ \Lambda^f F^* \end{matrix} \to E \to F \to \operatorname{coker}(u) \to 0 \tag{1.3.12}$$

are exact. We refer to the cited passages in [59] or [128, Appendix B] (from which these are taken) for other complexes of this type.

**Example 1.3.23 (Scrolls).** Let $E$ be an ample vector bundle of rank $n$ and degree $d$ on $\mathbf{P}^1$, so that

$$h^0(X, E) \;=\; d + n \;=_{\text{def}}\; r + 1.$$

(In the present setting, amplitude simply means that $E$ is a direct sum of line bundles of positive degree.) Let $\pi : S = \mathbf{P}(E) \longrightarrow \mathbf{P}^1$ be the corresponding projective bundle. Then $\mathcal{O}_S(1) = \mathcal{O}_{\mathbf{P}(E)}(1)$ is very ample, and this line bundle defines an embedding

$$S \;\subseteq\; \mathbf{P}^r.$$

Put $A = \pi^*\mathcal{O}_{\mathbf{P}^1}(1)$, and via extension by zero view $A$ a sheaf on $\mathbf{P}^r$ supported on $S$. Then $A$ admits a presentation

$$\mathcal{O}_{\mathbf{P}^r}^{r+1-n}(-1) \longrightarrow \mathcal{O}_{\mathbf{P}^r}^2 \longrightarrow A \longrightarrow 0,$$

and the Eagon–Northcott complex (1.3.11) gives a resolution of the ideal $\mathcal{I}_S \subseteq \mathcal{O}_{\mathbf{P}^r}$ of $S$. □

**Example 1.3.24 (Macaulay's theorem).** Consider homogeneous polynomials

$$F_0, \ldots, F_n \,\in\, S = \mathbf{C}[z_0, \ldots, z_n]$$

of degrees $d_0, \ldots, d_n$. Suppose that these polynomials have no common zeroes in $\mathbf{P}^n$, and let $I = (F_0, \ldots, F_n)$ be the ideal that they generate. The homogeneous Nullstellensatz implies that $I$ contains all polynomials of sufficiently large degree, but in the present situation a classical statement of Macaulay gives the best possible effective statement. Specifically,

$$\mathfrak{m}^k \,\subseteq\, I \iff k \,\geq\, (\Sigma\, d_i) - n.$$

(In fact, the $F_i$ form a regular sequence, and $S/I$ is resolved by the corresponding Koszul complex. On the other hand, $\widetilde{S/I} = 0$, and hence this complex determines a long exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-\Sigma\, d_i) \longrightarrow \ldots \longrightarrow \oplus\, \mathcal{O}_{\mathbf{P}}(-d_i) \longrightarrow \mathcal{O}_{\mathbf{P}} \longrightarrow 0 \tag{*}$$

of sheaves. The original Koszul resolution of $S/I$ is obtained from (*) by applying the functor $\Gamma_*$. So the question is equivalent to finding the least integer $k$ with the property that the map

$$\oplus\, H^0\big(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(k - d_i)\big) \longrightarrow H^0\big(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(k)\big)$$

is surjective, and this is computed by taking cohomology and chasing through (*).) □

## 1.4 Notes

We have ignored the rich – and historically important – connection between resolutions of graded ideals and their Hilbert functions. We refer to [60], especially Chapter 1, for an introduction to this story. Chapter 2 of that text, as well as the book [137] of Miller and Sturmfels, contain interesting discussions of resolutions of ideals of combinatorial origin. We have also neglected the computational side of the subject. For this we refer for instance to [44].

There has been recent interest in studying multigraded resolutions where projective space is replaced by a variety with Picard number $\geq 1$. See [23] for a recent contribution in this direction.

# Lecture 2

# An Introduction to Boij–Söderberg Theory

In 2006 the Swedish mathematicians Mats Boij and Jonas Söderberg stated some conjectures that proposed a remarkable picture of all Betti tables of Cohen–Macaulay modules of given dimension and codimension [25]. Their beautiful idea was that one can hope for a complete description *up to rational multiples*. Following contributions by Eisenbud, Fløystad, and Weyman [62] the Boij-Söderberg conjectures were established by Eisenbud and Schreyer in [68]. Since then the theory has developed quickly in several directions.

The present lecture aims to give a brief invitation to this circle of ideas. We will explain the statements of the main theorems, and we will say a word about some of the inputs to the proofs, but we won't actually establish the results. Besides the original papers, the reader interested in more details might consult one of the several expository accounts that have appeared, for instance [173] or [70]). We especially recommend the survey [77] of Fløystad, on which we have drawn very substantially.

For the most part, this material will not be used elsewhere in these lectures.

## 2.1 Preliminaries

### 2.1.A Set-up

Our goal is to understand the Betti numbers of graded Cohen–Macaulay modules of fixed dimension and codimension over a polynomial ring. Recall (Proposition 1.3.8) that these are unchanged upon modding out by a regular sequence of regular forms. Hence there is no loss in generality in focusing on modules of dimension zero.

Consider then the polynomial ring $S = \mathbf{C}[z_0, \ldots, z_n]$ and a finitely generated graded $S$-module $E$ of dimension $= 0$ over $S$. The assumption on $E$ is equivalent to requiring that

$E_m = 0$ for $m \gg 0$, or equivalently again that $\dim_{\mathbf{C}} E < \infty$. Thus $\operatorname{depth} E = 0$, so the minimal graded free resolution $P_\bullet$ of $E$ takes the form

$$0 \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow E \longrightarrow 0, \tag{2.1.1}$$

where

$$P_i \;=\; \oplus\, S(-j)^{b_{i,j}}.$$

We view the Betti numbers $b_{i,j} = b_{ij}(E)$ as the entries of the matrix

$$b(E) \;=\; \big(\, b_{i,j}(E)\,\big)$$

that one refers to as the Betti table or Betti diagram of $E$. Note that the homological degree $i$ lies in the range $0 \le i \le n+1$, but at the moment we do not impose any limitations on the grading index $j$.

We will often prefer to work with equivalent geometric data. Recall from Example 1.3.19 that (2.1.1) sheafifies to a long exact sequence $L_\bullet$ of vector bundles

$$0 \longrightarrow L_{n+1} \longrightarrow L_n \longrightarrow \ldots \longrightarrow L_1 \longrightarrow L_0 \longrightarrow 0 \tag{2.1.2}$$

on $\mathbf{P}^n$, where

$$L_i \;=\; \oplus\, \mathcal{O}_{\mathbf{P}^n}(-j)^{b_{i,j}}.$$

Conversely, $P_\bullet = \Gamma_*\big(\mathbf{P}^n, L_\bullet\big)$ is the minimal resolution of the finite length module

$$E \;=\; \operatorname{coker}\big(\Gamma_*(L_1) \longrightarrow \Gamma_*(L_0)\big).$$

We denote by $b(L_\bullet) \;=\; \big(\, b_{i,j}(L_\bullet)\,\big)$ the corresponding Betti matrix.

**Example 2.1.1 (Running example).** Following Fløystad [77] we consider as a running example the module

$$E \;=\; \mathbf{C}[x,y]\,/\,(x^2, xy, y^3) \tag{2.1.3}$$

over the polynomial ring $S = \mathbf{C}[x,y]$ in two variables. This has the resolution

$$0 \longleftarrow E \longleftarrow S \xleftarrow{\;u\;} S(-2)^2 \oplus S(-3) \xleftarrow{\;v\;} S(-3) \oplus S(-4) \longleftarrow 0,$$

where $u$ and $v$ are given by the matrices:

$$u \;=\; \begin{bmatrix} x^2 & xy & y^3 \end{bmatrix} \quad , \quad v \;=\; \begin{bmatrix} y & 0 \\ -x & y^2 \\ 0 & -x \end{bmatrix}.$$

The corresponding exact complex $L_\bullet$ of bundles on $\mathbf{P}^1$ is then:

$$0 \longleftarrow \mathcal{O}_{\mathbf{P}^1} \xleftarrow{\;u\;} \mathcal{O}_{\mathbf{P}^1}(-2)^2 \oplus \mathcal{O}_{\mathbf{P}^1}(-3) \xleftarrow{\;v\;} \mathcal{O}_{\mathbf{P}^1}(-3) \oplus \mathcal{O}_{\mathbf{P}^1}(-4) \longleftarrow 0.$$

Using the `Macaulay` display convention, these are summarized in the Betti table:

$$b \;=\; \begin{pmatrix} & \ldots & \\ 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \\ & \ldots & \end{pmatrix}. \qquad \square$$

Very roughly speaking, we would like to describe all possible Betti tables $b(L_\bullet)$. The first step is to render the question finite-dimensional by fixing upper and lower bounds on which Betti numbers are allowed to be non-zero.

To this end, fix two *degree sequences* $a^-$ and $a^+$, i.e. two strictly increasing sets of integers:

$$\begin{array}{lcl} a^- & : & a_0^- \ < \ a_1^- \ < \ \ldots \ < \ a_n^- \ < \ a_{n+1}^- \\ a^+ & : & a_0^+ \ < \ a_1^+ \ < \ \ldots \ < \ a_n^+ \ < \ a_{n+1}^+ \ , \end{array}$$

and assume that $a_i^- \leq a_i^+$ for each $i$. We say that a Betti table $b(L_\bullet) = \big(b_{i,j}(L_\bullet)\big)$, or an arbitrary matrix $\beta = (\beta_{i,j})$, lies in the *window* determined by $a^-$ and $a^+$ if the entries of all of its non-zero rows lie in the range

$$a_i^- \ \leq \ \beta_{i,j} \ \leq a_i^+ \tag{2.1.4}$$

for every $i \in [0, n+1]$.

**Definition 2.1.2** (**Tables in a window**). Denote by $\mathbf{D}(a^-, a^+)$ the finite-dimensional $\mathbf{Q}$-vector space of all matrices $\beta = (\beta_{i,j})$ of rational numbers satisfying (2.1.4).

Thus $\dim \mathbf{D}(a^-, a^+) = \sum(1 + a_i^+ - a_i^-)$.

**Example 2.1.3.** The Betti table $b(E)$ of the module $E$ from Example 2.1.1 lies in $\mathbf{D}(a^-, a^+)$ for

$$a^- \ = \ \big(0 \ < \ 2 \ < 3\big) \ \ , \ \ a^+ \ = \ \big(0 \ < \ 3 \ < 4\big).$$

Less efficiently, it also lies in the window specified by

$$a^- \ = \ \big(0 \ < \ 1 \ < 2\big) \ \ , \ \ a^+ \ = \ \big(0 \ < \ 10 \ < 20\big). \qquad \square$$

## 2.1.B The Herzog–Kühl equations

Consider a sequence $L_\bullet$ as in (2.1.2) lying in the window determined by degree sequences $a^-, a^+$. Herzog and Kühl [108] observed that the Betti numbers $b_{i,j} = b_{i,j}(L_\bullet)$ must satisfy some linear relations. For example, evidently

$$\operatorname{rank}(L_0) \ - \ \operatorname{rank}(L_1) \ + \ \ldots \ + (-1)^{n+1} \operatorname{rank}(L_{n+1}) \ = 0,$$

which leads to the equation

$$\sum_{i,j} (-1)^i \, b_{i,j} \ = \ 0.$$

Furthermore, the alternating sum of the first Chern classes of the $L_i$ must vanish, which implies

$$\sum_{i,j} (-1)^i \, j \cdot b_{i,j} \ = \ 0.$$

In general, the $b_{i,j}$ satisfy $n+1$ linearly independent equations:

**Proposition 2.1.4 (Herzog-Kühl).** *The Betti numbers $b_{i,j} = b_{i,j}(L_\bullet)$ satisfy the equations*

$$\begin{aligned} \sum_{i,j} (-1)^i \, b_{i,j} &= 0 \\ \sum_{i,j} (-1)^i \, j \cdot b_{i,j} &= 0 \\ &\cdots \\ \sum_{i,j} (-1)^i \, j^n \cdot b_{i,j} &= 0. \end{aligned} \tag{2.1.5}$$

Herzog and Kühl establish this by studying the Hilbert function of the module $E$. We will give below an alternative proof via Chern classes.

**Definition 2.1.5.** Given windows $a^-, a^+$, denote by

$$\mathbf{D}^{\mathrm{HK}}(a^-, a^+) \subseteq \mathbf{D}(a^-, a^+)$$

the subspace of $\mathbf{D}(a^-, a^+)$ consisting of tables $\beta = (\beta_{i,j})$ satisfying the Herzog-Kühl equations (2.1.5).

Thus the Betti tables $b(L_\bullet)$ of actual resolutions in the given window lie in the subspace $\mathbf{D}^{\mathrm{HK}}(a^-, a^+)$. (It turns out that they span it as a vector space, but the result for which we are aiming is much more precise.)

*Proof of Proposition 2.1.4.* We start by recalling some facts about Chern classes of vector bundles on $\mathbf{P}^n$. Given such a bundle $U$, we may identify the Chern classes $c_i(U)$ with integers via the isomorphism

$$c_i(U) \in H^{2i}\big(\mathbf{P}^n, \mathbf{Z}\big) = \mathbf{Z}.$$

As customary, we assemble these integers into the *Chern polynomial*

$$c_t(U) = 1 + c_1(U) \cdot t + c_2(U) \cdot t^2 + \ldots + c_n(U) \cdot t^n.$$

As $c_t(U)$ has constant term $= 1$, we may formally invert it to define

$$c_t(U)^{-1} \in \mathbf{Z}[t]/(t^{n+1}).$$

The essential point for us is that given a long exact sequence of vector bundles

$$0 \longrightarrow U_{n+1} \longrightarrow U_n \longrightarrow \ldots \longrightarrow U_1 \longrightarrow U_0 \longrightarrow 0,$$

the Whitney product formula implies the identity

$$c_t(U_0) \cdot c_t(U_1)^{-1} \cdot c_t(U_2) \cdot \ldots \cdot c_t(U_{n+1})^{(-1)^{n+1}} = 1 \pmod{t^{n+1}}. \tag{*}$$

We apply this to the exact sequence $L_\bullet$, so that

$$U_i \;=\; \oplus\, \mathcal{O}(-j)^{b_{i,j}}.$$

Then

$$c_t(U_i) \;=\; \prod_j \big(1 - j\cdot t\big)^{b_{i,j}} \quad (\mathrm{mod}\ t^{n+1}).$$

Plugging this into (*) and formally taking logarithms, one arrives at the identity

$$\sum_{i,j} (-1)^i\, b_{i,j} \cdot \log(1 - j\cdot t) \;=\; 0 \qquad (\mathrm{mod}\ t^{n+1}). \tag{**}$$

But

$$-\log(1 - j\cdot t) \;=\; j\cdot t \,+\, j^2 \cdot \tfrac{t^2}{2} \,+\, \dots \,+\, j^n \cdot \tfrac{t^n}{n} \qquad (\mathrm{mod}\ t^{n+1}).$$

The result then follows from (**) upon collecting coefficients of powers of $t$. □

## 2.1.C The cone of Betti tables

There are many situations in algebraic geometry where questions become greatly simplified by working only up to scaling. For example it often quite difficult to decide whether a given divisor on a projective variety $X$ is very ample, i.e. a hyperplane section of $X$ under an embedding $X \subseteq \mathbf{P}^N$. As geometers realized during the 1960s, things run much more smoothly if one focuses instead on the condition that this property holds for a positive multiple of a given divisor. One of the key new insights of Boij and Söderberg is that one should take the same perspective here: the object that has a clean description is the cone of positive rational multiples of Betti tables.

We formalize this in the

**Definition 2.1.6 (Cone of Betti tables).** Denote by

$$\mathbf{B}(a^-, a^+) \;\subseteq\; \mathbf{D}^{\mathrm{HK}}(a^-, a^+)$$

the set of all positive rational multiples of the tables $b(L_\bullet)$ for $L_\bullet$ a resolution in the specified window.

In other words, $\beta \in \mathbf{B}(a^-, a^+)$ if and only if

$$\beta \;=\; c \cdot b(L_\bullet) \quad \text{for some } \; L_\bullet \; \text{ and } c \,\geq\, 0.$$

We refer to $\mathbf{B}(a^-, a^+)$ as the *cone of Betti tables*, terminology justified by the following remark:

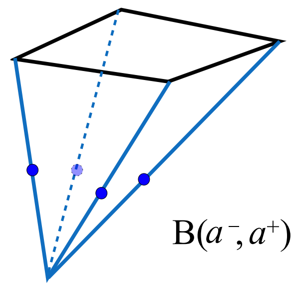



Figure 2.1: Schematic illustration of the Betti cone

**Lemma 2.1.7.** *The subset* $\mathbf{B}(a^-, a^+) \subseteq \mathbf{D}^{\mathrm{HK}}(a^-, a^+)$ *is a cone, i.e.*

$$\beta_1\,,\,\beta_2\ \in\ \mathbf{B}(a^-, a^+)\quad\Longrightarrow\quad \lambda_1\beta_1\,+\,\lambda_2\beta_2\,\in\,\mathbf{B}(a^-, a^+)$$

*for all rational numbers* $\lambda_1, \lambda_2 \geq 0$.

*Proof.* This boils down to the remark that

$$\ell_1 \cdot b(E_1)\ +\ \ell_2 \cdot b(E_2)\ =\ b\big(E_1^{\oplus \ell_1}\ \oplus\ E_2^{\oplus \ell_2}\big)$$

for all positive integers $\ell_1, \ell_2$. □

By way of preview, we close this section with a provisional statement of the main result of Eisenbud–Schreyer:

> Theorem. $\mathbf{B}(a^-, a^+)$ is the convex polyhedral cone whose edges are spanned by the Betti diagrams of all *pure resolutions* lying in the given window.

The statement is illustrated in Figure 2.1. These special generating rays are the subject of the next section.

## 2.2 Pure diagrams and the Boij–Söderberg fan

This section is devoted to a discussion of pure resolutions and an analysis of the simplicial fan they generate. We also give the statements of the main theorems.

### 2.2.A Pure resolutions

Fix a *degree sequence* $d$, i.e. a strictly increasing sequence of integers:

$$d\ :\ \ d_0\ <\ d_1\ <\ \ldots\ <\ d_n\ <\ d_{n+1}.$$

**Definition 2.2.1** (**Pure resolution**)**.** A *pure resolution* $L_\bullet$ with degree sequence $d$ is an exact complex of the form

$$0 \longleftarrow \mathcal{O}_{\mathbf{P}^n}(-d_0)^{b_0} \longleftarrow \mathcal{O}_{\mathbf{P}^n}(-d_1)^{b_1} \longleftarrow \ldots \longleftarrow \mathcal{O}_{\mathbf{P}^n}(-d_n)^{b_n} \longleftarrow \mathcal{O}_{\mathbf{P}^n}(-d_{n+1})^{b_{n+1}} \longleftarrow 0$$

in which each term $L_i$ is concentrated in degree $d_i$.

In other words, a resolution is pure if and only if its Betti table has a single non-zero entry in each column.

**Example 2.2.2** (**Some pure resolutions**)**.** Here are some examples on $\mathbf{P}^1$.

(i). As we have seen on several occasions, the (sheafified) resolution of $\mathbf{C}[x,y]/(x,y)^a$ has the form

$$0 \longleftarrow \mathcal{O}_{\mathbf{P}^1} \longleftarrow \mathcal{O}_{\mathbf{P}^1}(-a)^{a+1} \longleftarrow \mathcal{O}_{\mathbf{P}^1}(-a-1)^a \longleftarrow 0.$$

This is pure of type $0 < a < a+1$.

(ii). The Koszul resolution

$$0 \longleftarrow \mathcal{O}_{\mathbf{P}^1} \longleftarrow \mathcal{O}_{\mathbf{P}^1}(-a)^2 \longleftarrow \mathcal{O}_{\mathbf{P}^1}(-2a) \longleftarrow 0$$

determined by two relatively prime forms of degree $a$ is pure of type $0 < a < 2a$.

It was observed by Herzog and Kühl that the Betti numbers of a pure resolution are fixed up to scaling by its degree sequence:

**Proposition 2.2.3.** *Let $L_\bullet$ be a pure resolution with degree sequence*

$$d \;=\; (d_0 < \ldots < d_n).$$

*The the Betti table $b(L_\bullet)$ of $L_\bullet$ is determined up to scaling by $d$. In fact, writing $b_i = b_{i,d_i}(L_\bullet)$, for $i > 0$ one has*

$$b_i \;=\; b_0 \cdot \prod_{\substack{k \geq 1 \\ k \neq i}} \frac{d_k - d_0}{|d_k - d_i|}. \tag{2.2.1}$$

So in other words, each degree sequence $d$ within a given window determines a unique ray in the vector space $\mathbf{D}^{\mathrm{HK}}(a^-, a^+)$ of tables satisfying the Herzog–Hühl equations. For concreteness, we denote by

$$\pi(d) \;\in\; \mathbf{D}^{\mathrm{HK}}(a^-, a^+)$$

the (rational) table specified by (2.2.1), normalized so that $b_0 = 1$.

*Idea of Proof of Proposition 2.2.3.* The Betti table of a pure resolution has $n+2$ non-zero entries which have to satisfy the $(n+1)$ linearly independent Herzog–Kühl relations (2.1.5). So one expects it to be fixed up to scaling. This is indeed the case, and the expression (2.2.1) is the the solution to these equations. □

**Example 2.2.4** (**A decomposition**)**.** As noted at the end of last section, the main result (Theorem 2.2.8) will be that the Betti table of any resolution is a non-negative **Q**-linear combination of pure diagrams. Following [77], we illustrate this with the running Example 2.1.1, where $E = \mathbf{C}[x,y]/(x^2, xy, y^3)$ with Betti table

$$b \;=\; b(E) \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Here the Boij–Söderberg decomposition involves three of the pure resolutions appearing in Example 2.2.2:

$$\begin{aligned} b(E) \;&=\; \tfrac{1}{2}\cdot \pi(0,2,3) \;+\; \tfrac{1}{4}\cdot\pi(0,2,4) \;+\; \tfrac{1}{4}\cdot\pi(0,3,4) \\ &=\; \tfrac{1}{2}\cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \;+\; \tfrac{1}{4}\cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \;+\; \tfrac{1}{4}\cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 4 & 3 \end{bmatrix}. \end{aligned}$$

Note that this is purely a numerical relation: there is no claim that the complex resolving $E$ decomposes in any particular way. □

So far we have said nothing about the existence of pure resolutions having a given degree sequence. Indeed, this was not known at the time of [25], and the existence was conjectured by Boij and Söderberg. It was established by Eisenbud, Fløystad and Weyman:

**Theorem 2.2.5** ([62])**.** *For any degree sequence*

$$d_0 \;<\; d_1 \;<\; \ldots \;<\; d_n \;<\; d_{n+1},$$

*there exists an exact complex $L_\bullet$ on $\mathbf{P}^n$ that is pure of the required type.*

The construction in [62] used representation-theoretic ideas to produce resolutions that are actually $\mathrm{SL}(n+1)$-equivariant. Eisenbud and Schreyer [68] subsequently found a somewhat quicker cohomological approach in the spirit of Kempf. We will not write out the proofs here, but the following example gives at least a plausibility argument in the first case $n = 1$.

**Example 2.2.6** (**Pure resolutions on $\mathbf{P}^1$**)**.** Note quite generally that if $L_\bullet$ is a pure resolution of type $(d_0, \ldots, d_{n+1})$, then $L_\bullet \otimes \mathcal{O}_{\mathbf{P}^n}(-a)$ is pure of type $(d_0 + a, \ldots, d_{n+1} + a)$. So we are free to assume for instance that $d_0 = 0$. Focusing now on $\mathbf{P}^1$, this means that we'd like to produce a pure resolution of type $(0 < d_1 < d_2)$. For this, let $\delta$ be a general $d_1 \times d_2$ matrix of forms of degree $d_2 - d_1$ on $\mathbf{P}^1$, defining a map $\mathcal{O}_{\mathbf{P}^1}(d_1)^{d_2} \longrightarrow \mathcal{O}_{\mathbf{P}^1}(d_2)^{d_1}$. This homomorphism will be surjective for a sufficient general choice of $\delta$, so one arrives at an exact sequence:

$$0 \longrightarrow U \longrightarrow \mathcal{O}_{\mathbf{P}^1}(d_1)^{d_2} \xrightarrow{\;\delta\;} \mathcal{O}_{\mathbf{P}^1}(d_2)^{d_1} \longrightarrow 0, \qquad (*)$$

where $U$ is a bundle on $\mathbf{P}^1$ of rank $d_2 - d_1$ and degree 0. Now $U$, like any bundle on $\mathbf{P}^1$, splits as a sum of line bundles, and for sufficiently general $\delta$ one expects that $U$ slits as evenly as possible. In other words, one *hopes* that

$$U \;\cong\; \mathcal{O}_{\mathbf{P}^1}^{d_2-d_1}, \tag{**}$$

in which case the required pure complex is at hand. The isomorphism (**) is equivalent to the assertion that the homomorphism

$$H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d_1-1)^{d_2}\big) \longrightarrow H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d_2-1)^{d_1}\big)$$

given by $\delta$ is an isomorphism, and one can show that this is indeed the case for generic (or equivalently for some) $\delta$. See for example [77, p. 26] for a construction involving decompositions of SL(2)-modules. □

## 2.2.B The Boij–Söderberg fan

After one more definition, we will be ready to state the main theorem of Eisenbud–Schreyer.

**Definition 2.2.7 (Boij–Söderberg fan).** Having fixed a bounding window $a^-, a^+$ denote by

$$\Sigma(a^-, a^+) \;\subseteq\; \mathbf{D}^{\mathrm{HK}}(a^-, a^+)$$

the convex cone (over $\mathbf{Q}$) spanned by the diagrams $\pi(d)$ of all pure resolutions corresponding to degree sequences in the given window. We call $\Sigma(a^-, a^+)$ the *Boij–Söderberg* fan. □

Note that $\Sigma(a^-, a^+)$ is thus a purely combinatorial object. As we will see, it is in fact a simplicial fan.

The basic result of the theory is then:

**Theorem 2.2.8 (Eisenbud–Schreyer).** *The cone of Betti tables in* $\mathbf{D}^{\mathrm{HK}}(a^-, a^+)$ *is exactly the Boij–Söderberg fan, i.e.*

$$\mathbf{B}(a^-, a^+) \;=\; \Sigma(a^-, a^+)$$

*as subsets of* $\mathbf{D}^{\mathrm{HK}}(a^-, a^+)$.

In other words, a table $\beta = (\beta_{i,j})$ satisfying the Herzog–Kühl equations is (up to positive multiples) the Betti table of an actual resolution $L_\bullet$ if and only if $\beta$ can be written as a positive $\mathbf{Q}$-linear combination of the Betti diagrams of pure resolutions. We refer again to the schematic illustation in Figure 2.1: the points on the rays generating $\mathbf{B}(a^-, a^+)$ indicate pure diagrams.

One of the inclusions of the Theorem is a restatement of 2.2.5: since every degree sequence $d$ is realized by a pure resolution, $\pi(d) \in \mathbf{B}(a^-, a^+)$ and hence $\Sigma(a^-, a^+) \subseteq \mathbf{B}(a^-, a^+)$. Thus

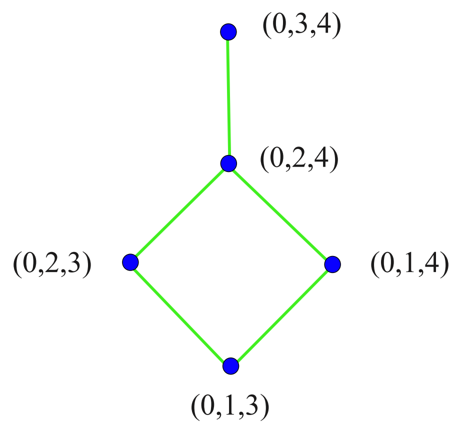



Figure 2.2: Poset of degree sequences between $(0,1,3)$ and $(0,3,4)$

the essential point is to prove that $\mathbf{B}(a^-,a^+)$ is contained in the Boij–Söderberg fan. Here the argument of Eisenbud–Schreyer proceeds in two stages. The first step is to understand in detail the combinatorial structure of $\Sigma(a^-,a^+)$: this study was already initiated by Boij and Söderberg. The proof is then completed by showing that $\Sigma(a^-,a^+)$ is cut out by linear functionals that are non-negative on all Betti tables. The novel and beautiful idea here is to use vector bundles (or coherent sheaves) on projective space to produce these functionals.

We now turn to the analysis of the Boij–Söderberg fan. The starting point is to define a partial ordering on degree sequences by declaring that

$$d \leq d' \quad \text{if} \quad d_i \leq d'_i \text{ for every } 0 \leq i \leq n+1.$$

Thus

$$[a^-,a^+] \; =_{\text{def}} \; \{\text{degree sequences } d \mid a^- \leq d \leq a^+\}$$

has the structure of a poset. Figure 2.2 displays this poset in the (canonical) example where $a^- = (0,1,3)$ and $a^+ = (0,3,4)$: in this case, $[a^-,a^+]$ contains five degree sequences.

The partially ordered set $[a^-,a^+]$ – like any poset – in turn determines a simplicial complex $\Delta = \Delta(a^-,a^+)$ whose simplices are increasing chains in $[a^-,a^+]$. This complex is pictured in Figure 2.3 when $a^- = (0,1,3)$ and $a^+ = (0,3,4)$. In this case, $\Delta$ contains two three-simplices, corresponding to the maximal chains

$$(0,1,3) < (0,1,4) < (0,2,4) < (0,3,4)$$
$$(0,1,3) < (0,2,3) < (0,2,4) < (0,3,4).$$

These meet along the (interior) two-simplex given by $(0,1,3) < (0,2,4) < (0,3,4)$.

The following proposition asserts in effect that $\Sigma(a^-,a^+)$ is the cone over $\Delta(a^-,a^+)$.

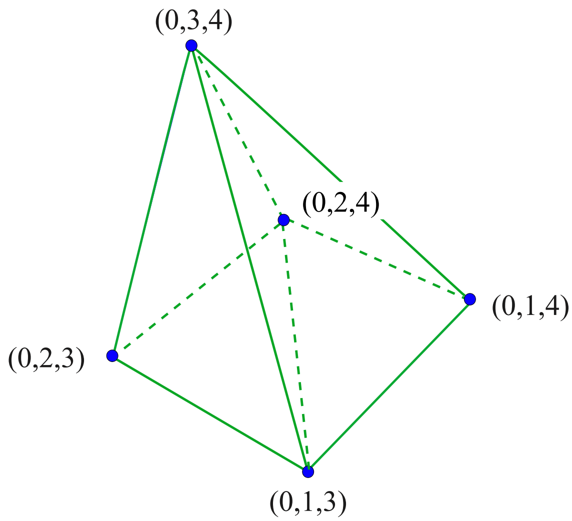



Figure 2.3: The simplicial complex $\Delta\big((0,1,3),(0,3,4)\big)$

**Proposition 2.2.9.** *The Boij–Söderberg fan*

$$\Sigma(a^-,a^+) \;\subseteq\; \mathbf{D}^{\mathrm{HK}}(a^-,a^+)$$

*is isomorphic to the geometric realization of the simplicial fan determined by* $\Delta(a^-,a^+)$*. More precisely, for any chain* $D$

$$d^1 \;<\; \dots \;<\; d^p$$

*of degree sequences in* $[a^-,a^+]$*, the corresponding vectors*

$$\pi(d^1)\,,\dots,\;\pi(d^p)\;\in\;\mathbf{D}^{\mathrm{HK}}(a^-,a^+)$$

*are linearly independent and span a simplicial cone* $\sigma(D)$*. As* $D$ *varies over all such chains, the* $\sigma(D)$ *form a fan whose intersection with an affine hyperplane is* $\Delta(a^-,a^+)$*.*

We refer to [77] for a sketch of the proof.

Since the top-dimensional simplices of $\Sigma(a^-,a^+)$ correspond to maximal chains $D \subseteq [a^-,a^+]$, its *facets* – i.e. the codimension one faces of $\sigma(D)$ – are given by chains of the form $D - \{f\}$ for some $f \in D$. One says that such a facet $F$ is *exterior* if it lies on a unique top simplex: in this case, $F$ is on the boundary of $\Sigma(a^-,a^+)$. For example, $\Sigma\big((0,1,3),(0,3,4)\big)$ has six exterior facets and one interior facet. One can analyze combinatorially the condition on $f \in D$ in order that $D - \{f\}$ determine an exterior facet. (See [25, Proposition 2.12], [68, Proposition 2.1] or [77, Proposition 2.2].)

Since $\Sigma(a^+,a^-)$ is a simplicial fan, given an exterior facet $F = \sigma\big(D - \{f\}\big)$ there is a linear functional

$$\phi_F\;:\;\mathbf{D}^{\mathrm{HK}}(a^-,a^+) \longrightarrow \mathbf{Q},$$

unique up to multiplication by a positive scalar, such that

$$\phi_F \mid F \ \equiv 0 \quad , \quad \phi_F\big(\pi(f)\big) \ > \ 0.$$

As $F$ is exterior, it follows that $\phi_F$ is non-negative on all of $\Sigma(a^-, a^+)$, and that in fact $\Sigma(a^-, a^+)$ is the intersection of all the non-negative half-spaces cut out by the $\phi_F$.

Theorem 2.2.8 then follows immediately from the main technical result of [68]:

**Theorem 2.2.10.** *If*

$$b \ = \ b(L_\bullet) \ \in \ \mathbf{D}^{\mathrm{HK}}(a^-, a^+)$$

*is the Betti table of a resolution, then* $\phi_F(b) \ \geq \ 0$ *for every exterior facet* $F$.

We remark that sometimes $\phi_F$ is the restriction of a coordinate function on $\mathbf{D}(a^-, a^+)$, in which case the conclusion of the Theorem is clear. For example, the face corresponding to the chain

$$(0,1,4) \ < \ (0,2,4) \ < \ (0,3,4)$$

in $\Sigma\big((0,1,3),(0,3,4)\big)$ is defined by the vanishing of the Betti number $b_{2,3}$, which is positive on the rest of the fan. However the supporting hyperplanes of the remaining exterior facets do not have such a simple description. Instead, as we have already hinted, the idea of Eisenbud–Schreyer is to show that the $\phi_F$ arise from a pairing between resolutions $L_\bullet$ and vector bundles $U$ on $\mathbf{P}^n$. We explain the basic idea of this pairing in the next section, but we will not go through the actual construction of the $\phi_F$.

## 2.3 Non-negative functionals on Betti tables

In this section we explain the idea of Eisenbud and Schreyer for producing functionals on $\mathbf{D}(a^-, a^+)$ that are non-negative on Betti tables. At the end we will say a word about the specific choices that come into the proof of Theorem 2.2.10.

By way of warm-up, consider a long exact sequence $V_\bullet$ of finite-dimensional vector spaces over a field:

$$0 \longrightarrow V_0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \ldots \longrightarrow V_{\ell-1} \longrightarrow V_\ell \longrightarrow 0.$$

Then it is elementary that the truncated Euler characteristics

$$\chi_k(V_\bullet) \ =_{\mathrm{def}} \ \sum_{i=0}^{k} (-1)^i \cdot \dim V_i$$

have a sign. In fact,

$$\chi_k(V_\bullet) \quad \text{is} \quad \begin{cases} \geq 0 & \text{if } k \text{ is even} \\ \leq 0 & \text{if } k \text{ is odd} \end{cases} .$$

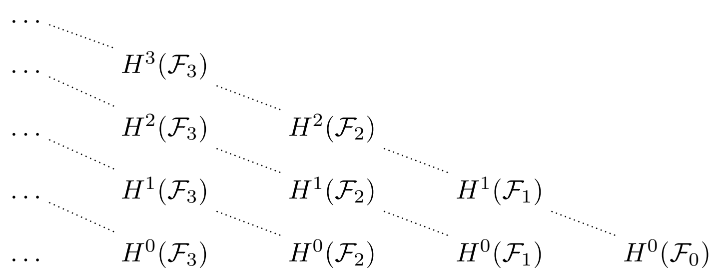



Figure 2.4: Diagram illustrating the definition of $\chi_0(\mathcal{F}_\bullet)$

The first observation is that there is a similar statement starting from a long exact sequence of sheaves.

Specifically, let $X$ be an irreducible projective variety, and consider a long exact sequence $\mathcal{F}_\bullet$ of coherent sheaves on $X$:

$$0 \longrightarrow \mathcal{F}_\ell \longrightarrow \mathcal{F}_{\ell-1} \longrightarrow \ldots \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_0 \longrightarrow 0.$$

Define

$$\chi_0(\mathcal{F}_\bullet) \; = \; \sum_{\substack{i,k \\ k \le i}} (-1)^{i-k} \cdot \dim H^k(\mathcal{F}_i) \tag{2.3.1}$$

The meaning of this expression is illustrated graphically in Figure 2.4: one starts by summing the dimensions of the cohomology groups appearing along each diagonal, and then combines the sums in an alternating manner. The remark of Eisenbud–Schreyer is that this expression likewise has a sign:

**Proposition 2.3.1.** *For any long exact sequence $\mathcal{F}_\bullet$ of sheaves, one has*

$$\chi_0(\mathcal{F}_\bullet) \; \geq 0.$$

As an example, suppose that $\mathcal{F}_\bullet$ is a short exact sequence

$$0 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_0 \longrightarrow 0$$

of sheaves. Denote by $V_\bullet$ the resulting long exact sequence of cohomology groups:

$$0 \longrightarrow H^0(\mathcal{F}_2) \longrightarrow H^0(\mathcal{F}_1) \longrightarrow H^0(\mathcal{F}_0) \longrightarrow H^1(\mathcal{F}_2) \longrightarrow H^1(\mathcal{F}_1) \longrightarrow \ldots.$$

Then we see that

$$\chi_0(\mathcal{F}_\bullet) \; = \; \chi_4(V_\bullet) \, + \, \dim H^2(\mathcal{F}_2)$$

which is indeed $\geq 0$.

In general, the Proposition is most efficiently established via spectral sequences:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^3(\mathcal{F}_3) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\cdots \longrightarrow & H^2(\mathcal{F}_3) & \longrightarrow & H^2(\mathcal{F}_2) & \longrightarrow & 0 & \longrightarrow & 0 \\
\cdots \longrightarrow & H^1(\mathcal{F}_3) & \longrightarrow & H^1(\mathcal{F}_2) & \longrightarrow & H^1(\mathcal{F}_1) & \longrightarrow & 0 \\
\cdots \longrightarrow & H^0(\mathcal{F}_3) & \longrightarrow & H^0(\mathcal{F}_2) & \longrightarrow & H^0(\mathcal{F}_1) & \longrightarrow & H^0(\mathcal{F}_0)
\end{array}$$

Figure 2.5: $E_1$ page of truncated spectral sequence

*Proof of Proposition 2.3.1.* Recall that there is a second-quadrant hypercohomology spectral sequence

$$E_1^{p,q} \;=\; H^q\big(X, \mathcal{F}_{-p}\big) \;\Rightarrow\; 0$$

converging to zero. (In general this sequence abuts to the hypercohomology groups of a complex, and in the case at hand these vanish since $\mathcal{F}_\bullet$ is exact.) Now define

$$'E_1^{p,q} \;=\; \begin{cases} E_1^{p,q} & \text{if } \; p+q \,\leq\, 0 \\ 0 & \text{if } \; p+q \,>\, 0. \end{cases}$$

(See Figure 2.5.) This is still a spectral sequence, and its abutment is zero except possibly in total degree zero. The Proposition then follows by the conservation of Euler characteristic throughout a spectral sequence. □

Now return to an exact complex $L_\bullet$ as in (2.1.2), and let $U$ be any vector bundle (or coherent sheaf) on $\mathbf{P}^n$. We define

$$\big\langle\, L_\bullet \,,\, U \,\big\rangle \;=\; \chi_0(L_\bullet \otimes U).$$

Since $L_\bullet \otimes U$ is an exact complex, Proposition 2.3.1 implies

**Corollary 2.3.2.** *For any exact complex $L_\bullet$ as above and vector bundle $U$ on $\mathbf{P}^n$,*

$$\big\langle\, L_\bullet \,,\, U \,\big\rangle \;\geq\; 0. \qquad \square$$

The next step is to explicate the expression appearing in the Corollary. Fixing $U$, observe that if $L_i = \oplus\, \mathcal{O}_{\mathbf{P}^n}(-j)^{b_{i,j}}$ then

$$L_i \otimes U \;=\; \oplus_j\, U(-j)^{b_{i,j}}.$$

Therefore:

$$\begin{aligned}
\big\langle\, L_\bullet\,,\, U \,\big\rangle \;&=\; \sum_{\substack{i,k\\ k\le i}} (-1)^{i-k}\cdot \dim H^k(L_i\otimes U) \\
&=\; \sum_{\substack{i,k\\ k\le i}} (-1)^{i-k}\cdot \Big(\, \sum_j \dim H^k\big(U(-j)^{b_{i,j}}\big)\Big) \\
&=\; \sum_{i,j}\,\Big(\,\sum_{k\le i} (-1)^{i-k}\cdot \dim H^k\big(U(-j)\big)\Big)\cdot b_{i,j}\;. \qquad (2.3.2)
\end{aligned}$$

The point now, as the notation suggests, is that for fixed $U$ we can view this as a linear functional on the Betti table of $L_\bullet$. Specifically, define

$$\phi_U : \mathbf{D}(a^-, a^+) \longrightarrow \mathbf{Q}$$

by the rule

$$\phi_U(\beta) \;=\; \sum_{i,j}\,\Big(\,\sum_{k\le i} (-1)^{i-k}\cdot \dim H^k\big(U(-j)\big)\Big)\cdot b_{i,j}\;.$$

Then we arrive at:

**Theorem 2.3.3.** *For any vector bundle $U$ on $\mathbf{P}^n$, the linear functional $\phi_U$ is non-negative on the cone of Betti tables of exact complexes $L_\bullet$.*

To a first approximation, this is the source of the functionals that Eisenbud–Schreyer use to prove Theorem 2.2.10. However this construction hasn't yet used the fact that $L_\bullet$ is a *minimal* complex. The actual functionals defining the faces of $\Sigma(a^-, a^+)$ arise by slightly modifying (2.3.2) in a manner tuned to the combinatorics describing the exterior facet $F$ without destroying the non-negativity.

There also remains the question of choosing the vector bundles $U$ with which to apply Theorem 2.3.3. Eisenbud and Schreyer show that the boundary facets of the Boij–Söderberg fan are cut out by the (modified) functionals associated to *supernatural bundles*: these are bundles $U$ whose cohomology tables $\{h^k\big(U(-j)\big)\}$ are as sparse as possible, much as pure resolutons are those whose Betti tables are particularly simple. We refer to [68], [173] or [77] for more details.

## 2.4 Notes

As we indicated in the text, since the original paper [68] of Eisenbud–Schreyer, the theory has developed in several directions. For example, in [69] Eisenbud and Schreyer extend the theory to the non Cohen–Macaulay setting. Eisenbud and Erman present a categorified perspective in [61]. We refer to the surveys cited in the body of the lecture for further information and references.

Inspired by the viewpoint of Boij–Söderberg theory, Berkesch–Erman–Kummini–Sam [22] characterize up to scaling the possible sequences of Betti numbers that can appear in resolutions of modules over a local ring. They find in particular that these can behave in quite surprising ways.

# Lecture 3

# Castelnuovo-Mumford Regularity: Definition, Examples and Applications

This is the first of two lectures focused on Castelnuovo–Mumford regularity. Introduced by Mumford in [140], regularity is a fundamental measure of the algebraic complexity of a sheaf or module. It controls the overall shape of Betti tables of resolutions, and the theory comes up in many other questions as well. Establishing bounds on regularity is a very interesting problem that has sparked a great deal of activity.

The current lecture presents an introduction to the theory and some of its applications. The first section, which is central to much of what follows, gives the basic definition and results. The remaining two sections are somewhat more specialized. Section 3.2 surveys several geometric and algebraic questions where Castelnuovo–Mumford plays a natural role. Finally we discuss in §3.3 a theorem of Bayer and Stillman relating the regularity of an ideal to that of its generic initial ideal.

Regularity bounds and constructions are the focus of Lecture 4.

## 3.1 Regularity for sheaves and modules

This section presents the definition and essential properties of Castelnuovo–Mumford regularity.

The theory has its origins in a classical argument on algebraic curves known as the "basepoint-free pencil trick." This asserts that if $B$ is a line bundle of positive degree on an algebraic curve $X$ that moves in a basepoint-free linear series, and if $\mathcal{F}$ is a sheaf on $X$ with the property that $H^1\big(X, \mathcal{F} \otimes B^*\big) = 0$, then $\mathcal{F}$ is itself globally generated and moreover the multiplication map $H^0(\mathcal{F}) \otimes H^0(B) \longrightarrow H^0(\mathcal{F} \otimes B)$ is surjective. See for example [11, p. 126] for a proof, which is an elementary application of the Koszul complex associated to $B$.

Mumford realized that this statement extends in a natural way to all dimensions. The hypothesis of the classical statement is replaced by the vanishing of one higher cohomology group in each degree. We could – and eventually will in §3.2.B – indicate the result for a basepoint-free ample line bundle on an arbitrary projective variety $X$. However it is simpler in the first instance to work with coherent sheaves on a projective space $\mathbf{P}$, where the hyperplane line bundle $\mathcal{O}_{\mathbf{P}}(1)$ plays the role of the divisor $B$.

The discussion proceeds in three parts. The first subsection deals with arbitrary coherent sheaves on projective space, while in the second we focus on the particularly important case of ideal sheaves. A parallel theory for graded modules, due to Eisenbud and Goto [63], is outlined in Section 3.1.C.

### 3.1.A Definition and first properties.

Fix a finite dimensional space $V$ of dimension $r+1$ over $\mathbf{C}$, and denote by $\mathbf{P} = \mathbf{P}(V)$ the $r$-dimensional projective space of one-dimensional quotients of $V$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}$.

We start with a definition and a theorem.

**Definition 3.1.1 (Castelnuovo–Mumford regularity).** Given an integer $m$, one says that $\mathcal{F}$ is *m-regular* in the sense of Castelnuovo–Mumford if

$$H^i\big(\mathbf{P}, \mathcal{F}(m-i)\big) \;=\; 0 \quad \text{for } i > 0. \tag{3.1.1}$$

**Theorem 3.1.2 (Mumford).** *Assume that $\mathcal{F}$ is $m$-regular. Then:*

*(i). $\mathcal{F}(m)$ is globally generated.*

*(ii). For every $k \geq 1$, the mapping*

$$H^0\big(\mathbf{P}, \mathcal{F}(m)\big) \otimes H^0\big(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(k)\big) \longrightarrow H^0\big(\mathbf{P}, \mathcal{F}(m+k)\big)$$

*is surjective.*

*(iii). $\mathcal{F}$ is $(m+k)$-regular for every $k \geq 1$.*

Observe that (i) and (iii) imply that $\mathcal{F}(m+k)$ is globally generated for every $k \geq 0$.

Serre's theorems guarantee that a high enough twist of any coherent sheaf on $\mathbf{P}$ is cohomologically well-behaved, and the Theorem suggests that one can think of Castelnuovo-Mumford regularity as an effective estimate of when this starts. Very vaguely speaking, the smaller the regularity of $\mathcal{F}$, the closer $\mathcal{F}$ is to already being "sufficiently twisted" in this sense. However a more important perspective for us is that regularity exerts overall control on degrees of generators of various syzygy modules associated to $\mathcal{F}$: see Theorem 3.1.8 and Corollary 3.1.9.

We'll prove the Theorem shortly, but first we give an additional definition and a couple of examples.

**Definition 3.1.3 (Regularity of a sheaf).** Unless $\mathcal{F}$ is supported on a finite set of points – in which case we say that $\operatorname{reg}(\mathcal{F}) = -\infty$ – there is a smallest integer $m$ for which Definition 3.1.1 is satisfied. We define this to be the *regularity* $\operatorname{reg}(\mathcal{F})$ of $\mathcal{F}$. □

**Example 3.1.4 (Line bundles on projective space).** It is important when dealing with regularity to keep in mind the basic facts about the higher cohomology of line bundles on projective space. Specifically, recall that if $0 < i < r$ then

$$H^i\big(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(\ell)\big) \ = \ 0 \ \text{ for all } \ \ell \in \mathbf{Z},$$

whereas

$$H^r\big(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(\ell)\big) \ = \ 0 \iff \ \ell > -(r+1).$$

Therefore one finds the prototypical fact that

$$\operatorname{reg}(\mathcal{O}_{\mathbf{P}}) \ = \ 0$$

on any projective space $\mathbf{P}$. This leads to several related computations:

(i). For any integer $a \in \mathbf{Z}$, $\operatorname{reg}(\mathcal{O}_{\mathbf{P}}(a)) = -a$ for any integer $a \in \mathbf{Z}$. In general,

$$\operatorname{reg}\big(\mathcal{F}(a)\big) \ = \ \operatorname{reg}(\mathcal{F}) - a$$

for any coherent sheaf $\mathcal{F}$.

(ii). Given any integers $a, b \in \mathbf{Z}$,

$$\operatorname{reg}\big(\, \mathcal{O}_{\mathbf{P}}(a) \oplus \mathcal{O}_{\mathbf{P}}(b)\,\big) \ = \ \max\{-a\,,\,-b\}.$$

In general, $\operatorname{reg}(\,\mathcal{F}_1 \oplus \mathcal{F}_2\,) \ = \ \max\{\operatorname{reg}(\mathcal{F}_1)\,,\,\operatorname{reg}(\mathcal{F}_2)\,\}$.

(iii). Using the Euler sequence

$$0 \longrightarrow \Omega^1_{\mathbf{P}} \longrightarrow V \otimes \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow \mathcal{O}_{\mathbf{P}} \longrightarrow 0$$

to compute its higher cohomology, one finds that the cotangent bundle $\Omega^1_{\mathbf{P}}$ of projective space is 2-regular. (Beware however that in general the regularity of the kernel of a surjective homomorphism of sheaves is not controlled by the regularities of the source and target: see Example 3.1.6 (iii).) □

**Example 3.1.5 (Complete intersections).** Let $X \subseteq \mathbf{P}^2$ be the complete intersection of two curves of degrees $d_1$ and $d_2$. Then the Koszul resolution

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-d_1 - d_2) \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-d_1) \oplus \mathcal{O}_{\mathbf{P}^2}(-d_2) \longrightarrow \mathcal{I}_X \longrightarrow 0$$

shows that $H^1\big(\mathbf{P}^2, \mathcal{I}_X(k)\big) = 0$ if and only of $k \geq d_1 + d_2 - 2$. Hence $\operatorname{reg}(\mathcal{I}_X) = d_1 + d_2 - 1$. More generally, if $X \subseteq \mathbf{P}$ is the complete intersection of $e$ hypersurfaces of degrees $d_1, \ldots, d_e$,then

$$\operatorname{reg}(\mathcal{I}_X) = (d_1 + \ldots + d_e) - (e-1). \qquad \square$$

**Example 3.1.6** (**Regularity in exact sequences**)**.** Let

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

be an exact sequence of sheaves on $\mathbf{P}$.

(i). If $\mathcal{F}'$ and $\mathcal{F}''$ are $m$-regular, then so is $\mathcal{F}$.

(ii). If $\mathcal{F}$ is $m$-regular and $\mathcal{F}'$ is $(m+1)$-regular, then $\mathcal{F}''$ is $m$-regular.

(iii). In general one cannot control the regularity of $\mathcal{F}'$ from the regularities of $\mathcal{F}$ and $\mathcal{F}''$. For instance, it can happen that $\mathcal{F}$ and $\mathcal{F}''$ are 0-regular while the regularity of $\mathcal{F}'$ is arbitrarily large. To construct a concrete example, take

$$X \;=\; X_k \;\subset\; \mathbf{P}^3$$

to be the union of $k$ disjoint lines $L_1, \ldots, L_k$ in $\mathbf{P}^3$, and consider the exact sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbf{P}^3} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Then $\text{reg}(\mathcal{O}_{\mathbf{P}^3}) = 0$, and $\mathcal{O}_X = \oplus\, \mathcal{O}_{L_i}$, so that also $\text{reg}(\mathcal{O}_X) = 0$. On the other hand, we claim that the regularity of the ideal sheaf $\mathcal{I}_X = \mathcal{I}_{X_k}$ must go to infinity with $k$. In fact, it follows from the exact sequence

$$H^0\big(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(m-1)\big) \longrightarrow H^0\big(X, \mathcal{O}_X(m-1)\big) \longrightarrow H^1\big(\mathbf{P}^3, \mathcal{I}_X(m-1)\big) \longrightarrow 0$$

that the group on the right is non-vanishing if

$$h^0(\mathcal{O}_{\mathbf{P}^3}(m-1)) \;=\; \binom{m+2}{3} \;<\; m \cdot k \;=\; h^0(\mathcal{O}_X(m-1)).$$

Therefore $r = \text{reg}(\mathcal{I}_X)$ must satisfy

$$\frac{1}{r} \cdot \binom{r+2}{3} \;\geq\; k.$$

(Concerning this phenomenon, compare Example 3.1.37.)

(iv). Suppose that $\mathcal{F}$ sits at the end of a (possibly infinite) long exact sequence

$$\ldots \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\mathcal{F}_i$ is $(m+i)$-regular. Then $\mathcal{F}$ is $m$-regular. (Chop into short exact sequences: see Lemma 3.1.7 below.) Corollary 3.1.9 gives a converse. □

We next give the proof of Mumford's theorem.

*Proof of Theorem 3.1.2.* We claim to begin with that statement (i) follows from (ii). In fact, consider the evaluation mapping

$$\mathrm{ev}_{\mathcal{F}(m)} : H^0\big(\mathbf{P}, \mathcal{F}(m)\big) \otimes_{\mathbf{k}} \mathcal{O}_{\mathbf{P}} \longrightarrow \mathcal{F}(m).$$

The assertion of (i) is that this is surjective as a homomorphism of sheaves. Fixing any $k$, this is equivalent to the surectivity of its twist

$$\mathrm{ev}_{\mathcal{F}(m)}(k) : H^0\big(\mathbf{P}, \mathcal{F}(m)\big) \otimes_{\mathbf{k}} \mathcal{O}_{\mathbf{P}}(k) \longrightarrow \mathcal{F}(m+k).$$

On the other hand, take $k \gg 0$ and consider the commutative diagram:

$$\begin{array}{ccc} H^0\big(\mathbf{P}, \mathcal{F}(m)\big) \otimes H^0\big(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(k)\big) \otimes_{\mathbf{k}} \mathcal{O}_{\mathbf{P}} & \longrightarrow & H^0\big(\mathbf{P}, \mathcal{F}(m)\big) \otimes_{\mathbf{k}} \mathcal{O}_{\mathbf{P}}(k) \\ \downarrow & & \downarrow {\scriptstyle \mathrm{ev}_{\mathcal{F}(m)}(k)} \\ H^0\big(\mathbf{P}, \mathcal{F}(m+k)\big) \otimes_{\mathbf{k}} \mathcal{O}_{\mathbf{P}} & \xrightarrow{\mathrm{ev}_{\mathcal{F}(m+k)}} & \mathcal{F}(m+k). \end{array}$$

Since in any event $\mathcal{F}(m+k)$ is globally generated when $k$ is large, we can suppose that the bottom horizontal map is surjective. Statement (ii) implies that the vertical map on the left is surjective, yieldng the surjectivity of $\mathrm{ev}_{\mathcal{F}(m)}(k)$, as required.

It remains to prove statements (ii) and (iii), and by induction it suffices to treat the case $k = 1$. For this, one starts with the (exact) Koszul complex $K_\bullet$ resolving $\mathcal{O}_{\mathbf{P}}$:

$$0 \longrightarrow \Lambda^{r+1} V \otimes \mathcal{O}_{\mathbf{P}}(-r-1) \longrightarrow \ldots \longrightarrow \Lambda^2 V \otimes \mathcal{O}_{\mathbf{P}}(-2) \longrightarrow V \otimes \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow \mathcal{O}_{\mathbf{P}} \longrightarrow 0.$$

To prove (ii), tensor through by $\mathcal{F}(m+1)$ to obtain a long exact sequence:

$$0 \longrightarrow \Lambda^{r+1} V \otimes \mathcal{F}(m-r) \longrightarrow \ldots \longrightarrow \Lambda^2 V \otimes \mathcal{F}(m-1) \longrightarrow V \otimes \mathcal{F}(m) \longrightarrow \mathcal{F}(m+1) \longrightarrow 0.$$

The $m$-regularity of $\mathcal{F}$ gives:

$$H^1\big(\Lambda^2 V \otimes \mathcal{F}(m-1)\big) \;=\; \ldots \;=\; H^r\big(\Lambda^{r+1} V \otimes \mathcal{F}(m-r)\big) \;=\; 0.$$

Chasing through the complex, it follows from Lemma 3.1.7 below that

$$V \otimes H^0\big(\mathbf{P}, \mathcal{F}(m)\big) \longrightarrow H^0\big(\mathbf{P}, \mathcal{F}(m+1)\big)$$

is surjective. Recalling that $V = H^0\big(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(1)\big)$, this proves (ii). Statement (iii) is established similarly upon twisting $K_\bullet$ by $\mathcal{F}(m)$. □

In the proof just completed, as well as at many other points in this Lecture, it is useful to have at hand the following elementary

**Lemma 3.1.7 (Diagram chasing).** *Suppose given a long exact sequence*

$$0 \longrightarrow \mathcal{F}_\ell \longrightarrow \mathcal{F}_{\ell-1} \longrightarrow \ldots \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_0 \xrightarrow{\varepsilon} \mathcal{F} \longrightarrow 0$$

*of coherent sheaves on a projective variety $X$.*

*(i). Assume that*

$$H^i(\mathcal{F}_0) \;=\; H^{i+1}(\mathcal{F}_1) \;=\; \ldots \;=\; H^{i+\ell}(\mathcal{F}_\ell) \;=\; 0.$$

*Then* $H^i(\mathcal{F}) = 0$.

*(ii). Assume that*

$$H^1(\mathcal{F}_1) \;=\; H^2(\mathcal{F}_2) \;=\; \ldots \;=\; H^\ell(\mathcal{F}_\ell) \;=\; 0.$$

*Then the homomorphism*

$$H^0\big(X, \mathcal{F}_0\big) \longrightarrow H^0\big(X, \mathcal{F}\big)$$

*induced by* $\varepsilon$ *is surjective.*

*Proof.* Chop the given sequence into short exact sequences and chase through the resulting diagram. □

For our purposes, the most important feature of regularity is that it controls the overall shape of the syzygies of a sheaf. Specifically, let $S = \mathrm{Sym}(V)$ be the homogeneous coordinate ring of $\mathbf{P}$, and as in Section 1.3.C denote by $E = E_{\mathcal{F}}$ the finitely generated graded $S$-module

$$E \;=\; \oplus_{k \gg -\infty}\, H^0\big(\mathbf{P}, \mathcal{F}(k)\big)$$

associated to $\mathcal{F}$. It follows from statement (ii) of Mumford's Theorem 3.1.2 that all the minimal generators of $E$ appear in degrees $\le \mathrm{reg}(\mathcal{F})$. In general the regularity of $\mathcal{F}$ is not computed simply by the degrees of the generators of $M$ (Example 3.1.5), but it is determined by the generating degrees of all of its syzyzy modules:

**Theorem 3.1.8 (Regularity and syzygies).** *Consider the minimal graded free resolution of* $E = E_{\mathcal{F}}$*:*

$$\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow E \longrightarrow 0, \tag{3.1.2}$$

*where* $P_i = \oplus\, S(-a_{i,j})$*. Then* $\mathcal{F}$ *is* $m$*-regular if and only if*

$$a_{i,j} \le i + m \quad \text{for all} \;\; i, j.$$

**Corollary 3.1.9.** *Keeping the notation of the Theorem, assume that* $\dim \mathrm{Supp}(\mathcal{F}) > 0$*, so that* $\mathrm{reg}(\mathcal{F}) > -\infty$*. Then*

$$\mathrm{reg}(\mathcal{F}) \;=\; \max\{\, \mathrm{reg}(P_i) - i \,\},$$

*where* $\mathrm{reg}(P_i) = \max_j\{a_{i,j}\}$*.* □

*Proof of Theorem 3.1.8.* Suppose first that $\mathcal{F}$ is $m$-regular. Then $E = E_{\mathcal{F}}$ is generated in degrees $\le m$ by Theorem 3.1.2, yielding a surjection

$$P_0 \;=_{\mathrm{def}}\; \oplus S(-a_{0,j}) \longrightarrow E \longrightarrow 0$$

with all $a_{0,j} \le m$. Consider the short exact sequence obtained by sheafifying:

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \oplus\, \mathcal{O}_{\mathbf{P}}(-a_{0,j}) \longrightarrow \mathcal{F} \longrightarrow 0.$$

By construction $H^1\big(\mathbf{P}, \mathcal{F}_1(k)\big) = 0$ for every $k \gg -\infty$,[1] and one finds that $\mathcal{F}_1$ is $(m+1)$-regular. Therefore $E_{\mathcal{F}_1}$ is generated in degrees $\leq (m+1)$ and repeating the argument one eventually arrives at (3.1.2).

Conversely, suppose that $E_{\mathcal{F}}$ admits a resolution 3.1.2. Sheafifying leads to a locally free resolution of $\mathcal{F}$:

$$\ldots \longrightarrow \mathcal{P}_2 \longrightarrow \mathcal{P}_1 \longrightarrow \mathcal{P}_0 \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\mathcal{P}_i = \oplus\, \mathcal{O}_{\mathbf{P}}(-a_{i,j})$ with all $a_{i,j} \leq m+i$. Using this to compute the cohomology of twists of $\mathcal{F}$, as in Example 3.1.6 (iv), one finds that $\mathcal{F}$ is $m$-regular. □

**Example 3.1.10 (Complete intersections, II).** Corollary 3.1.9 gives a quick way of computing the regularity of a sheaf if one happens to know its resolution. For example, consider as in Example 3.1.5 a complete intersection $X \subseteq \mathbf{P}$ of hypersurfaces of degrees $d_1, \ldots, d_e$. Then the homogeneous ideal of $X$ has the length $(e-1)$ Koszul resolution

$$0 \longrightarrow S(-\Sigma\, d_i) \longrightarrow \ldots \longrightarrow \oplus\, S(-d_i) \longrightarrow I_X \longrightarrow 0,$$

so Corollary 3.1.9 shows that

$$\operatorname{reg}(\mathcal{I}_X) \;=\; \operatorname{reg}\big(S(-d_1 - \ldots - d_e)\big) - (e-1).$$

Thus one recovers the assertion of Example 3.1.5. □

**Remark 3.1.11 (Regularity and complexity).** Theorem 3.1.8 gives a first explanation for why the regularity of a sheaf $\mathcal{F}$ is often considered as a measure of its algebraic complexity. Specifically, algorithms for computing suyzygies – such as those implemented in `Macaulay2` – typically work degree by degree. The theorem shows that $\operatorname{reg}(\mathcal{F})$ controls how many defrees need to be checked. A more precise algorithmic interpretation of regularity appears in a theorem of Bayer and Stillman, discussed in §3.3. □

Theorem 3.1.8 has a pleasant reformulation in terms of Betti tables. Specifically, the result asserts that $\mathcal{F}$ is $m$-regular if and only if all the non-zero entries of the Betti table of $E_{\mathcal{F}}$, displayed as described in Lecture 1 according to the `Macaulay` convention, vanish after the row with label $m$.

For example, consider the curve $C \subseteq \mathbf{P}^3$ arising as the complete intersection of a quadric and a cubic surface. Then $C$ is projectively normal, and the module $S/I_C = \Gamma_*\big(\mathbf{P}^3, \mathcal{O}_C\big)$ corresponding to $\mathcal{O}_C$ has a Koszul resolution of the shape:

$$0 \longleftarrow S/I_C \longleftarrow S \longleftarrow S(-2) \oplus S(-3) \longleftarrow S(-5) \longleftarrow 0.$$

This is summarized by the Betti table:

[1] If $\mathcal{F}$ has no associated points of dimension $= 0$, then $E_{\mathcal{F}} = H^0_*\big(\mathbf{P}, \mathcal{F}\big)$ and $H^1\big(\mathbf{P}, \mathcal{F}_1(k)\big) = 0$ for every $k$; in general $H^1$ only appears, in very negative twists, if one had to truncate to construct $E$.

| | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 1 | – | – |
| 1 | – | 1 | – |
| 2 | – | 1 | – |
| 3 | – | – | 1 |

So we see that $\operatorname{reg}(\mathcal{O}_C) = 3$. On the other hand, the Betti table of the homogeneous ideal $I_C$ of $C$ has the Betti table:

| | 0 | 1 |
|---|---|---|
| 2 | 1 | – |
| 3 | 1 | – |
| 4 | – | 1 |

.

So we see that the ideal sheaf $\mathcal{I}_C$ of $C$ satisfies $\operatorname{reg}(\mathcal{I}_C) = 4$. (Compare Examples 3.1.5 and 3.1.6.)

**Remark 3.1.12.** This example shows that one has to exercise a little care in specifying exactly which sheaf a Betti diagram refers to. For example, one often describes a sheaf as the image of a map $\mathcal{O}_{\mathbf{P}}^q(-b) \xrightarrow{u} \mathcal{O}_{\mathbf{P}}^p(-a)$ whereas computer programs such as `Macaulay2` may display the Betti tables of $\operatorname{coker}(u)$. In this case there is a shift by 1 in the indexing. □

We restate for emphasis a sheafified version of the previous result and its proof:

**Corollary 3.1.13.** *A coherent sheaf $\mathcal{F}$ on $\mathbf{P}$ is $m$-regular if and only if it sits in a (possibly infinite) long exact sequence*

$$\ldots \longrightarrow \oplus\, \mathcal{O}_{\mathbf{P}}(-a_{1,j}) \longrightarrow \oplus\, \mathcal{O}_{\mathbf{P}}(-a_{0,j}) \longrightarrow \mathcal{F} \longrightarrow 0,$$

*with $a_{i,j} \leq m + i$ for all $j$.* □

The following variant of Theorem 3.1.8 and Corollary 3.1.8 is often useful.

**Proposition 3.1.14 (Linear resolutions).** *Let $\mathcal{F}$ be a coherent sheaf on the projective space $\mathbf{P}$. Then $\mathcal{F}$ is $m$-regular if and only if there exist finite-dimensional vector spaces $W_i$ such that $\mathcal{F}$ admits a (finite or infinite) locally free resolution of the form*

$$\ldots \longrightarrow W_2 \otimes \mathcal{O}_{\mathbf{P}}(-m-2) \longrightarrow W_1 \otimes \mathcal{O}_{\mathbf{P}}(-m-1) \longrightarrow W_0 \otimes \mathcal{O}_{\mathbf{P}}(-m) \longrightarrow \mathcal{F} \longrightarrow 0.$$

*Moreover when $\mathcal{F}$ is $m$-regular, one can take $W_0 = H^0\big(\mathbf{P}, \mathcal{F}(m)\big)$.*

Such a resolution is called *linear* owing to the fact that the maps are defined by matrices whose entries are linear forms.

*Proof of Proposition 3.1.14.* The argument is very similar to the proof of Theorem 3.1.8. Given (3.1.3), the $m$-regularity of $\mathcal{F}$ follows from Corollary 3.1.13. Conversely, suppose that $\mathcal{F}$ is $m$-regular, and set $W_0 = H^0(\mathcal{F}(m))$. Thanks to the global generation of $\mathcal{F}(m)$ we get an exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow W_0 \otimes \mathcal{O}_{\mathbf{P}} \xrightarrow{ev} \mathcal{F} \longrightarrow 0,$$

where $ev$ is (a twist of) the evaluation map. Then $H^1\big(\mathbf{P}, \mathcal{F}_1(m)\big) = 0$ by construction, and as before the $m$-regularity of $\mathcal{F}$ then implies the $(m+1)$-regularity of $\mathcal{F}_1$. One continues by induction. (We leave it to the interested reader to check that this process eventually ends.) □

An important consequence of Proposition 3.1.14 is that regularity of vector bundles behaves well in tensor products:

**Theorem 3.1.15** (**Regularity of tensor products**). *Let $U_1$ and $U_2$ be locally free sheaves on the projective space $\mathbf{P}$. Assume that*

$$U_1 \ \text{ is } \ m_1\text{-regular} \ \ , \ \ U_2 \ \text{ is } \ m_2\text{-regular.}$$

*Then $U_1 \otimes U_2$ is $(m_1+m_2)$-regular. Moreover, the natural mapping*

$$H^0\big(\mathbf{P}, U_1(m_1)\big) \otimes H^0\big(\mathbf{P}, U_2(m_2)\big) \longrightarrow H^0\big(\mathbf{P}, (U_1 \otimes U_2)(m_1+m_2)\big) \qquad (*)$$

*is surjective.*

*Proof.* By Proposition 3.1.14, $U_1(m_1)$ admits a linear resoluton having the shape:

$$\ldots \longrightarrow W_2 \otimes \mathcal{O}_{\mathbf{P}}(-2) \longrightarrow W_1 \otimes \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow H^0\big(U_1(m_1)\big) \otimes \mathcal{O}_{\mathbf{P}} \longrightarrow U_1(m_1) \longrightarrow 0.$$

This being an exact sequence of locally free sheaves, the sequence obtained upon tensoring through by $U_2(m_2)$ remains exact. Both assertions then follow from Lemma 3.1.7. □

**Remark 3.1.16.** The proof shows that it is sufficient to assume that one of $U_1$ or $U_2$ be locally free. □

**Corollary 3.1.17** (**Regularity of symmetric and exterior powers**). *Working as always in characteristic zero, let $U$ be an $m$-regular locally free sheaf on $\mathbf{P}$. Then for any $p > 0$:*

$$\Lambda^p U \quad \text{and} \quad \mathrm{Sym}^p(U)$$

*are $pm$-regular. Moreover the natural maps*

$$\begin{aligned} \Lambda^p\, H^0\big(\mathbf{P}, U(m)\big) &\longrightarrow H^0\big(\,\mathbf{P}\,,\, (\Lambda^p U)(pm)\,\big) \\ \mathrm{Sym}^p\, H^0\big(\mathbf{P}, U(m)\big) &\longrightarrow H^0\big(\,\mathbf{P}\,,\, (\mathrm{Sym}^p U)(pm)\,\big) \end{aligned} \qquad (*)$$

*are surjecive.*

*Proof.* Theorem 3.1.15 implies that the $p$-fold tensor power $T^p(U) = \otimes^p U$ is $pm$-regular, and that the map

$$T^p H^0\big(\mathbf{P}, U(m)\big) \longrightarrow H^0\big(\mathbf{P}, (T^p U)(pm)\big) \tag{**}$$

is surjective. But in characteristic zero, $\Lambda^p U$ and $\mathrm{Sym}^p U$ are summands of $T^p U$, and the maps in (*) are summands of (**). □

**Remark 3.1.18 (Generalizations).** One can generalize Definition 3.1.1 and Theorem 3.1.2 to discuss regularity with respect to a globally generated ample bundle on any projective variety, as well as to a relative setting. These extensions arise naturally in a number of geometric questions. See Section 3.1.14 for one example. We refer to [128, Chapter 1.8] for a more detailed discussion. □

## 3.1.B Regularity of a subvariety.

As above, write $\mathbf{P} = \mathbf{P}^r = \mathbf{P}(V)$. From a geometric perspective, ideal sheaves of subvarieties $X \subseteq \mathbf{P}$ are the most interesting examples to consider. These merit a special definition:

**Definition 3.1.19.** A subvariety (or subscheme) $X \subseteq \mathbf{P}$ is *$m$-regular* if its ideal sheaf $\mathcal{I}_X \subseteq \mathbf{P}$ is so. The *regularity* of $X$ is the regularity of $\mathcal{I}_X$.

Theorem 3.1.8 yields:

**Corollary 3.1.20.** *$X$ is $m$-regular if and only if its homogeneous ideal $I_X$ is generated in degrees $\leq m$, and all the generators of the $i^{th}$ module of syzygies of $I_X$ occur in degrees $\leq m + i$.* □

Suppose that $\dim X = n$ and $\operatorname{codim} X = r - n \geq 2$. From the exact sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbf{P}} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

we see that

$$H^{i+1}\big(\mathbf{P}, \mathcal{I}_X(k)\big) \ = \ H^i\big(X, \mathcal{O}_X(k)\big) \ \text{ for all } \ i \ > \ 0 \tag{3.1.3}$$

and all $k$. Therefore, to begin with:

**Lemma 3.1.21 (Vanishings for regularity of a subvariety).** *The $n$-dimensional variety (or subscheme) $X \subseteq \mathbf{P}$ is $m$-regular if and only if*

$$H^i\big(\,\mathbf{P}\,,\, \mathcal{I}_X(m-i)\,\big) \ = \ 0 \quad \textit{for all} \ \ 1 \ \leq \ i \ \leq \ n+1. \quad \square$$

The criteria for regularity of $X$ have classical geometric interpretations. In fact, observe that $H^1\big(\mathbf{P}, \mathcal{I}_X(k)\big) = 0$ if and only if the restriction mapping

$$\rho_k \ : \ H^0\big(\mathbf{P}, \mathcal{O}_{\mathbf{P}^r}(k)\big) \longrightarrow H^0\big(X, \mathcal{O}_X(k)\big)$$

is surjective, i.e. hypersurfaces of degree $k$ trace out a complete linear series on $X$. Therefore:

**Proposition 3.1.22.** *A subvariety $X \subseteq \mathbf{P}$ of dimension $n$ is $m$-regular if and only if*

*(i). The restriction $\rho_{m-1}$ is surjective; and*

*(ii). $H^i\big(X, \mathcal{O}_X(m-i-1)\big) \ = \ 0 \quad$ for $i \geq 1$.* □

The two conditions in the Proposition differ somewhat in flavor. The vanishing in (ii) depends only on the line bundle defining the embedding $X \subseteq \mathbf{P}$, whereas (i) typically also involves the specific subspace of $H^0\big(X, \mathcal{O}_X(1)\big)$ cut out by hyperplanes in $\mathbf{P}$. In practice, verifying (i) is often the main difficulty in estimating regularity: for example, even for a smooth rational curve $C \subseteq \mathbf{P}^r$ it can be quite tricky to determine when $\rho_k$ is surjective. On the other hand, sometimes the condition in (i) is automatic, for instance if one is dealing with the complete embedding associated to a normally generated divisor. In these cases the regularity of $X$ can be quite easy to compute. See Proposition 3.1.28 for an illustration.

Here are some examples.

**Example 3.1.23 (Four points in the plane).** Let $X \subseteq \mathbf{P}^2$ consist of four distinct points. In Example 1.3.6 we saw that there are three possibilities for the shape of the resolution of the ideal of $X$. Specifically:

(a). No three points of $X$ are collinear. In this case $\mathcal{I}_X$ admits a resolution

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-4) \longrightarrow \mathcal{O}_{\mathbf{P}}(-2)^2 \longrightarrow \mathcal{I}_X \longrightarrow 0,$$

and $\text{reg}(X) = 3$.

(b). Three but not all four of the points of $X$ are collinear. Now $\mathcal{I}_X$ is no longer generated by conics, and its minimal resolution has the shape

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-3) \oplus \mathcal{O}_{\mathbf{P}}(-4) \longrightarrow \mathcal{O}_{\mathbf{P}}(-2)^2 \oplus \mathcal{O}_{\mathbf{P}}(-3) \longrightarrow \mathcal{I}_X \longrightarrow 0.$$

But here again $\text{reg}(X) = 3$: in other words, regularity does not pick up the difference between this case ands the previous one.

(c). If all four points of $X$ are collinear, then $X$ is the complete intersection of a line and a quartic, and $\text{reg}(X) = 4$.

**Example 3.1.24 (Finite sets).** Let $X \subseteq \mathbf{P}^r$ be a finite subset consisting of $d$ distinct points. Then:

(i). $X$ is $d$-regular. (This is equivalent to the assertion that $H^1\big(\mathbf{P}^r, \mathcal{I}_X(d-1)\big) = 0$, i.e. that the mapping

$$\rho_{d-1} \, : \, H^0\big(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(d-1)\big) \longrightarrow H^0\big(X, \mathcal{O}_X(d-1)\big)$$

is surjective. But this follows from the observation that for any point $x \in X$, a general union of hyperplanes through the remaining points is a hypersurface of degree $d-1$ vanishing on $X - \{x\}$ but not at $x$.)

(ii). $X$ fails to be $(d-1)$-regular if and only if it consists of $d$ collinear points.

(iii). Now suppose that $X \subseteq \mathbf{P}^r$ is a collection of $d$ *general* points. By choosing the points one at a time, one sees that for every $k \geq 0$ the restriction

$$\rho_k : H^0\big(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(k)\big) \longrightarrow H^0\big(X, \mathcal{O}_X(k)\big)$$

has *maximal rank*, i.e. $\rho_k$ is either injective or surjective. Therefore $X$ is $m$-regular if and only if

$$\binom{r+(m-1)}{r} \;=\; h^0\big(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(m-1)\big) \;\geq\; h^0\big(X, \mathcal{O}_X(m-1)\big) \;=\; d.$$

In particular, as a function of $d$, $\operatorname{reg}(X)$ grows like $d^{1/r}$. □

**Remark 3.1.25 (Failure of minimum resolution property).** The computations in part (iii) of the previous example might lead one to suspect that if $X \subseteq \mathbf{P}^r$ is a collection of $d$ general points, then $X$ should satisfy the *minimal resolution property*, meaning roughly speaking that the resolution of $I_X$ is as close as numerically possible to being pure. However it turns out that this is not the case: counter-examples were given by Eisenbud and Popescu [67] using the Gale transform. □

**Example 3.1.26 (Rational normal curves).** Let $C \subseteq \mathbf{P}^r$ be a rational normal curve of degree $r$. Then $\operatorname{reg}(C) = 2$, and $I_C$ has a linear resolution. □

**Example 3.1.27 (Regularity of curves on a quadric).** Curves on a quadric already provide some interesting examples.

(i). Fix $d$, and let $C \subseteq \mathbf{P}^3$ be the image of the embedding

$$\mathbf{P}^1 \hookrightarrow \mathbf{P}^3 \quad , \quad [s,t] \mapsto [s^d, s^{d-1}t, st^{d-1}, t^d].$$

Then $\operatorname{reg}(C) = d-1$. (It turns out that this example is extremal: see Theorem 4.2.2.) .

(ii). Let $C$ be a curve of type $(a,b)$ on a smooth quadric $Q = \mathbf{P}^1 \times \mathbf{P}^1 \subseteq \mathbf{P}^3$, with $a \geq b+1 \geq 3$. Then $\operatorname{reg}(C) = a$ □

As another example, we compute the regularity of a sufficiently positive embedding of an arbitrary variety. Specifically, let $X$ be an irreducible projective variety of dimension $n$, and let $B$ be a very ample line bundle on $X$. Given $d \geq 1$, set $L_d = B^{\otimes d}$ and consider the embedding

$$X \;\subseteq\; \mathbf{P}\big(H^0(L_d)\big) \;=\; \mathbf{P}^{r_d}, \tag{*}$$

defined by the complete linear series $|L_d|$, so that $r_d = h^0\big(X, L_d\big) - 1$. At least for sufficiently large $d$ this is the image of $X$ under the $d$-fold Veronese embedding of $\mathbf{P} = \mathbf{P}H^0(B)$. It turns out that for $d \gg 0$ the regularity of $X$, considered as a subvariety of $\mathbf{P}^{r_d}$, is completely determined:

**Proposition 3.1.28.** *Under the embedding* $(*)$, *if* $d \gg 0$ *then*

$$\operatorname{reg}(X) \;=\; \begin{cases} (n+1) & \text{if } H^n\big(X, \mathcal{O}_X\big) = 0 \\ (n+2) & \text{if } H^n\big(X, \mathcal{O}_X\big) \neq 0 \end{cases} .$$

*Proof.* Recall to begin with that if $d \gg 0$, then $L_d$ is normally generated, i.e. the maps

$$\operatorname{Sym}^k H^0\big(X, L_d\big) \longrightarrow H^0\big(X, (L_d)^{\otimes k}\big)$$

are surjective for all $k \geq 2$.[2] This implies that

$$H^1\big(\mathbf{P}^{r_d}, \mathcal{I}_{X/\mathbf{P}^{r_d}}(k)\big) \;=\; 0$$

for every $k$. On the other hand, under the embedding defined by $|L_d|$, $\mathcal{O}_X(1) = L_d$, and if $d \gg 0$ then $H^i\big(X, (L_d)^{\otimes k}\big) = 0$ for $i > 0$ and every $k \geq 1$. The assertion then follows from Proposition 3.1.22 (ii). □

**Example 3.1.29 (Veronese varieties).** Let $V_d \subseteq \mathbf{P}$ be the $d$-fold Veronese variety, i.e. the image of the $d$-fold Veronese embedding of $\mathbf{P}^n$. It is elementary that $V_d$ is projectively normal, and therefore

$$\operatorname{reg}(V_d) \;=\; (n+1)$$

for every $d \geq n+1$. (If $2 \leq d \leq n+1$, then $3 \leq \operatorname{reg}(V_d) \leq n+1$.) As the homogeneous ideal of $V_d$ is generated by quadrics, this means that high degree Veroneses do not have linear resolutions when $n \geq 2$. In fact, it will emerge in Lecture 8 that in a sense to be made precise the resolutions are "as non-linear as possible." □

## 3.1.C Arithmetic regularity of a graded module.

We have so far defined the Castelnuovo-Mumford regularity of a coherent sheaf $\mathcal{F}$ on a projective space $\mathbf{P}$, and have shown that it controls the minimal resolution of the corresponding graded module $E = E_{\mathcal{F}}$ over the polynomial ring $S$. There is a parallel theory, developed by Eisenbud and Goto [63], starting instead with an arbitrary finitely generated graded $S$-module $E$. The present subsection is devoted to a quick outline.

As above, let $V$ be a finite-dimensional vector space and let $S = \operatorname{Sym}(V)$ be the symmetric algebra on $V$. Fix a finitely generated graded $S$-module $E$.

[2]This is a classical fact that can be established for instance by working with the diagonal $\Delta \subseteq X \times X$. Writing $\mathrm{pr}_1, \mathrm{pr}_2 : X \times X \longrightarrow X$ for the projections, Serre vanishing implies that if $d \gg 0$ then

$$H^1\Big(\, X \times X \,,\, \mathcal{I}_{\Delta/X\times X} \,\otimes\, \mathrm{pr}_1^* L_d \,\otimes\, \mathrm{pr}_2^*(L_d)^{\otimes k-1}\Big) \;=\; 0.$$

It follows that $H^0\big(L_d\big) \otimes H^0\big((L_d)^{\otimes k-1}\big) \longrightarrow H^0\big((L_d)^{\otimes k}\big)$ is surjective, as required.

**Definition 3.1.30 (Arithmetic regularity).** We say that $E$ is *arithmetically $m$-regular* if the graded Betti numbers $b_{i,j}$ of $E$ vanish in degrees $j \geq m+i$:

$$b_{i,j}(E) \ =_{\text{def}}\ \dim \operatorname{Tor}_i^S(M, \mathbf{k})_j \ = \ 0 \ \text{ when } j \geq m+i.$$

The *arithmetic regularity* $\operatorname{arithreg}(E)$ of $E$ is the least such integer $m$. □

In other words, consider the minimal graded free resolution of $E$:

$$\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow E \longrightarrow 0,$$

where $P_i = \oplus S(-a_{i,j})$. The condition for arithmetic $m$-regularity is that $a_{i,j} \leq m+i$ for every $i$ and $j$.

**Remark 3.1.31 (Terminology).** In the literature, the property just defined is called simply the regularity of $E$. As we will be dealing with regularity for both sheaves and modules, we prefer to add an adjective to distinguish the two.

For modules arising from coherent sheaves, arithmetic regularity doesn't give anything new. Indeed, Theorem 3.1.8 implies:

**Proposition 3.1.32.** *Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P} = \mathbf{P}(V)$, and let*

$$E \ = \ E_{\mathcal{F}} \ = \ \oplus_{k \gg -\infty} \Gamma(\mathcal{F}(k))$$

*be the graded $S$-module it determines. Then $E$ is arithmetically $m$-regular if and only if $\mathcal{F}$ is $m$-regular.* □

However if one starts with a finitely generated $S$-module $E$ and takes $\mathcal{F} = \mathcal{F}_E = \widetilde{E}$ to be the coherent sheaf on $\mathbf{P}$ that it defines, then the arithmetic regularity of $E$ may diverge from the regularity of $\mathcal{F}$. This stems from the fact that different modules determine the same sheaf. Proposition 3.1.34 below makes precise the connection between the regularities of $E$ and $\mathcal{F}_E$.

The next result is the arithmetic analogue of Mumford's cohomological interpretation (Theorem 3.1.2) of regularity for sheaves. It involves the local cohomology modules $H^i_{\mathfrak{m}}(\cdot)$ with respect to the irrelevant ideal $\mathfrak{m} = S_+$. Recall that these are again graded $S$-modules that vanish in sufficiently large degrees.

**Theorem 3.1.33 (Eisenbud–Goto, [63]).** *Let $E$ be a finitely generated graded $S$-module. Then $E$ is arithmetically $m$-regular if and only if*

$$H^i_{\mathfrak{m}}(E)_{m-i+1} = 0 \quad \textit{for all } i \geq 0.$$

We refer to [63] or [60, Chapter 4] for the proof. The idea, which is inspired by Mumford's original proof of 3.1.2, is to use hyperplane sections to set up an induction on dimension.

The theorem allows one to explicate the connection between arithmetic and geometric regularity. Recall that for a graded $S$-module $E$, $H^0_{\mathfrak{m}}(E)$ is the graded submodule of $E$ consisting of elements annihilated by a power of the irrelevant ideal.

**Proposition 3.1.34** (**Arithmetic versus geometric regularity**). *Let $E$ be a finitely generated $S$-module, and denote by $\mathcal{F}_E$ the coherent sheaf on $\mathbf{P}$ that it determines. Then $E$ is arithmetically $m$-regular if and only if:*

*(i). $\mathcal{F}_E$ is $m$-regular;*

*(ii). The natural map $E_m \longrightarrow H^0\big(\mathbf{P}, \mathcal{F}_E(m)\big)$ is surjective; and*

*(iii). $H^0_{\mathfrak{m}}(E)_{m+1} \ = \ 0$.*

*Proof.* This follows from Theorem 3.1.33 thanks to the exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(E) \longrightarrow E \longrightarrow \Gamma_*(\mathbf{P}, \mathcal{F}_E) \longrightarrow H^1_{\mathfrak{m}}(E) \longrightarrow 0, \tag{3.1.4}$$

together with the fact that $H^i\big(\mathbf{P}, \mathcal{F}_E\big) = H^{i+1}_{\mathfrak{m}}\big(E\big)$ for $i \geq 1$. □

Proposition 3.1.34 takes a particularly clean form for homogeneous ideals $I \subseteq S$. Recall that $I$ is said to be *$m$-saturated* if $I_d = I^{\text{sat}}_d$ for $d \geq m$, where

$$I^{\text{sat}} \ = \ \big(I \,:\, \mathfrak{m}^\infty\big) \ = \ \{f \in S \mid f \cdot \mathfrak{m}^k \subseteq I \ \text{ for some } k \geq 0 \ \}$$

is the saturation of $I$. This saturation is the homogeneous ideal of the subscheme of projective space defined by $I$. Since $I^{\text{sat}}/I = H^1_{\mathfrak{m}}(I)$ and $H^0_{\mathfrak{m}}(I) = 0$, one finds:

**Corollary 3.1.35.** *A homogeneous ideal $I$ is arithmetically $m$-regular if and only if $I$ is $m$-saturated, and the corresponding ideal sheaf $\widetilde{I} \subseteq \mathcal{O}_{\mathbf{P}}$ is $m$-regular.* □

Example 3.1.39 outlines some constructions of homogeneous ideals with regularity considerably worse that the subvarieties they define.

We conclude with a number of examples and additional results.

**Example 3.1.36** (**Modules of finite length**). Let $E$ be a graded $S$-module of finite length. Then

$$\text{arithreg}(E) \ = \ \max \big\{\, k \mid E_k \neq 0 \,\big\}. \qquad \square$$

**Example 3.1.37** (**Arithmetic regularity in exact sequences**). Consider a short exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

of finitely generated graded $S$-modules.

(i). If $E'$ and $E''$ are arithmetically $m$-regular, then so is $E$.

(ii). If $E$ is arithmetically $m$-regular and $E'$ is arithmetically $(m+1)$-regular, then $E''$ is arithmetically $m$-regular.

(iii). If $E''$ is arithmetically $(m-1)$-regular and $E$ is arithmetically $m$-regular, then $E'$ is arithmetically $m$-regular.

(All three statements follow from the long exact sequence of graded Tor.) Recall (Example 3.1.6) that the analogue of (iii) fails in the geometric setting. One might say that this is one of the biggest differences between regularity for sheaves and modules. □

**Example 3.1.38** (**Linear resolution of truncations**). Let $E$ be an arithmetically $m$-regular graded $S$ module, and define $E_{\geq m} = \oplus_{k \geq m} E_k$. Then $E_{\geq m}$ has a linear resolution:

$$\ldots \longrightarrow \oplus S(-m-2) \longrightarrow \oplus S(-m-1) \longrightarrow \oplus S(-m) \longrightarrow E_{\geq m} \longrightarrow 0.$$

(It suffices to show that $E_{\geq m}$ is arithmetically $m$-regular. In view of Example 3.1.36, this follows from the exact sequence

$$0 \longrightarrow E_{\geq m} \longrightarrow E \longrightarrow E/E_{\geq m} \longrightarrow 0$$

together with Example 3.1.37.) □

**Example 3.1.39** (**Some unsaturated ideals**). It is quite easy to construct ideals defining nice varieties $X$ with regularity worse than $\operatorname{reg}(X)$.

(i). Let $C \subseteq \mathbf{P}^r$ be the rational normal curve of degree $r \geq 4$. Then $\operatorname{reg}(C) = 2$, and the homogeneous ideal $I_C$ of $C$ is generated by $\binom{r}{2}$ quadratic generators. However $C$ is also cut out scheme-theoretically by any $(r+1)$ general quadrics $Q_0, \ldots, Q_r \in I_C$. In other words, if

$$J \;=\; (Q_0, \ldots, Q_r) \;\subseteq\; S$$

is the homogeneous ideal they generate, then $\widetilde{J} = \mathcal{I}_C$. However in view of Corollary 3.1.35, $J$ is not arithmetically 2-regular

(ii). More generally, given any subvariety $X \subseteq \mathbf{P}$, let

$$J_m \;=\; \big(I_X\big)_{\geq m} \;\subseteq\; S$$

be the homogeneous ideal generated by all forms of degrees $\geq m$ vanishing on $X$. If $m \gg 0$ then

$$\widetilde{J_m} \;=\; \mathcal{I}_X \quad \text{and} \quad \operatorname{arithreg}(J_m) = m. \qquad \square$$

**Example 3.1.40** (**Realizing arithmetic regularity geometrically**). Denote by $S' = S[t]$ the polynomial ring obtained from $S$ by adding a new variable, and let $\mathbf{P}' = \operatorname{Proj}(S')$, so that $\mathbf{P}'$ is a projective space of dimension $r+1$. Given a homogeneous ideal $I \subseteq S$, consider the ideal $I' = I \cdot S'$ generated by $I$, and write $\mathcal{I}' = \widetilde{I'}$ for the corresponding ideal sheaf on $\mathbf{P}'$. Then

$$\operatorname{reg}(\mathcal{I}') \;=\; \operatorname{arithreg}(I).$$

**Example 3.1.41** (**Caviglia's example**). Let $S = \mathbf{C}[x, y, z, w]$ and let $I \subseteq S$ be the homogeneous ideal generated by the three quartics

$$f_1 \;=\; x^4 \;,\; f_2 \;=\; y^4 \;,\; h \;=\; xz^3 - yw^3.$$

A computer calculation shows that the module of syzygies among these quartics has three (Koszul) minimal generators in degree 8, together with one generator each in degrees $10, 12, 14$ and 16, the last being given by the relation

$$z^{12} \cdot f_1 \;-\; w^{12} \cdot f_2 \;-\; \left(x^3 z^9 + x^2 y z^6 w^3 + x y^2 z^3 w^6 + y^3 w^9\right) \cdot h \;=\; 0,$$

and in fact $\text{arithreg}(I) = 15$. On the other hand, $I^{\text{sat}}$ contains $(x, y)^4$, and the corresponding ideal sheaf has regularity only 6. (Caviglia [37] shows that the homogeneous ideal in $S$ generated by $x^d, y^d, xz^{d-1} - yw^{d-1}$ has (the exceptionally large) arithmetic regularity $= d^2 - 1$.) □

**Remark 3.1.42** (**Asymptotic regularity of powers**)**.** The difference between arithmetic and geometric regularity is strikingly illustrated by a circle of results concerning asymptotic regularity of powers of an ideal. Specifically, fix a homogeneous ideal $I \subseteq S$ in the polynomial ring, and let $\mathcal{I} \subseteq \mathcal{O}_{\mathbf{P}}$ be the corresponding ideal sheaf. Cutkosky–Herzog–Trung [47] and Kodiyalam [119] established that there are integers $a$ and $b$ with the property that

$$\text{arithreg}(I^n) \;=\; an + b \quad \text{for} \;\; n \gg 0.$$

In particular, the limit

$$\lim_{n \to \infty} \frac{\text{arithreg}(I^n)}{n}$$

exists and is an integer. By contrast, Cutkosky [45] showed that the asymptotic regularity $\text{reg}(\mathcal{I}^n)$ of an ideal sheaf can have irrational growth: given an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{\mathbf{P}}$, the limit

$$\lim_{n \to \infty} \frac{\text{reg}(\,\mathcal{I}^n\,)}{n}$$

exists, but it may be an irrational number. In fact, it was established by Cutkosky and the authors in [46] that this limit is governed by invariants of positivity on the blowing up of $\mathbf{P}$ along $\mathcal{I}$. □

# 3.2 Castelnuovo–Mumford regularity in nature

Castelnuovo–Mumford regularity arises naturally in several geometric and algebraic questions. In this section we briefly survey a few of these. Most of this material will not be used in the sequel.

## 3.2.A Hilbert polynomials and Gotzmann's bound.

Definition 3.1.1 was originally introduced in Mumford's lectures [140] as a tool for the construction of Grothendieck's Hilbert schemes. These parameterize all subschemes $X \subseteq \mathbf{P}$ of a fixed projective space $\mathbf{P}$ having a given Hilbert polynomial $P(t) = P_X(t)$. Grothendieck's idea was to associate to $X$ the space of all forms of suitable degree vanishing on $X$, viewed as a point in a Grassmannian. In order for this to get off the ground, one needs to know that there is an integer $m_0$, depending only on $P(t)$, such that $\mathcal{I}_X(m_0)$ is globally generated with vanishing higher cohomology. To this end, Mumford proved that in fact one can bound the regularity $\operatorname{reg}(X)$ of $X$ in terms of $P(t)$. While his argument could have been made effective, it wasn't intended to be sharp. Subsequently, the optimal statement was established by Gotzmann. We sketch this here following the approach and exposition of Green [91].

Gotzmann's result is the following:

**Theorem 3.2.1.** *Let $X \subseteq \mathbf{P}$ be a closed subscheme with ideal sheaf $\mathcal{I}_X \subseteq \mathcal{O}_{\mathbf{P}}$, and denote by*

$$P(t) \;=\; P_X(t) \;=\; \chi\big(\mathbf{P}, \mathcal{O}_X(t)\big)$$

*the Hilbert polynomial of $X$. There are unique integers*

$$a_1 \;\geq\; a_2 \;\geq\; \ldots \;\geq\; a_s \;\geq\; 0$$

*such that $P(t)$ can be written in the form*

$$P(t) \;=\; \binom{t+a_1}{a_1} + \binom{t+a_2-1}{a_2} + \ldots + \binom{t+a_s-(s-1)}{a_s}, \tag{3.2.1}$$

*and then* $\operatorname{reg}(\mathcal{I}) \leq s$.

**Example 3.2.2 (One-dimensional schemes).** Suppose that $X \subseteq \mathbf{P}$ is a one-dimensional scheme of degree $d$ and arithmetic genus $p$, so that $P_X(t) = dt + (1-p)$. The Gotzmann representation is gotten by taking

$$s \;=\; \binom{d}{2} + (1-p)$$
$$a_1 \;=\; \ldots \;=\; a_d \;=\; 1 \;\;,\;\; a_{d+1} \;=\; \ldots \;=\; a_s \;=\; 0.$$

In particular, $X$ is $\binom{d}{2} + (1-p)$ regular. This is optimal for suitable monomial schemes. □

Theorem 3.2.1 is deduced from a result of Macaulay concerning multiplication in the polynomial ring, which in turn involves special representations of integers in terms of binomial coefficients. Specifically, fix an integer $d > 0$. Given $c > 0$, there are unique integers $k_d > k_{d-1} > \ldots > k_1 \geq 0$ such that

$$c \;=\; \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \ldots + \binom{k_1}{1}. \tag{3.2.2}$$

This is called the $d^{\text{th}}$ Macaulay expansion of $c$, and the $k_i$ are its Macaulay coefficients. For example, when $d = 3$, the Macaulay coefficients of $c = 24$ are $k_3 = 6$, $k_2 = 3$ and $k_1 = 1$.

Given $d$ and $c$ as above, put

$$c^{<d>} \;=\; \binom{k_d+1}{d+1} + \binom{k_{d-1}+1}{d} + \ldots + \binom{k_1+1}{2}.$$

So for instance when $d=3$ and $c=24$, one finds that $c^{<3>}=40$. Observe that $(c+1)^{<d>} > c^{<d>}$, and hence the function $c \mapsto c^{<d>}$ is strictly increasing in $c$.

Consider now a linear series

$$W \;\subseteq\; H^0\big(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(d)\big)$$

of polynomials of degree $d$, and denote by

$$W_1 \;\subseteq\; H^0\big(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(d+1)\big)$$

the subspace spanned by $W$ via multiplication by linear forms. Macaulay showed that $W_1$ cannot be too small:

**Theorem 3.2.3 (Macaulay's Theorem).** *Set*

$$c \;=\; \operatorname{codim}\big(W\,,\, H^0(\mathcal{O}_{\mathbf{P}}(d))\big) \quad , \quad c_1 \;=\; \operatorname{codim}\big(W_1\,,\, H^0(\mathcal{O}_{\mathbf{P}}(d+1))\big).$$

*Then* $c_1 \;\leq\; c^{<d>}$.

Green gave an elegant proof of this result by proving an analogous statement for the restriction of $W$ to a general hyperplane: see [91] or [93]. As a sample of these ideas, we indicate how Macaulay's Theorem leads to Gotzmann's bound.

*Sketch of Proof of Theorem 3.2.1.* We grant the existence of the representation (3.2.1), and establish the stated bound on regularity. The proof is by induction on the dimension of $\mathbf{P}$, the statement being trivial for $\mathbf{P}=\mathbf{P}^1$. (A similar argument is used to show the possibility of expressing $P(t)$ in the form (3.2.1).)

Referring to (3.2.1), denote by $r$ the largest index such that $a_r > 0$. Fix a general hyperplane $H \subseteq \mathbf{P}$, let $X_H = X \cap H$, and denote by $P_H(t) = P_{X\cap H}(t)$ the Hilbert polynomial of $X_H$. Then $P_H(t) = P_X(t) - P_X(t-1)$, so $P_H(t)$ has the Gotzmann representation

$$P_H(t) \;=\; \binom{t+b_1}{b_1} + \binom{t+b_2-1}{b_2} + \ldots + \binom{t+b_r-(r-1)}{b_r},$$

where $b_i = a_i - 1$ for $1 \leq i \leq r$. In particular, we can assume by induction that $\operatorname{reg}(X_H) \leq r$. It then follows from the exact sequence $0 \longrightarrow \mathcal{I}_X(-1) \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{I}_{X\cap H} \longrightarrow 0$ that $H^i\big(\mathbf{P}, \mathcal{I}_X(s-i)\big) = 0$ for $i \geq 2$. So it remains only to prove that $H^1\big(\mathbf{P}, \mathcal{I}_X(s-1)\big) = 0$, or equivalently that

$$f_d \;=_{\text{def}}\; \operatorname{codim}\Big(H^0\big(\mathbf{P}, \mathcal{I}_X(d)\big), H^0\big(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(d)\big)\Big) \;=\; P_X(d).$$

when $d = s-1$. Now suppose to the contrary that $f_{s-1} < P_X(s-1)$. Writing

$$P_X(t) \;=\; \binom{t+a_1}{t} + \binom{t+a_2-1}{t-1} + \ldots + \binom{t+a_s-(s-1)}{t-(s-1)},$$

this implies to begin with that

$$(f_{s-1})^{<s-1>} \;<\; P_X(s).$$

(Consider separately the cases $a_s = 0$ and $a_s > 0$). By repeated applications of Macaulay's theorem, one finds first that $f_s < P_X(s)$ and then that

$$f_d \;\leq\; (f_{d-1})^{<d-1>} \;<\; P_X(d-1)^{<d-1>} \;=\; P_X(d)$$

for all $d \geq s+1$. On the other hand, $f_d = P_X(d)$ for $d \gg 0$ thanks to Serre vanishing. This contradiction completes the proof. □

## 3.2.B Regularity, vanishing theorems, and positivity

Castelnuovo-Mumford regularity is a powerful geometric tool in the presence of Kodaira-type vanishing theorems, where it leads to various sorts of positivity statements. In fact, special cases of Theorem 3.1.2 have been rediscovered several times in this context.

To give the flavor, we sketch here two applications. The first, a simple application of vanishing, concerns the geometry of adjoint-type divisors. The second gives some positivity statements for direct images of suitably positive line bundles. The present section is somewhat more advanced algebro-geometrically than the others in this lecture.

A word on notation: it is traditional to discuss questions of this sort using the language of divisors rather than line bundles. We follow this custom here.

**Regularity with respect to a very ample divisor.** So far we have developed the theory of Castelnuovo–Mumford regularity in the context of sheaves on projective space. However it is convenient to rephrase the main results without explicit reference to a projective embedding.

Consider then an irreducible projective variety $X$, and suppose that $B$ is a fixed very ample divisor on $X$.

**Definition 3.2.4.** We say that a coherent sheaf $\mathcal{F}$ on $X$ is *m-regular with respect to B* if it satisfies the vanishings

$$H^i\big(X, \mathcal{F} \otimes \mathcal{O}_X\big((m-i)B\big)\big) \;=\; 0 \quad \text{for} \;\; i > 0. \qquad \square$$

One then has:

**Theorem 3.2.5** (**Generalization of Mumford's Theorem**). *Assume that $\mathcal{F}$ is $m$-regular with respect to the very ample divisor $B$. Then*

*(i). $\mathcal{F} \otimes \mathcal{O}_X(mB)$ is globally generated.*

*(ii). For every $k \geq 1$, the mapping*

$$H^0\big(X, \mathcal{F} \otimes \mathcal{O}_X(mB)\big) \otimes H^0\big(X, \mathcal{O}_X(kB)\big) \longrightarrow H^0\big(X, \mathcal{F} \otimes \mathcal{O}_X((m+k)B)\big)$$

*is surjective.*

*(iii). $\mathcal{F}$ is $(m+k)$-regular with respect to $B$ for every $k \geq 1$.*

*Proof.* This is a formal consequence of Theorem 3.1.2: one considers the embedding $X \subseteq \mathbf{P}$ defined by $|B|$, so that $\mathcal{O}_X(B) = \mathcal{O}_{\mathbf{P}}(1) \mid X$, and views $\mathcal{F}$ as a coherent sheaf on $\mathbf{P}$. □

**Example 3.2.6** (**Regularity and syzygies**). Theorem 3.1.2 and the results growing out of it relating regularity to syzygies do not extend to the more general setting of Definition 3.2.4. This is because the trivial line bundle $\mathcal{O}_X$ need not be 0-regular with respect to $B$ and then $\mathcal{O}_X(-kB)$ is not $k$-regular. However there is a variant, due to Arapura [8] taking into account the regularity of $\mathcal{O}_X$:

> **Proposition 3.2.7.** *Assume that $\mathcal{O}_X$ is $a$-regular with respect to a very ample divisor $B$ for some $a \geq 1$. If a coherent sheaf $\mathcal{F}$ on $X$ is $m$-regular with respect to $B$, then it admits a (possibly infinite) resolution having the shape*
>
> $$\ldots \longrightarrow W_2 \otimes \mathcal{O}_X(-b_2B) \longrightarrow W_1 \otimes \mathcal{O}_X(-b_1B) \longrightarrow W_0 \otimes (-b_0B) \longrightarrow \mathcal{F} \longrightarrow 0,$$
>
> *where $b_i = ia + m$, and the $W_i$ are finite dimensional vector spaces.* □

**Remark 3.2.8** (**Regularity with respect to a globally generated ample divisor**). The interested reader may check that it suffices to assume in the definition that $B$ is ample and globally generated. To see this, one can either repeat the proof of Mumford's theorem, or else apply that result to the direct image of $\mathcal{F}$ under finite the morphism

$$\phi = \phi_B : X \longrightarrow \mathbf{P}H^0(B)$$

determined by $B$, and use the surjection $\phi^*\phi_*\mathcal{F} \twoheadrightarrow \mathcal{F}$. In this strengthened form, the Theorem 3.2.5 becomes a direct generalization to all dimensions of the base-point free pencil trick on curves that we discussed at the beginning of this Lecture.

**Geometry of adjoint bundles.** The first results involve adjoint bundles on a smooth complex projective variety $X$. These are bundles of the form $\mathcal{O}_X(K_X + L)$, where $K_X$ is a canonical divisor of $X$ and $L$ is a divisor satisfying a suitable positivity condition. One thinks of these as the higher-dimensional analogues of line bundles of degree $\geq 2g - 1$ on a curve of genus $g$, which are particularly well behaved.

Adjoint divisors derive their importance from the the celebrated:

**Theorem 3.2.9 (Kodaira vanishing theorem).** *Let $X$ be a smooth complex projective variety. If $A$ is any ample divisor on $X$, then*

$$H^i\big(X, \mathcal{O}_X(K_X + A)\big) \ = \ 0 \quad \textit{for} \ \ i > 0.$$

We refer for example to [99, Chapter 0] or [128, Chapter 4] for further discussion and proofs of this very basic result.

The point now is that Kodaira vanishing works very well as input to Theorem 3.2.5. As a simple example, here is a criterion the for global generation and very ampleness of suitable adjoint bundles.

**Proposition 3.2.10.** *Let $X$ be a smooth complex projective variety of dimension $n$, let $B$ be a very ample divisor on $X$, and set*

$$L_d \ = \ \mathcal{O}_X(K_X + dB).$$

(*i*). *If $d \geq n+1$, then $L_d$ is globally generated.*

(*ii*). *If $d \geq n+2$, then $L_d$ is very ample.*

Observe that the bounds on $d$ are optimal when $X = \mathbf{P}^n$ and $B$ is the hyperplane divisor. In Section 6.4 we will study the syzygetic properties of embeddings defined by bundles of this type.

*Proof of Proposition 3.2.10.* Thanks to Kodaira vanishing, $L_d$ is 0-regular with respect to $B$ when $d \geq n+1$, and hence (i) follows from Theorem 3.2.5. For (ii) one observes that the sum of a free and a very ample divisor is very ample. □

**Example 3.2.11 (Extensions).** Several extensions are possible. To begin with, one can work more generally with $\mathcal{O}_X(K_X + dB + P)$ where $P$ is any nef divisor, since Kodaira still gives the requisite vanishings. (Recall that a divisor $P$ is *nef* if $\big(C \cdot P\big) \geq 0$ for every irreducible curve $C$ on $P$. By a theorem of Kleiman, the sum of a nef and an ample divisor is ample: cf [128, Chapter 1.4].) More generally still, it suffices that $B$ be ample and globally generated: for (i) this follows from Remark 3.2.8, and in general one can argue with Seshadri constants via [128]. □

**Remark 3.2.12 (Reider's theorem and Fujita's conjecture).** It is a very interesting question whether analogous statements hold for divisors of the form $K_X + dA$ assuming only that $A$ is ample. For surfaces the situation is well-understood, but in higher dimensions much less is known. We refer for instance to [170], [127] and [128, Chapter 10.4] for an overview. □

**Positivity of direct images.** The second application asserts the positivity of the direct images of certain divisors on $Y$ under a nice morphism $Y \longrightarrow X$ So we start with a few words about notions of positivity for vector bundles.

Let $F$ be a vector bundle on an irreducible projective variety $X$. One says that $F$ is *ample* if the Serre line bundle $\mathcal{O}_{\mathbf{P}(F)}(1)$ is ample, and $F$ is *nef* if $\mathcal{O}_{\mathbf{P}(F)}(1)$ is so. If $\text{rank}(F) = 1$ then $\mathbf{P}(F) = X$ and $\mathcal{O}_{\mathbf{P}(F)}(1) = F$, so at least this recovers the situation for line bundles. While the intuition behind the definition might not be completely transparent, it turns out to lead to all the properties one would like. For instance:

(i). A direct sum of ample (or nef) line bundles is ample (or nef).

(ii). A quotient of an ample (or nef) vector bundle is ample (or nef).

(iii). A vector bundle $F$ is ample if any only if the symmetric power $\text{Sym}^k(F)$ is ample for any (or all) $k > 0$. The same holds with "ample" replaced by "nef."

(iv). Assume that $F$ is ample, and let $D$ be an arbitrary divisor on $X$. If $k \gg 0$ then $\text{Sym}^k(F) \otimes \mathcal{O}_X(D)$ is globally generated. In particular, a sufficiently large symmetric power of $F$ can be expressed as a quotient of a direct sum of very ample line bundles.

(v). Conversely, suppose that $F$ is a vector bundle on $X$, and suppose that there is a fixed divisor $D$ having the property that $\text{Sym}^k(F) \otimes \mathcal{O}_X(D)$ is globally generated for all $k \gg 0$. Then $F$ is nef.

We refer to [128, Chapter 6] for proofs and an overview of positivity for bundles.

The following statement generalizes a result of Laytimi and Nahm [123, Theorem 2.2].

**Theorem 3.2.13.** *Let $f : Y \longrightarrow X$ be a flat morphism of irreducible projecective varieties, and let $L$ be an ample divisor on $Y$. Then for $k \gg 0$*

$$F_k \ =_{\text{def}} \ f_*\big(\mathcal{O}_Y(kL)\big)$$

*is an ample vector bundle on $X$.*

*Proof.* Observe to begin with that if $k$ is sufficiently large then $\mathcal{O}_Y(kL)$ has no higher cohomology along the fibres thanks to the amplitude of $L$. Therefore it follows from the theorems on cohomology and base-change that the higher direct images of the line bundle in question vanish for $k \gg 0$ and that $F_k$ is indeed locally free. Moreover

$$H^i\big(X, F_k \otimes \mathcal{O}_X(N)\big) \ = \ H^i\big(Y, \mathcal{O}_Y(kL + f^*N)\big) \tag{*}$$

for any divisor $N$ on $X$ and every $i$ provided that $k$ is sufficiently large as then the Leray spectral sequence degenerates. Fix now a very ample divisor $B$ on $X$. Observe that to prove the Theorem, it suffices to show that

$$H^i\Big(X, F_k \otimes \mathcal{O}_X\big(-(i+1)B\big)\Big) \ = \ 0 \ \text{ for } \ i > 0 \text{ and } k \gg 0.$$

Indeed, this implies that $F_k$ is a quotient of a sum of copies of $\mathcal{O}_X(B)$ thanks to Theorem 3.2.5, and hence is ample by virtue of properties (i) and (ii). But the stated vanishing follows from (*) and Serre vanishing on $Y$, and we are done. □

One thinks of Theorem 3.2.13 as being a statement of Serre-type in that the assertion holds for a sufficiently large multiple of an ample divisor. A theorem of Mourougane gives a more precise statement under additional hypotheses:

**Theorem 3.2.14** (**Mourourgane**, [138])**.** *Suppose that*

$$f : Y \longrightarrow X$$

*is a smooth morphism of non-singular complex projective varieties, and denote by $K_{Y/X} = K_Y - f^*K_X$ the relative canonical divisor of $f$. Let $L$ be any ample divisor on $X$. Then the direct image*

$$F \;=\; f_*\big(\mathcal{O}_Y(K_{Y/X} + L)\big)$$

*either vanishes or is an ample vector bundle on $Y$*

*Proof.* The argument mirrors the proof of a theorem of Kollár (c.f. [128, 6.3.61]), but the result is more elementary as one uses Kodaira in place of Kollár vanishing. We will show that $F$ (if non-zero) is nef; amplitude is established by replacing $F$ with a small perturbation by a $\mathbf{Q}$-divisor, as in [128]. Fix a very ample divisor $B$ on $X$ which is sufficiently positive so that $B - K_X$ is ample. Writing $n = \dim X$, we will prove:

$$F \otimes \mathcal{O}_X\big((n+1)B\big) \;\text{ is globally genenerated,} \tag{3.2.3}$$

and in particular this bundle is nef.

Granting (3.2.3) for the time being, the idea is to bootstrap by applying it to a product. Fix $k \geq 2$ and consider the $k$-fold fibre product

$$f^{(k)} : Y^{(k)} =_{\text{def}} Y \times_X \;\ldots\; \times_X Y \longrightarrow X.$$

Since $f$ is smooth, so too is is $f^{(k)}$. Write $L^{(k)}$ for the sum of the pull-backs of $L$ under the projections $\text{pr}_i : Y^{(k)} \longrightarrow Y$. Then $K_{Y^{(k)}/X} = (K_{Y/X})^{(k)}$, and by Künneth:

$$f^{(k)}_*\big(K_{Y^{(k)}/X} + L^{(k)}\big) \;=\; F^{\otimes k}.$$

Now apply (3.2.3) to $f^{(k)}$. It follows that $F^{\otimes k} \otimes \mathcal{O}_X\big((n+1)B\big)$ is globally generated for all $k > 0$, and hence so to is $\text{Sym}^k(F) \otimes \mathcal{O}_X\big((n+1)B\big)$. This implies $F$ is nef by Property (v).

It remains to prove (3.2.3). Thanks to the amplitude of $B$ and $B - K_X$, $\big(L + f^*(kB - K_X)\big)$ is an ample divisor on $Y$ for all $k > 0$. Therefore

$$H^i\Big(Y, \mathcal{O}_Y\big(K_{Y/X} + L + f^*(kB)\big)\Big) \;=\; 0 \quad \text{for all } \; i,\, k > 0$$

by Kodaira vanishing. Lemma 3.2.15 below then yields the vanishing of the higher direct images of $\mathcal{O}_Y(K_{Y/X} + L)$ and hence also an isomorphism

$$H^i\big(X, F \otimes \mathcal{O}_X(kB)\big) \;=\; H^i\big(Y, \mathcal{O}_Y\big(K_{Y/X} + L + f^*(kB)\big)\big) \;=\; 0$$

for $i$ , $k > 0$. Taking $k = (n+1-i)$, (3.2.3) follows from Theorem 3.2.5. □

We record for the convenience of the reader the fact invoked in the argument just completed. See [128, 4.3.10] for the proof.

**Lemma 3.2.15** (**Criterion for vanishing of higher direct images**). *Let $f : V \longrightarrow W$ be a morphism of irreducible projective varieties, and let $\mathcal{F}$ be a coherent sheaf on $V$. Suppose that $A$ is an ample divisor on $W$ with the property that*

$$H^i\Big(V, \mathcal{F} \otimes f^*\mathcal{O}_W(kA)\Big) \;=\; 0$$

*for all $i > 0$ and all $k \gg 0$. Then $R^j f_*(\mathcal{F}) = 0$ for $j > 0$.* □

**Remark 3.2.16.** Needless to say, the smoothness hypothesis is unrealistic in practice. Several authors, eg [**??**] have established much more subtle and powerful results along the same lines.

## 3.2.C Regularity, simplicial complexes, and graphs

Various combinatorial constructions give rise in a natural way to monomial ideals in a polynomial ring. It is then interesting to relate algebraic invariants of the resulting ideals – for instance their arithmetic regularity – to the underlying combinatorial geometry. Here we will state without proof a couple of results as a sample of the large literature in this area. We recommend the very nice papers [105] and [78] of Hà and Francisco–Mermin–Schweig, as well as Chapter 8 of the book [110] of Herzog and Hibi for references and much more information.

We start with Stanley–Reisner ideals. Let $\Delta$ be a simplicial complex on the vertex set $[n] = \{1, \ldots, n\}$. In other words, $\Delta$ consists of a collection of subsets $\sigma$ of $[n]$ – the *simplices* of $\Delta$ – having the property that if $\sigma \in \Delta$, then any subset $\sigma' \subseteq \sigma$ – a *face* of $\sigma$ – also lies in $\Delta$. A simplicial complex $\Delta$ on [4] is pictured on the left in Figure 3.1. It consists of four vertices, five 1-simplices and one 2-simplex.

A simplicial complex $\Delta$ on $[n]$ determines a square-free monomial ideal

$$I_\Delta \;\subseteq\; \mathbf{C}[x_1, \ldots, x_n] \;=\; S,$$

called its *Stanley-Reisner* ideal, as follows. For any subset $F \subseteq [n]$, denote by $x^F$ the product of the variables indexed by $F$. The Stanley–Reisner ideal of $\Delta$ is the ideal generated by all monomials $x^F$ such that $F$ is *not* a face of $\Delta$:

$$I_\Delta \;=\; \big\{\, x^F \;\mid\; F \not\in \Delta \,\big\}.$$

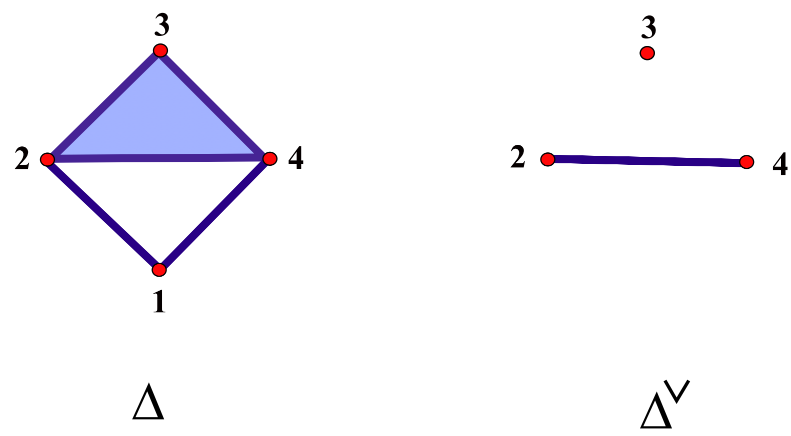



Figure 3.1: A simplicial complex and its Alexander dual

So for the complex $\Delta$ shown in Figure 3.1,

$$I_\Delta \;=\; \big(x_1x_3\,,\, x_1x_2x_4\big).$$

In general the correspondence $\Delta \leftrightarrow I_\Delta$ establishes a bijection between simplicial complexes and squarefree monomial ideals.

A theorem of Hochster (discussed in Section 5.4.B) computes the graded Betti numbers of $I_\Delta$ in terms of the homology of $\Delta$, so in principle the arithmetic regularity of $I_\Delta$ is determined topologically. However a very nice result of Terai [?] asserts that the regularity of $\Delta$ is computed by the projective dimension of the Stanley–Reisner ideal of another complex $\Delta^\vee$ determined by $\Delta$.

Specifically, given a simplicial complex $\Delta$ on $[n]$, the *Alexander dual* of $\Delta$ is the complex $\Delta^\vee$ on $[n]$ whose simplices consist of those subsets $F \subset [n]$ with the property that the complement of $F$ in $[n]$ is not a simplex of $\Delta$:

$$\Delta^\vee \;=\; \big\{\, F \;|\; ([n]-F) \notin \Delta \,\big\}.$$

For example, the dual of the complex $\Delta$ in Figure 3.1 is shown to its right. Note that only three of the four elements in $[4]$ appear as vertices of $\Delta^\vee$.

Terai uses Hochster's theorem (and Alexander duality for simplicial homology) to prove:

**Theorem 3.2.17** (Terai)**.** *For any simplicial compex* $\Delta$*,*

$$\operatorname{arithreg}(\, I_\Delta\,) \;=\; \operatorname{proj\,dim}\,(\, S/I_{\Delta^\vee}\,).$$

Observe that since $\Delta^{\vee\vee} = \Delta$, the theorem also implies that $\operatorname{proj\,dim}(S/I_\Delta) = \operatorname{arithreg}(I_{\Delta^\vee})$.

For example, returning to Figure 3.1, the resolution of $I_\Delta = (x_1x_3\,,\, x_1x_2x_4)$ has the shape

$$0 \longleftarrow I_\Delta \longleftarrow S(-2) \oplus S(-3) \longleftarrow S(-4) \longleftarrow 0, \tag{3.2.4}$$

and hence

$$\text{arithreg}(I_\Delta) = 3 \quad \text{and} \quad \text{proj dim}(S/I_\Delta) = 2.$$

On the other hand,

$$I_{\Delta^\vee} \;=\; \big(x_1\,,\, x_2x_3\,,\, x_3x_4\big),$$

with resolution

$$0 \longleftarrow S/I_{\Delta^\vee} \longleftarrow S \longleftarrow S(1) \oplus S(-2)^2 \longleftarrow S(-3)^3 \longleftarrow S(-4) \longleftarrow 0.$$

We see that

$$\text{arithreg}(I_{\Delta^\vee}) = 2 \quad \text{and} \quad \text{proj dim}(S/I_{\Delta^\vee}) = 3,$$

as Terai predicts.

Edge ideals of graphs are another interesting source of examples. Let $G$ be a simple graph with vertex set $V = [n]$ and edges $E$. The *edge ideal* $I(G) \subseteq S$ of $G$ is the monomial ideal in $S = \mathbf{k}[x_1, \ldots, x_n]$ generated by all quadratic monomials $x_ix_j$ where $\{i, j\}$ is an edge of $G$:

$$I \;=\; I(G) \;=\; \big\{\, x_ix_j \;\mid\; \{i,j\} \in E \,\big\}\;.$$

Edge ideals of graphs are in one-to-one correspondence with squarefree monomial ideals generated by monomials of degree 2.[3] For example, Figure 3.2 shows (in blue) the cyclic graph $C_5$ of order five. Here

$$I(C_5) \;=\; \big(x_1x_2\,,\, x_2x_3\;x_3x_4\,,\, x_4x_5\,,\, x_5x_1\big) \;\subseteq\; S$$

This has the resolution

$$0 \longleftarrow I(G) \longleftarrow S(-2)^5 \longleftarrow S(-3)^5 \longleftarrow S(-5) \longleftarrow 0,$$

and in particular $\text{arithreg}(I) = 3$.

A nice result originally due to Wegner and Froberg gives a criterion for $I(G)$ to have a linear resolution, or equivalently to have arithmetic regularity 2. Given a graph $G$ on $[n]$, the *complementary* graph $G^c$ is the graph on $[n]$ whose edge set is the complement of the edge set of $G$. For example, the complement of the cyclic graph $C_5$ is again cyclic of order 5. A graph is *chordal* if it contains no induced cyclic sub-graphs of order $\geq 4$. The cyclic graph $C_5$ is not chordal, but it becomes so if one removes one edge.

The result in question is the following:

[3] The edge ideal of $G$ coincides with the Stanley–Reisner ideal of the so-called independence complex of $G$ ([?]).

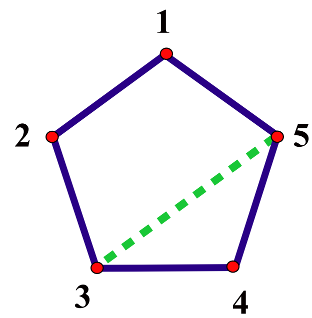



Figure 3.2: The cyclic graph $C_5$ of order five and an added edge

**Theorem 3.2.18.** *Let $G$ be a simple graph. Then*

$$\operatorname{arithreg}\big(I(G)\big) \;=\; 2$$

*if and only if the complementary graph $G^c$ is chordal.*

We refer to [105, §5] for details.

We have seen that the statement is verified (in the negative) for $C_5$. On the other hand, let $G$ be the graph obtained from $C_5$ by adding one edge, say $\{3,5\}$. Then $G^c$ a tree, hence chordal. Now

$$I(G) \;=\; \big(\, x_1x_2\,,\, x_2x_3\; x_3x_4\,,\, x_4x_5\,,\, x_5x_1\,, x_3x_5\,\big),$$

which has the resolution

$$0 \longleftarrow I(G) \longleftarrow S(-2)^6 \longleftarrow S(-3)^8 \longleftarrow S(-4)^3 \longleftarrow 0,$$

so indeed $\operatorname{arithreg}\big(I(G)\big) = 2$.

## 3.3 The theorem of Bayer and Stillman

The regularity of a sheaf or module is widely viewed as a measure of its algebraic complexity. While Theorem 3.1.8 already points in this direction, a result of Bayer and Stillman establishes a more precise connection. The present section is devoted to a presentation of their work. This material will not be used elsewhere.

Most algorithms for computing with ideals (or modules) proceed degree by degree starting with a Gröbner basis. In brief, denote by $S = \mathbf{C}[x_0, \ldots, x_r]$ the polynomial ring, with the variables ordered via $x_0 > x_1 > \ldots > x_r$. One extends this to a multilicative term ordering $>$ on all monomials, and for a homogeneous polynomial $f$ one writes $\operatorname{in}(f) = \operatorname{in}_>(f)$ for its highest term with respect to $>$. Given a homogeneous ideal $I \subseteq S$, the *initial ideal*

$$\operatorname{in}(I) \;=\; \operatorname{in}_>(I) \;\subseteq\; S$$

of $I$ is the monomial ideal generated by the initial forms $\mathrm{in}(f)$ of every homogeneous element $f \in I$. Monomial generators of $\mathrm{in}(I)$ are computed by an algorithm of Buchsburger. The initial ideal is a flat limit of $I$, and therefore every syzygy among these monomial generators can be lifted to a syzygy of $I$. These in turn can be trimmed to a system of minimal syzygies, and eventually one arrives at a computation of the whole resolution of $I$. We refer for instance to [93] or [59, Chapter 15] for a detailed account.

In general $\mathrm{arithreg}\big(\mathrm{in}(I)\big) \geq \mathrm{arithreg}\big(I\big)$, but strict inequality can hold. This means in effect that that in such cases the monomial-based algorithms involve wasted effort, since syzygies of $\mathrm{in}(I)$ in degrees above $\mathrm{arithreg}(I)$ cannot contribute to the minimal resolution of $I$. The theorem of Bayer and Stillman asserts that under suitable conditions the arithmetic regularities of $I$ and $\mathrm{in}(I)$ in fact coincide, and that one can read off the regularity of $I$ from the degrees of the generators of $\mathrm{in}(I)$.

Postponing until later the relevant definitions, we preview the result for which we are aiming:

**Theorem 3.3.1** (**Bayer–Stillman** [16])**.** *Let $I \subseteq S$ be a homogeneous ideal, and let* $\mathrm{in}(I)$ *be the initial ideal of $I$ with respect to reverse lexicographical order in generic coordinates. Then*

$$\mathrm{arithreg}\big(\mathrm{in}(I)\big) \ = \ \mathrm{arithreg}(I).$$

*Moreover,* $\mathrm{arithreg}(I)$ *coincides with the largest degree of a minimal generator of* $\mathrm{in}(I)$.

In other words, one might say that the arithmetic regularity of $I$ directly governs the computational complexity of its revlex generic initial ideal.

## 3.3.A Generic initial ideals

We start by reviewing the construction and properties of generic initial ideals. We work as usual over $\mathbf{C}$, but we note that the main results actual require hypotheses of characteristic zero.

Let $S = \mathbf{C}[x_0, \ldots, x_r]$ be the ring of polynomials, We order the variables by taking $x_0 > \ldots > x_r$, and extend this to an order $>$ on monomials compatible with multiplication. The most important for our purposes will be the *reverse lexicographic* (revlex) order: for monomials of the same degree specified by exponent vectors $A = (a_0, \ldots, a_r)$ and $B = (b_0, \ldots, b_r)$, one declares that $x^A > x^B$ if the last non-zero entry of $A - B$ is negative. So for example the revlex ordering on quadratic monomials in three variables is:

$$x_0^2 \ > \ x_0x_1 \ > \ x_1^2 \ > \ x_0x_2 \ > \ x_1x_2 \ > \ x_2^2.$$

However for the time being we work with any multiplicative order.

Having fixed a term order $>$, the initial form $\mathrm{in}(f)$ of a homogeneous polynomial $f \in S$ is the term of $f$ having highest weight. Given a homogeous ideal $I \subseteq S$, the *initial ideal* of $I$ is the monomial ideal $J = \mathrm{in}(I)$ generated by the initial forms of all elements $f \in I$:

$$\mathrm{in}(I) \ =_{\mathrm{def}} \big(\, \mathrm{in}(f) \ \mid f \in I \big) \ \subseteq \ S.$$

As usual it may not be enough to work only with generators of $I$, but there is a well-understood algorithm for computing generators of $\mathrm{in}(I)$. An important fact is that $\mathrm{in}(I)$ can be realized as a flat deformation of $I$. In paritular:

$$I \text{ and } \mathrm{in}(I) \text{ have the same Hilbert functions.} \tag{3.3.1}$$

Again we refer to [93] or [59] for further discussion and proofs.

The initial ideal of $I \subseteq S$ depends on the choice of coordinates $x_0, \ldots, x_r$. However making a general linear change of variables leads to an intrinsically defined monomial ideal. Specifically, let $G = \mathrm{GL}(r+1)$ be the general linear group, acting on $S$ via the rule

$$g \cdot x_j \;=\; \sum g_{ij} \cdot x_i,$$

where $g = (g_{ij})$. The first point one checks is:

**Proposition/Definition 3.3.2.** *Fix a multiplicative term order* $>$, *and let* $I \subseteq S$ *be a homogeneous ideal. Then there is a non-empty Zariski open subset* $U \subseteq G$ *with the property that for every* $g \in U$,

$$J \;=\; \mathrm{in}(g \cdot I)$$

*is a fixed monomial ideal* $J$. *This ideal is called the generic initial ideal of* $I$ :

$$J \;=\; \mathrm{gin}(I) \;=\; \mathrm{gin}_{>}(I). \qquad \square$$

Now let $B \subseteq G$ be the Borel subgroup of $G$ consisting of upper triangular invertible matrices. A very basic theorem of Galligo, Bayer and Stillman asserts that generic initial ideals are fixed by the action of $B$:

**Theorem 3.3.3.** *Let* $I \subset S$ *be a homogeneous ideal, and let* $J = \mathrm{gin}(I)$ *be its generic initial ideal with respect to any term order. Then* $J$ *is Borel-fixed, i.e.*

$$b \cdot J \;=\; J \quad \textit{for all } \; b \in B.$$

We refer to [93] for the proof.

The importance of the Theorem is that one can say quite a bit about Borel-fixed ideals.

**Proposition 3.3.4.** *Let* $J \subseteq S$ *be any Borel-fixed ideal. Then:*

*(i).* $J$ *is a monomial ideal.*

*(ii). If* $x_j \in J$, *then* $x_1, \ldots, x_j \in J$.

*(iii). Suppose that* $u = x_j^a \cdot m$ *is a monomial in* $J$ *with the property that* $x_j \not| \, m$. *Then also*

$$x_i x_j^{a-1} \cdot m \;\in\; J \quad \textit{when } \; i < j.$$

*(iv). If a monomial ideal $J' \subseteq S$ satisfies the property in (iii), then $J'$ is Borel-fixed.*

*(v). Every associated prime ideal of $S/J$ is also Borel-fixed.*

*Sketch of Proof.* Statement (i) follows from the fact that the diagonal matrices form a subgroup of $B$. For (ii), let $\Gamma_{ij}$ be the matrix with 1's along the diagonal and at the $(ij)$ postion, and zeroes elsewhere. Assume that $i < j$, so that $\Gamma_{ij} \in B$. Then $\Gamma_{ij}(x_j) = x_i + x_j \in J$, and hence $x_i \in J$. For (iii) note that

$$\begin{aligned} \Gamma_{ij}(u) &= (x_i + x_j)^a \cdot m \\ &= \left( \sum_{q=0}^{a} \binom{a}{q} x_i^q x_j^{a-q} \right) \cdot m \ \in \ J, \end{aligned}$$

and since we are in characteristic zero all the terms in the sum appear with non-zero coefficient. Thus each of $x_i^q x_j^{a-q} \cdot m \in J$. We leave (iv) to the reader, while for (v) observe that $B$ acts on $S/J$, and hence must take one associated prime to another. But a connected group cannot act non-trivially on a finite set. □

## 3.3.B Almost regular sequences and regularity

The plan is to reduce Theorem 3.3.1 to computing the regularities of several finite length modules, where Example 3.1.36 applies. We start with some general remarks about the construction, and then specialize to the case at hand.

**Definition 3.3.5 (Almost regular sequence).** Let $E$ be a finitely generated graded $S$-module, and let $\ell \in S_1$ be a linear form. One says that $\ell$ is an *almost non-zerodivisor* for $E$ if

$$\ker \left( E \xrightarrow{\cdot \ell} E(1) \right)$$

is a module of finite length. An ordered sequence $\{\ell_1, \ldots, \ell_p\}$ of linear forms is an *almost regular sequence* if each $\ell_i$ is an almost non-zerodivisor for $E/(\ell_1, \ldots, \ell_p)E$. □

Note that in the situation of the definition, a linear form $\ell \in S_1$ *fails* to be an almost non-zerodivisor of $E$ if and only if $\ell$ lies in an associated prime of $E$ other than the irrelevant ideal $\mathfrak{m} = (x_n, \ldots, x_0)$. It follows that if

$$(\ell_1, \ldots, \ell_p) \ \in \ S_1^{\times p}$$

is a *general* $r$-tuple, then $(\ell_1, \ldots, \ell_p)$ is an almost regular sequence. In the case of Borel-fixed ideals, one can say more:

**Proposition 3.3.6.** *Let $J \subseteq S$ be a Borel-fixed monomial ideal. Then $\{x_r, \ldots, x_0\}$ is an almost regular sequence for $E = S/J$.*

*Proof.* Let
$$N \;=\; \ker\big(\,(S/J) \xrightarrow{\cdot x_r} (S/J)(1)\,\big),$$
and assuming $N_1 \neq 0$ let $P$ be one of its associated primes. Then $P$ is Borel-fixed and $x_n \in P$. Hence $P = \mathfrak{m}$ thanks to Proposition 3.3.4 (ii), i.e. $\mathfrak{m}$ is the unique associated prime of $N$. Hence $N$ has finite length. Now consider
$$N' \;=\; \ker\Big(\,(S/\,(J, x_r)) \xrightarrow{\cdot x_{r-1}} (S/\,(J,x_r))\,(1)\,\Big),$$
and put $B' = \big\{ b \in B \mid b \cdot x_r = x_r \big\}$. This acts on $S/(J, x_r)$, hence any non-zero associated prime $P'$ of the $S$-module $N'$ is $B'$-fixed. But $(x_r, x_{r-1} \in P'$, and hence as above $P' = \mathfrak{m}$. Continuing in this fashion, the Proposition follows. □

Returning to the setting of Definition 3.3.5, we next explain how to calculate arithmetic regularity in terms of almost regular sequences.

**Lemma 3.3.7.** *Let $\{\ell_1, \ldots, \ell_p\}$ be an almost regular sequence for $E$, and set*
$$\begin{aligned} E_{i-1} \;&=\; E/\,(\ell_1, \ldots, \ell_{i-1}) \cdot E \\ N_i \;&=\; \ker\big(\,E_{i-1} \xrightarrow{\cdot \ell_i} E_{i-1}(1)\,\big). \end{aligned} \tag{3.3.2}$$
*Then*
$$\operatorname{arithreg}(E) \;=\; \max\Big\{\operatorname{arithreg}(N_1)\,,\,\ldots\,,\,\operatorname{arithreg}(N_p)\,,\,\operatorname{arithreg}(E_p)\Big\}.$$

Note that if $\dim E \leq p$, then $E_p$ has finite length, so this reduces the computation of arithmetic regularity to the case of modules of finite length.

*Proof of Lemma 3.3.7.* By induction it suffices to treat the case $r = 1$. For this, consider the two exact sequences
$$\begin{gathered} 0 \longrightarrow N_1 \longrightarrow E \xrightarrow{\cdot \ell_1} \big(\ell_1 \cdot E\big)(1) \longrightarrow 0 \\ 0 \longrightarrow \big(\ell_1 \cdot E\big)(1) \longrightarrow E(1) \longrightarrow \big(E/\,\ell_1 E\big)(1) \longrightarrow 0. \end{gathered}$$
Keeping in mind Example 3.1.37, the first gives
$$\operatorname{arithreg}(E) \;=\; \max\big\{\,\operatorname{arithreg}(N_1)\,,\,\operatorname{arithreg}\big(\,(\ell_1 E)(1)\big)\,\big\},$$
while the second implies that
$$\operatorname{arithreg}\big(\,(\ell_1 E)(1)\big) \;\leq\; \max\,\Big\{\operatorname{arithreg}(E) - 1\,,\,\operatorname{arithreg}(E/\ell_1 E)\,\Big\}.$$
The assertion follows. □

We will apply the previous Lemma when $E = S/I$ for a homogeneous ideal $I \subseteq S$. Note that in this case

$$E_{i-1} \;=\; S/\big(I\,,\,\ell_1\,,\,\ldots\,,\,\ell_{i-1}\big) \quad , \quad N_i \;=\; \frac{\big((I\,,\,\ell_1\,,\,\ldots\,,\,\ell_{i-1}) : \ell_i\big)}{(I\,,\,\ell_1\,,\,\ldots\,,\,\ell_{i-1})}.$$

This being so, the following crucial statement is where revlex order enters the picture.

**Proposition 3.3.8.** *Let $I \subseteq S$ be a homogeneous ideal, and let*

$$J \;=\; \mathrm{in}(I)$$

*be the initial ideal of $I$ with respect to reverse lexocographic order. Then for every $i$:*

*(i).* $\mathrm{in}\big(\,(I, x_r \ldots, x_i)\big) \;=\; (J, x_r, \ldots, x_i)$

*(ii).* $\mathrm{in}\big(\,(I, x_r \ldots, x_{i+1}) : x_i\,\big) \;=\; \big(\,(J, x_r, \ldots, x_{i+1}) : x_i\,\big)$,

*where both initial ideals are again taken with respect to revlex order.*

*Proof.* We will prove (ii), for which the inclusion $\subseteq$ is clear. For the other direction, fix a monomial

$$u \;\in\; \big((J, x_r, \ldots, x_{i+1}) \;:\; x_i\,\big),$$

i.e. $u$ is a monomial with the property that

$$x_i \cdot u \;\in\; (J, x_r, \ldots, x_{i+1}).$$

If $u \in \mathbf{C}[x_r, \ldots, x_{i+1}]$, then clearly $u \in \mathrm{in}\big(\,(I, x_r \ldots, x_{i+1}) \;:\; x_i\,\big)$, so we may assume that $u \in \mathbf{C}[x_1, \ldots, x_i]$. Write

$$u \;=\; x_i^a \cdot u' \quad \text{with} \;\; u' \in \mathbf{C}[x_1, \ldots, x_{i-1}].$$

Then

$$x_i \cdot u \;=\; x_i^{a+1} \cdot u' \;\in\; \big(J, x_r, \ldots, x_{i+1}\big),$$

and since the since the left-hand side only involves the variables $x_1, \ldots, x_i$, this forces

$$x_i^{a+1} \cdot u' \;\in\; J$$

i.e. $x_i^{a+1} \cdot u' \;=\; \mathrm{in}(f)$ for some $f \in I$. Write

$$f \;=\; \sum_{j=i+1}^{r} x_j \cdot f_j \;+\; f' \quad \text{where} \;\; f' \in \mathbf{k}[x_1, \ldots, x_i].$$

Then $f' \in (I, x_r, \ldots, x_{i+1})$. Now $x_i^{a+1} \cdot u'$ is $>$ all terms involving $x_{i+1}, \ldots, x_r$ since we are using reverse lex order. Hence $x_i^{a+1} \cdot u' = \mathrm{in}(f')$. Recalling that $f'$ only involves $x_1, \ldots, x_i$, reverse lex order then forces $x_i^{a+1} \mid f'$, say $f' = x_i^{a+1} \cdot g$. Then

$$x_i^a \cdot g \;\in\; \big((I, x_r, \ldots, x_{i+1}) \;:\; x_i\,\big),$$

and $\mathrm{in}(x_i^a \cdot g) = u$. □

## 3.3.C Proof of the theorem of Bayer and Stillman

In this section we prove two results that combine to yield Theorem 3.3.1. To begin with:

**Theorem 3.3.9.** *Let $I \subseteq S$ be a homogeneous ideal, and let $J = \operatorname{gin}(I)$ be the generic initial ideal of $I$ with respect to reverse lex order. Then*

$$\operatorname{arithreg}(S/I) \ = \ \operatorname{arithreg}(S/J).$$

*Proof.* Since $J$ is Borel-fixed, we know that $\{x_r, \ldots, x_0\}$ is an almost regular sequence for $S/J$ (Proposition 3.3.6). We claim that

$$\{\, x_r\, ,\ \ldots\ ,\ x_0 \,\} \ \text{ is an almost regular sequence for } S/I, \tag{*}$$

and that for every $i$:

$$\operatorname{arithreg} \frac{\big((I, x_r \ldots, x_{i+1}) : x_i\big)}{(I, x_r \ldots, x_i)} \ = \ \operatorname{arithreg} \frac{\big((J, x_r \ldots, x_{i+1}) : x_i\big)}{(J, x_r \ldots, x_i)}. \tag{**}$$

In fact, granting this the Theorem follows from Lemma 3.3.7.

As for the claim, Proposition 3.3.8 implies that the ideals appearing on the right in (**) are the initial ideals of the corresponding terms on the left. But then thanks to (3.3.1) they have the same Hilbert functions, i.e.

$$\begin{aligned} \big((I, x_r \ldots, x_{i+1}) : x_i\big)_d \ &= \ \big((J, x_r \ldots, x_{i+1}) : x_i\big)_d \\ (I, x_r \ldots, x_i)_d \ &= \ (J, x_r \ldots, x_i)_d \end{aligned} \tag{***}$$

for every $d$. On the other hand, note that if $K \subseteq S$ is any ideal, and $\ell \in S_1$ is a linear form, then $\ell$ is an almost non-zerodivisor for $S/K$ if and only if

$$(\, K : \ell\,)_d \ = \ K_d \quad \text{for} \quad d \gg 0.$$

Hence (*) follows from (***), and likewise (***) implies (**) thanks to Example 3.1.36. □

The remaining statement in Theorem 3.3.1 now follows from

**Theorem 3.3.10.** *Assume that $J \subseteq S$ is a Borel-fixed ideal with $\dim S/J = r$.*

(*i*). *Let $S' = \mathbf{C}[x_r, \ldots, x_{r-p+1}]$. Then $S/J$ is a finite $S'$-module.*

(*ii*). *Assume that $J$ is generated by elements of degree $\leq m$ and has at least one minimal generator of degree $m$. Then*

$$\operatorname{arithreg}(J) \ = \ m$$

*Proof.* Since $J$ is Borel-fixed, $\{x_r \dots, x_{r-p+1}\}$ is an almost regular sequence for $S/J$, and

$$S\,/\,(J, x_r, \dots, x_{r-p+1})$$

is of finite length. Then the usual proof of Nakayama's lemma gives assertion (i).

For (ii) we ned to show that $\operatorname{arithreg}(S/J) \leq m-1$. To this end, set

$$\begin{aligned} N' &= \frac{S}{(J, x_n, \dots, x_{r-p+1})} \\ N_i &= \frac{\big((J, x_r, \dots\, x_{i+1}) : x_i\big)}{(J, x_r, \dots\, x_{i+1})} \quad (\text{for } r-p+1 \leq i \leq r-1). \end{aligned}$$

It suffices to show that

$$(N')_d \;=\; 0 \;\text{ and }\; (N_i)_d \;=\; 0$$

for $d \geq m$. For this, note to begin with that for $s \gg 0$ the elements $x_{r-p}^m, \dots, x_{r-p}^s$ are $\mathbf{C}$-linearly dependent in the finite dimensional vector space $N'$. Taking homogeneous parts of a relation of linear dependence, and recalling that if a sum of monomials lies in a monomial ideal then so too does each summand, we find that

$$x_{r-p}^{\ell} \;\in\; J \quad \text{for some } \ell \geq m.$$

Since $J$ is generated in degrees $\leq m$, this implies that $x_{r-p}^m \in J$. But Proposition 3.3.4 (iii) then implies that $(x_0, \dots, x_{r-p})^m \subseteq J$, and hence $(N')_d = 0$ when $d \geq m$.

Finally, we verify that $(N_i)_d = 0$ when $d \geq m$; we will treat the case $d = m$. Supposing that $u$ is a monomial of degree $m$ with the property that

$$x_i \cdot u \;\in\; (J, x_r, \dots, x_{i+1}), \tag{*}$$

the issue is to show that $u \in (J, x_r, \dots, x_{i+1})$. If $u \in (x_r, \dots, x_{i+1})$ this is clear, so we may assume that $u \in \mathbf{C}[x_1, \dots, x_i]$. Then (*) forces $x_i \cdot u \in J$. But as $J$ is generated in degrees $\leq m$, this means that

$$x_i \cdot u \;=\; x_\ell \cdot v$$

for some monomial $v \in J$ of degree $m$ and some index $\ell \leq i$. If $\ell = i$ then $u = v \in J$. So we can assume $\ell < i$. Then $x_i | v$, so

$$u \;=\; \left(\frac{x_\ell}{x_i}\right) \cdot v \;\in\; J$$

thanks to Proposition 3.3.4 (iii). □

## 3.4 Notes

As noted in the text, Castelnuovo-Mumford regularity was introduced (more or less in passing) by Mumford in [140] in the course of constructing Hilbert schemes: he showed that the regularity of a subscheme of projective space could be bounded in terms of its Hilbert polynomial. Since then, the theory has attracted a great deal of activity, particularly in the commutative algebra community. Besides the references cited at the beginning of Section 3.2.C, we recommend Green's notes [93] for a development of the theory from a more geometric viewpoint. Chapter 1.8 of [128] surveys the algebro-geometric side of the story. Our account of the theorem of Bayer and Stillman draws on [?] and [?].

Several authors have studied extensions of Castelnuovo–Mumford regularity to the multigraded or toric setting. We refer in particular to [107, 131, 176, 177]. Here the regularity of a sheaf or module is not a single integer but rather a region in the appropriate space of multi-indices.

In another direction, Pareschi and Popa [151], [152], [153] have developed an analogue of regularity for sheaves on an abelian variety, and give many interesting applications. Some of these will be discussed in Section 6.3 (when it is written).

# Lecture 4

# Regularity Bounds and Constructions

Castelnuovo–Mumford regularity is the basic measure of algebraic complexity for a subvariety or subscheme $X \subseteq \mathbf{P}^r$, and therefore it is of considerable interest to establish bounds on this invariant. However as Bayer and Mumford stressed in their influential article [15], there is a striking dichotomy between "nice" and arbitrary ideals. Namely, the regularity of smooth complex projective varieties turns out to be at worst linear in the natural input parameters, even if the best-possible statements aren't always known. By contrast regularity can grow doubly exponentially in general. For a long time it wasn't clear what to expect for reduced but possibly singular varieties, but recent work of McCullough and Peeva has shown that any pathological behavior can be reproduced in prime ideals. In particular, it emerges that non-singularity is the natural setting for geometrically meaningful regularity bounds.

The present lecture is devoted to this circle of ideas. We start with a theorem from [24] showing that complete intersections have the worst regularity among all smooth varieties defined by equations of given degree. In the second section we turn to Castelnuovo-type bounds involving the degree of a variety. Here the optimal expected statement is currently established only for curves and surfaces, but a weaker linear bound due to Mumford covers smooth varieties of all dimensions. The third section surveys constructions of ideals with "bad" regularity, in particular work of Ullery and the Rees-like algebras of McCullough and Peeva.

## 4.1 Vanishing theorems for subvarieties of projective space

This section is devoted to a sharp regularity bound due to Bertram and the authors [24] concerning smooth varieties defined by equations of given degrees.

By way of background, recall from Examples 3.1.5 and 3.1.10 the computation of the regularity of a complete intersection. We saw that if $X \subseteq \mathbf{P}^{n+e}$ is the transversal intersection of hypersurfaces of degrees $d_1, d_2, \ldots, d_e$, then

$$\mathrm{reg}(X) \ = \ (d_1 + \ldots + d_e - e + 1).$$

In particular, if $d_1 = \ldots = d_e = d$, then $\mathrm{reg}(X) = (ed - e + 1)$. It is natural to ask whether an analogous statement holds for an arbitrary smooth subvariety $X \subseteq \mathbf{P}^r$: can we bound the regularity of $X$ knowing something about the degrees of its defining equations?

The theorems presented in this section state that this is indeed the case. Specifically, suppose that $X$ has codimension $e$ and is cut out scheme-theoretically by hypersurfaces of degrees

$$d_1 \ \geq \ d_2 \ \geq \ \ldots \ \geq \ d_m$$

(for some $m \geq e$). Then

$$\mathrm{reg}(X) \ \leq \ (d_1 + \ldots + d_e - e + 1).$$

Moreover equality holds if and only if $X$ is a complete intersection. When all the defining degrees $d_i$ coincide, this is established by a rather quick proof using a strengthening of Kodaira vanishing due to Kawamata and Viehweg. We will explain the argument in detail. The general case involves in addition some ideas involving linkage, and this we will sketch only briefly.

Turning to details, the main results for which are aiming – from [24] – are the following:

**Theorem 4.1.1** (**Vanishing for smooth subvarieties, I**). *Let $X \subseteq \mathbf{P}^r$ be a smooth complex projective variety of dimension $n$ and codimension $e = r - n$. Assume that $X$ is cut out scheme-theoretically by hypersurfaces of degree $d$. Then*

$$H^i\big(\mathbf{P}^r, \mathcal{I}_X(k)\big) \ = \ 0 \tag{4.1.1}$$

*for $i > 0$ and $k \geq de - r$. In particular, $X$ is $(ed - e + 1)$-regular.*

The hypothesis on the defining equations of $X$ is that $\mathcal{I}_X(d)$ should be generated by its global sections, where as usual $\mathcal{I}_X \subseteq \mathcal{O}_{\mathbf{P}^r}$ denotes the ideal sheaf of $X$. In particular, we do not assume that the homogeneous ideal $I_X$ of $X$ is known to be generated in degrees $\leq d$.

More generally:

**Theorem 4.1.2** (**Vanishing for smooth subvarieties, II**). *With $X \subseteq \mathbf{P}^r$ as above, assume that $X$ is cut out scheme-theoretically by hypersurfaces of degrees*

$$d_1 \ \geq \ d_2 \ \geq \ \ldots \ \geq d_m.$$

*Then*

$$H^i\big(\mathbf{P}, \mathcal{I}_X(k)\big) \ = \ 0 \ \ \textit{for} \ \ i \geq 1$$

*provided that* $k \geq d_1 + d_2 + \ldots + d_e - r$.

Observe that only the largest $e = \mathrm{codim}(X, \mathbf{P}^r)$ degrees of defining equations enter into the hypothesis.

These statements lead to:

**Corollary 4.1.3.** *Keep the hypotheses of Theorem 4.1.2. Then:*

$$\operatorname{reg}(X) \;\leq\; (d_1 + \ldots + d_e + 1 - e).$$

*Moreover equality holds if and only if $X$ is the complete intersection of hypersurfaces of degrees $d_1, \ldots, d_e$.*

So one can say that complete intersections have the "worst" regularities of smooth varieties defined by equations of given degrees.

We have stated Theorem 4.1.1 separately because it is quicker to prove than 4.1.2, while at the same time illustrating most of the essential tools that go into the more general result. Consequently we focus here on 4.1.1, leaving the more general Theorem 4.1.2 to some brief indications.

**Remark 4.1.4 (Arithmetic regularity).** It is natural to ask whether these statements extend to the setting of arithmetic regularity in the sense of Definition 3.1.30. Suppose then that $F_1, \ldots, F_m \subseteq \mathbf{P}^r$ are hypersurfaces of degrees $d_1 \geq \ldots \geq d_m$ whose scheme-theoretic intersection is a smooth variety $X \subseteq \mathbf{P}^r$ of dimension $n$. Denote by

$$J \;=\; (F_1\,,\,\ldots\,,F_m) \;\subseteq\; \mathbf{C}[z_0, \ldots, z_r]$$

the ideal that they generate in the polynomial ring. It is established in [50] that $J$ is saturated in degrees $\geq d_1 + \ldots + d_{r+1} - r$. In particular, it then follows from Theorems 4.1.2 and Corollary 3.1.35 that

$$\operatorname{arithreg}(J) \;\leq\; d_1 + \ldots + d_{r+1} - r.$$

Contrary to what one might hope by extrapolating the statement of 4.1.2, simple examples show that one cannot replace the $(r+1)$-fold sum of degrees with a smaller one. □

## 4.1.A Proof of Theorem 4.1.1.

We would like to establish Theorem 4.1.1 using Kodaira-type vanishing theorems, but these deal with line bundles rather than ideal sheaves. One gets around this via the time-tested idea of blowing up $X$ to reduce to a statement about divisors. We will start by explaining how one goes about this. It will emerge that the classical Kodaira vanishing theorem is not quite enough to do the job; we will eventually use instead an extension due to Kawamata and Viehweg.

**The set-up.** Suppose then $X \subseteq \mathbf{P}^r$ is a smooth irreducible subvariety of dimension $n$ and codimension $e = r - n$, and as always, write $\mathcal{I}_X \subseteq \mathcal{O}_{\mathbf{P}^r}$ for the ideal sheaf of $X$. Now consider the blowing up

$$\mu \,:\, \mathbf{P}' \,=\, \operatorname{Bl}_X(\mathbf{P}^r) \longrightarrow \mathbf{P}^r$$

of $\mathbf{P} = \mathbf{P}^r$ along $X$. Observe that $\mathbf{P}'$ is smooth thanks to the non-singularity of $X$. Denote by $E \subseteq \mathbf{P}'$ the exceptional divisor, and by $H$ the pullback of a hyperplane. It follows from the universal property of blowing up that $\mathcal{I}_X$ pulls back to the ideal sheaf of $E$ in the sense that $\mathcal{I}_X \cdot \mathcal{O}_{\mathbf{P}'} = \mathcal{O}_{\mathbf{P}'}(-E)$, and therefore $\mathcal{I}_X(k) \cdot \mathcal{O}_{\mathbf{P}'} = \mathcal{O}_{\mathbf{P}'}(kH - E)$. More importantly, the smoothness of $X$ implies that line bundle $\mathcal{O}_{\mathbf{P}'}(-E)$ pushes down to $\mathcal{I}_X$ with vanishing higher direct images:

$$\mu_* \mathcal{O}_{\mathbf{P}'}(-E) \ = \ \mathcal{I}_X \ \text{ and } \ R^j \mu_* \mathcal{O}_{\mathbf{P}'}(-E) \ = \ 0 \ \text{ for } j \ > \ 0. \tag{4.1.2}$$

(See for instance [?] or prove directly by studying the push forward of the exact sequence $0 \longrightarrow \mathcal{O}_{\mathbf{P}'}(-E) \longrightarrow \mathcal{O}_{\mathbf{P}'} \longrightarrow \mathcal{O}_E \longrightarrow 0$.) In particular, the projection formula then implies that

$$\mu_* \, \mathcal{O}_{\mathbf{P}'}(kH - E) \ = \ \mathcal{I}_X(k),$$

while the Leray spectral sequence degenerates to yield

$$H^j\big(\mathbf{P}', \mathcal{O}_{\mathbf{P}'}(kH - E)\big) \ = \ H^j\big(\mathbf{P}, \mathcal{I}_X(k)\big) \tag{4.1.3}$$

for all $j$. So we are reduced to studying higher cohomology of line bundles of the form $\mathcal{O}_{\mathbf{P}'}(kH - E)$.

Recall next that since $\operatorname{codim} X = e$, the canonical divisor of $\mathbf{P}'$ is given by

$$K_{\mathbf{P}'} \ \equiv_{\text{lin}} \ \mu^*(K_{\mathbf{P}^r}) + (e-1)E \ \equiv_{\text{lin}} \ -(r+1)H \ + \ (e-1)E.$$

As we want to prove vanishings for divisors of the form $kH - E$, the positive coefficient of the exceptional divisor in $K_{\mathbf{P}'}$ is problematic. To circumvent this, we should add a non-negative divisor in which $E$ appears negatively. This is where the hypothesis on $X$ comes in.

Specifically, since $\mathcal{I}_X(d)$ is globally generated, and since

$$\mathcal{I}_X(d) \cdot \mathcal{O}_{\mathbf{P}'} \ = \ \mathcal{O}_{\mathbf{P}'}(dH - E),$$

it follows that the line bundle $\mathcal{O}_{\mathbf{P}'}(dH - E)$ is globally generated. Geometrically, the linear series $|\mathcal{O}_{\mathbf{P}'}(dH - E)|$ defines a morphism $\mathbf{P}' \longrightarrow \mathbf{P}^N$ that resolves the rational mapping $\mathbf{P} \dashrightarrow \mathbf{P}^N$ given by hypersurfaces of degree $d$ passing through $X$.

So we are led to consider the divisor

$$K_{\mathbf{P}'} \ + \ e \cdot (dH - E) \ + \ H \ \equiv_{\text{lin}} \ \big(ed - r\big) \cdot H \ - \ E. \tag{4.1.4}$$

Now imagine – which is not true – that $H$ were an ample divisor on $\mathbf{P}'$. Then when $X$ is cut out by hypersurfaces of degree $d$ – so that $(dH - E)$ is globally generated on $\mathbf{P}'$ – it would follow that the left hand side of (4.1.4) is of the form $K_{\mathbf{P}'} + (\text{ample})$, and Kodaira vanishing (Theorem 3.2.9) together with (4.1.3) would give exactly what we need.

Unfortunately the divisor $H$ is trivial along the fibres of $E \longrightarrow X$, and hence it isn't ample. But while one cannot directly invoke the classical Kodaira vanishing theorem, the argument just sketched goes through perfectly using instead an improvement of Kodaira's result due to Kawamata and Viehweg. In the next paragraph we state this result, which we will apply here on the blow-up $\mathbf{P}'$.

**Kawamata–Viehweg vanishing for nef and big line divisors.** Following earlier work of Mumford [142], Kawamata [114] and Viehweg [186] found in the early 1980s that one could relax the hypothesis of amplitude in the classical Kodaira vanishing theorem. (At the same time they also proved vanishing theorems for $\mathbf{Q}$-divisors, but we do not need this here.) In the present subsection we explain the statement of their result.

Suppose then that $M$ is a smooth complex projective variety of dimension $m$. Recall that a divisor $L$ on $M$ is said to be *numerically effective* or *nef* if

$$\big(L \cdot C\big) \;\geq\; 0 \quad \text{for all irreducible curves } C \subseteq M.$$

Evidently an ample divisor is nef, as is any divisor moving in a base-point free linear series. One thinks of nef classes as being limits of ample ones. For example, an important result of Kleiman [118] asserts that if $L$ is nef and $H$ is ample, then $mL + H$ is ample for every $m \geq 0$. Equivalently, in the language of $\mathbf{Q}$-divisors, $L + \frac{1}{m}H$ is ample for all $m > 0$. We refer to [128, Chapter 1.4] for a detailed discussion.

One cannot expect the vanishing of the higher cohomology groups $H^i\big(M, \mathcal{O}_M(K_M + L)\big)$ for an arbitrary nef divisor $L$ on $M$: the zero divisor is nef, but of course it need not be the case that $H^i\big(M, \mathcal{O}_M(K_M)\big) = 0$ for $i > 0$. What's required is a condition ruling out, for example, the possibility that $L$ is the pull-back of an ample divisor under a morphism $f : M \longrightarrow N$ where $\dim N < \dim M$.

To this end, one says that a nef divisor is *big* if its top self-intersection number is positive:

$$\big(L^m\big) \;>\; 0,$$

where as above $m = \dim M$. This turns out to be equivalent to asking that $h^0\big(M, \mathcal{O}_M(kL)\big)$ grows (maximally) like $k^m$. An ample divisor is evidently nef and big. More importantly for us, if $\mathcal{O}_M(L)$ is globally generated and defines a generically finite mapping

$$\phi \;=\; \phi_L \;:\; M \longrightarrow \mathbf{P}^N,$$

then $L$ is nef and big. (A warning: bigness for an arbitrary divisor $D$ is defined by requiring that $h^0\big(M, \mathcal{O}_M(kD)\big)$ grow like $k^m$; absent nefness this is not controlled by the top self-intersection number of $D$.) We again refer to [128, Chapters 1.4 and 2.2] for a detailed discussion.

The result of Kawamata and Viehweg is that the statement of Kodaira vanishing remains true for divisors that are only required to be nef and big.

**Theorem 4.1.5** (**Vanishing for nef and big divisors**)**.** *Let $M$ be a smooth complex projective variety, and $L$ a nef and big divisor on $M$. Then*

$$H^i\big(M, \mathcal{O}_M(K_M + L)\big) \;=\; 0 \quad \textit{for} \;\; i > 0.$$

When $\mathcal{O}_M(L)$ is globally generated – which is the main case that we will need – this was established earlier by Mumford [142] along the lines of Kodaira's original argument. The proofs of Kawamata and Viewheg use covering arguments to reduce to the classical statement. The reader may consult [128, Chapter 4] for a detailed account. While it may seem at first blush that this represents only a technical strengthening of Theorem 3.2.9, the result of Kawamata and Viehweg has a considerably wider range of applications.

**Remark 4.1.6. (Grauert–Riemenschneider Vanishing Theorem).** Theorem 4.1.5 leads to a quick proof of a fundamental result of Grauert and Riemenschneider concerning pushforwards of the dualizing sheaf. Specifically, let $M$ be a smooth quasi-projective variety, and let $f : M \longrightarrow \overline{M}$ be a generically finite projective morphism from $M$ onto some (possibly singular) variety $\overline{M}$. Then the higher direct images of $\omega_M$ vanish:

$$R^j f_* \omega_M \;=\; 0 \quad \text{for} \quad j > 0.$$

When $M$ is projective this follows directly from 4.1.5 by an argument with the Leray spectral sequence. In general one compactifies $M$ and $f$ and reduces to this case. We refer for instance to [128] for a detailed account. □

**Completion of proof of Theorem 4.1.1.** We start by completing the proof of Theorem 4.1.1, which follows at once from the calculations above together with the theorem of Kawamata and Viehweg. Then we prove the equi-generated analogue of Corolary 4.1.3.

*Proof of Theorem 4.1.1.* Recall (Example 3.1.21) that to establish the $m$-regularity of an $n$-dimensional variety $X \subseteq \mathbf{P}^r$ one only needs to show that $H^i\big(\mathbf{P}^r, \mathcal{I}_X(m-i)\big) = 0$ for $1 \leq i \leq n+1$. Therefore the regularity bound follows from (4.1.1).

To prove this vanishing, we return to the blowing up $\mathbf{P}' = \mathrm{Bl}_X(\mathbf{P})$, and rewrite (4.1.4) in the form:

$$K_{\mathbf{P}'} + e \cdot (dH - E) + (i+1)H \;\equiv_{\text{lin}}\; \big(ed - r + i\big) \cdot H \,-\, E.$$

Assuming that $(dH - E)$ is globally generated and $i \geq 0$, the required vanishing will follow from 4.1.5 and (4.1.3) as soon as we verify the

> Claim: If $B$ is a globally generated divisor on $\mathbf{P}'$ and $H$ as before is the pullback of a hyperplane on $\mathbf{P}$, then
>
> $$B + aH \quad \text{is nef and big when} \quad a \geq 1.$$

In fact, the divisor in question is clearly nef since it basepoint free. It remains to show that its self-intersection number $\big((B + aH)^r\big)$ is strictly positive. But this follows by expanding out the product: $\big(B^\ell \cdot H^{r-\ell}\big) \geq 0$ for every $\ell$ since the integer in question computes the degree of $B$ on the pullback of an $\ell$-plane, and $\big(H^r\big) = 1$. □

**Remark 4.1.7 (Regularity via multiplier ideals).** In recent years, the language of multiplier ideals has emerged as a convenient way to package ideas around vanishing theorems. In particular, once the machinery has been erected, Theorem 4.1.1 pops out immediately from the Kawamata–Viehweg–Nadel vanishing theorem [?]. We refer the reader to [?] or [?] for an introduction to this theory. □

Next we show that equality holds in 4.1.1 if and only if $X$ is a complete intersection.

**Proposition 4.1.8.** *In the situation of Theorem 4.1.1, $X$ fails to be $(ed-e)$-regular if and only if it is the complete intersection of $e$ hypersurfaces of degree $d$.*

*Proof of Proposition 4.1.8.* We have already seen that Theorem 4.1.1 is optimal if $X$ is a complete intersection. Conversely, if $X$ fails to be $(ed-e)$-regular, then necessarily

$$H^{n+1}\big(\mathbf{P}^r, \mathcal{I}_X(de-e-n-1)\big) \;=\; H^n\big(X, \mathcal{O}_X(de-r-1)\big) \;\neq\; 0,$$

since the remaining vanishings are covered by 4.1.1. Equivalently,

$$H^0\big(X, \mathcal{O}_X(K_X)\otimes\mathcal{O}_X(-de+r+1)\big) \;\neq\; 0. \tag{*}$$

It remains to show that this forces $X$ to be a complete intersection.

To this end choose $e$ general hypersurfaces $D_1, \ldots, D_e$ of degree $d$ passing through $X$. Then

$$D_1 \cap \ldots \cap D_e \;=\; X \cup Y.$$

where $Y$ is another variety of dimension $n$. It suffices to show that $Y = \varnothing$. Assuming for the moment that $n \geq 1$, $X \cup Y$ – like any complete intersection – is connected. So the question is reduced to proving that $X \cap Y = \varnothing$. Denote by $N^*_{X/\mathbf{P}^r}$ the conormal bundle to $X$ in $\mathbf{P}^r$: this is a vector bundle of rank $= e$ on $X$ with

$$\det N^*_{X/\mathbf{P}^r} \;=\; \mathcal{O}_X(-r-1)\otimes\mathcal{O}_X(-K_X),$$

The $D_i$ give rise to a mapping $\mathcal{O}_{\mathbf{P}}(-d)^e \longrightarrow \mathcal{I}_X$ whose restriction to $X$ is a vector bundle homomorphism

$$u : \mathcal{O}_X(-d)^e \longrightarrow N^*_{X/\mathbf{P}^r}.$$

This map drops rank exactly on the locus where the $D_i$ fail to generate the ideal of $X$, i.e. on $X \cap Y$. But

$$0 \neq \det(u) \;\in\; \Gamma\big(X\,,\,\mathcal{O}_X(de+r+1)\otimes\mathcal{O}_X(-K_X)\big). \tag{**}$$

Comparing (*) and (**), it follows that the bundle in question is trivial, hence $\det(u)$ is nowhere zero and $X \cap Y = \varnothing$.

Finally suppose that $n = 0$ and $r = e$. Then $X$ is a finite subset of a complete intersection of type $(d, \ldots, d)$. But a proper subset of such a complete intersection has regularity $\leq ed-e$ (Example 4.1.9), and we are done. □

**Example 4.1.9** (**Subsets of finite complete intersections**). Let $Z \subseteq \mathbf{P}^e$ be a complete intersection of hypersurfaces of degrees $d_1, \ldots, d_e$. Then $h^1\big(\mathbf{P}^r, \mathcal{I}_Z(d_1 + \ldots + d_e - e - 1)\big) = 1$, but if $X \subsetneq Z$ is a proper subset then $H^1\big(\mathbf{P}^r, \mathcal{I}_X(d_1 + \ldots + d_e - e - 1)\big) = 0$. In particular $\operatorname{reg}(X) < \operatorname{reg}(Z)$. In particular, $X$ is $(d_1 + \ldots + d_e - e + 1)$-regular. □

We conclude this subsection with some applications, examples and remarks.

**Example 4.1.10** (**Criterion for projective normality**). In the situation of Theorem 4.1.1, suppose that $ed \leq r + 1$. Then $X$ is projectively normal. If $ed \leq r$, then $X$ is projectively Cohen–Macaulay. For example the hypothesis applies, with $d = 2$, to the Segre embedding

$$\mathbf{P}^1 \times \mathbf{P}^{n-1} \ \subseteq \ \mathbf{P}^{2n-1},$$

although of course in this case the conclusion is already well-known. (In the first case the Theorem implies that $H^1\big(\mathbf{P}^r, \mathcal{I}_X(k)\big) = 0$ for all $k \geq 1$, and hence $X$ is projectively normal. If $de \leq r$, then in addition

$$H^i\big(X, \mathcal{O}_X(k)\big) \ = \ H^{i+1}\big(\mathbf{P}^r, \mathcal{I}_X(k)\big) \ = \ 0$$

for $k \geq 0$ and all $i > 0$, while $H^i\big(X, \mathcal{O}_X(k)\big) = 0$ for $k < 0$ and $i < n$ thanks to Kodaira vanishing.) Needless to say an analogous statement follows from Theorem 4.1.2. □

**Example 4.1.11** (**Vanishing for powers of $\mathcal{I}_X$**). Consider $X \subseteq \mathbf{P}^r$ as in Theorem 4.1.1, and fix any integer $a \geq 1$. Then

$$H^i\big(\mathbf{P}^r, \, \mathcal{I}_X^a(k)\big) \ = \ 0$$

for $i > 0$ provided that $k \geq (e + a)d - r$. (In the setting of the proof, observe by induction on $a$ that $\mu_* \mathcal{O}_{\mathbf{P}'}(-aE) = \mathcal{I}_X^a$, and that $R^j \mu_* \mathcal{O}_{\mathbf{P}'}(-aE) \ = \ 0 \quad \text{for} \quad j > 0$.) □

**Remark 4.1.12** (**Singular varieties**). One can ask to what extent the non-singularity hypothesis on $X$ is actually necessary. It follows from the recent work of McCullough and Peeva (Section 4.3.B) that Theorem 4.1.1 (and hence also Theorem 4.1.2) can fail badly if one assumes only that $X$ is reduced and irreducible. On the other hand, the statement remains true if $X$ has only isolated singularities ([128, Ex. 10.5.1]), and several authors have studied extensions of these results where $X$ is allowed to have mild singularities of higher dimension: see for [?], [?], [?]. It would be very interesting to know whether reduced and integral subvarieties satisfy a regularity bound that is singly exponential in the defining degree $d$. □

## 4.1.B Inputs to the proof of Theorem 4.1.2

In this section we indicate the additional ideas coming into the proof of Theorem 4.1.2. We state the results that are used, but do not write out proofs.

Theorem 4.1.2 can be seen as a generalization of a statement due to Severi that appears as the last result quoted in the book of Semple and Roth [174, XIII.9.8]. Specifically, suppose that

$$X\,,\,Y\ \subseteq\ \mathbf{P}^r$$

are smooth surfaces whose union is the complete intersection of hypersurfaces of degrees $d_1,\ldots,d_{r-2}$. Then:

> Hypersurfaces of degrees $k\geq\sum d_i-r$ cut out a complete linear series on $X$ and on $Y$. Moreover the canonical series on $Y$ is cut out by hypersurfaces of degree $\sum d_i-(r+1)$ that pass through $X$.

Assuming for simplicity of notation that $r=4$, one can prove this by arguing that the Koszul complex determined by the two equations gives rise to an exact sequence

$$0\longrightarrow\mathcal{O}_{\mathbf{P}^4}(-d_1-d_2)\longrightarrow\mathcal{O}_{\mathbf{P}^4}(-d_1)\oplus\mathcal{O}_{\mathbf{P}^4}(-d_2)\longrightarrow\mathcal{I}_X\longrightarrow\omega_Y(5-d_1-d_2)\longrightarrow 0.\qquad(*)$$

The assertion can be read off from (*) using Kodaira vanishing on $Y$. This suggests that Theorem 4.1.2 is closely related to linkage. Indeed, the proof of 4.1.2 involves linking $X$ to another variety $Y$ on which one applies vanishing for nef and big divisors.

**Linkage.** We start with some general remarks about linkage. The main results are Theorem 4.1.15 which computes the canonical bundle of a generic link (generalizing (*)), and Theorem 4.1.16 which establishes a global vanishing.

Let $M$ be a smooth variety of dimension $r$ – that for the moment we do not require to be complete – and let $X\subseteq M$ be a smooth subvariety of dimension $n$ and codimension $e=r-n$. Consider $e$ divisors

$$D_1\,,\,\ldots\,,D_e\ \subseteq\ M$$

passing through $X$. If the $D_i$ are sufficiently general, one expects that their intersection will have codimension $e$, and hence will contain $X$ as an irreducible component. In other words,

$$D_1\cap\ldots\cap D_e\ =\ X\cup Y,\tag{4.1.5}$$

where $Y\subseteq M$ also has pure dimension $n$. One says that $Y$ is *linked* to $X$ by the $D_i$. We wish to study the properties of $Y$ under suitable genericity hypotheses on the $D_i$.

In general one cannot expect that $Y$ or the $D_i$ will be smooth ($Y$ is typically singular in codimension 4). The natural condition for our purposes is that $Y$ be the image of a smooth variety on the blowing up of $M$ along $X$. Specifically, consider the blow-up

$$\mu: M'\ =\ \mathrm{Bl}_X(M)\ \longrightarrow M.$$

Denote by $E\subseteq M'$ the exceptional divisor, so that $E=\mathbf{P}(N^*)$ where $N^*=N^*_{X/M}$ is the conormal bundle to $X$ in in $M$.[1] Since each $D_i$ contains $X$, its inverse image $\mu^*D_i$ vanishes

[1]Recall our convention that $\mathbf{P}(E)$ denotes the space or bundle of one-dimensional *quotients* of a vector space or bundle $E$.

along $E$. Therefore

$$D_i' =_{\text{def}} \mu^* D_i - E \in |\mu^* D_i - E|$$

is an effective divisor, the *proper transform* of $D_i$.

**Definition 4.1.13 (Generic linkage).** We will say that the linkage (4.1.5) is *generic* if the $D_i'$ meet transversely along a smooth variety

$$Y' = D_1' \cap \ldots \cap D_e',$$

and if no component of $Y'$ is contained in $E$. Note that this implies in particular that $Y'$ is a resolution of singularities of $Y$. □

**Example 4.1.14.** Suppose that $M$ is projective, and let $D$ be a divisor on $M$ that is sufficiently positive so that $\mathcal{I}_X \otimes \mathcal{O}_M(D)$ is globally generated. Then $e$ general divisors $D_1, \ldots, D_e \in |D|$ passing through $X$ give a generic linkage. (The linear series $|\mu^* D - E|$ is free, and the assertion follows from Bertini.) □

The importance of a generic linkage in the present setting is that one can relate the ideal sheaf of $X$ to the canonical bundle of the link, where vanishing theorems apply.

**Theorem 4.1.15 (An exact sequence for linkage).** *Assume that $D_1, \ldots, D_e \subseteq M$ define a generic linkage of $X$ with $Y$, and set $D = \sum D_i$. Then the Koszul complex determined by the $D_i$ gives rise to a long exact sequence*

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_M(K_M) \longrightarrow \oplus\, \mathcal{O}_M(K_M + D_i) \longrightarrow \ldots \\ \ldots \longrightarrow \oplus\, \mathcal{O}_M(K_M + D - D_i) \longrightarrow \mathcal{I}_{X/M} \otimes \mathcal{O}_M(K_M + D) \longrightarrow \mu_* \omega_{Y'} \longrightarrow 0 \end{aligned}$$

*of sheaves on $M$.*

**Theorem 4.1.16 (Global vanishing for linkage).** *In the situation of Theorem 4.1.15, assume that $M$ is projective and that each $D_i$ is ample. Then for any ample divisor $A$ on $X$ one has the vanishing*

$$H^i\big( M\,,\, \mathcal{I}_{X/M} \otimes \mathcal{O}_M(K_X + D + A) \big) = 0 \quad \textit{for} \quad i > 0.$$

The first statement is established by pushing down the evident Koszul complex on $M'$. For the second one applies Kawamata–Viehweg vanishing on $Y'$ and Grauert–Riemenschneider (Remark 4.1.6).

**Remark 4.1.17.** Linkage of varieties has been extensively studied in algebraic geometry and commutative algebra. We refer for instance to [158], [162], [136], [143] or [76, Chapter 7] for a sampling of this work and further references.

**Remark 4.1.18 (Higher powers of $\mathcal{I}_X$).** Both of the Theorems admit generalizations involving higher powers of the ideal of $X$. For instance, in the situation of Theorem 4.1.15, fix $a > 0$, and write

$$\mathcal{O}_{Y'}(E) = \mathcal{O}_{M'}(E) \otimes \mathcal{O}_{Y'}.$$

Then there is a long exact sequence

$$0 \longrightarrow \mathcal{I}^{a-e}_{X/M}(K_M) \longrightarrow \oplus\, \mathcal{I}^{a-e+1}_{X}(K_M + D_i) \longrightarrow \dots$$
$$\dots \longrightarrow \oplus\, \mathcal{I}^{a-1}_{X/M}(K_M + D - D_i) \longrightarrow \mathcal{I}^{a}_{X/M}(K_M + D) \longrightarrow \mu_* \omega_{Y'}(-(a-1)E) \longrightarrow 0,$$

with the convention that $\mathcal{I}^{b}_{X/M} = \mathcal{O}_X$ if $b < 0$. □

**Application to Theorem 4.1.2.** Suppose now that $X \subseteq \mathbf{P}^r$ is a smooth subvariety of codimension $e$ that can be realized as the scheme-theoretic intersection of hypersurfaces $F_1, \dots, F_m$ of degrees $d_1 \geq \dots \geq d_m$. The hypothesis on the ordering of the $d_i$ is used in the proof of the crucual:

**Lemma 4.1.19** ([24], Claim 1.5 on p. 593)**.** *There exist hypersurfaces*

$$D_1\,,\ \dots\ ,\, D_e \ \subseteq\ \mathbf{P}^r,$$

*containing $X$, with $\deg D_i = d_i$, that define a generic linkage of $X$ with a variety $Y \subseteq \mathbf{P}^r$.*

Theorem 4.1.2 now follows immediately from the global Theorem 4.1.16. In fact

$$\deg\big(K_{\mathbf{P}^r} + \sum D_i\big) \ = \ (d_1 + \dots + d_e) - (r+1),$$

so if $k \geq (d_1 + \dots + d_e) - r$, then 4.1.16 gives the required vanishing.

It remains to say a word about the proof of Corollary 4.1.3. The regularity bound follows from the vanishing theorem, so the issue is to prove the second assertion. This is established by an evident modification of the proof of Proposition 4.1.8, whose details leave details to the reader.

**Example 4.1.20 (Higher powers of the ideal).** Just as in Example 4.1.11, a variant of Theorem 4.1.2 gives a vanishing for powers of $\mathcal{I}_X$. Specifically, with hypotheses as in 4.1.2, one has:

$$H^i\big(\mathbf{P}^r, \mathcal{I}^a_X(k)\big) \ = \ 0 \text{ for } \ i \geq 1$$

provided that $k \geq ad_1 + d_2 + \dots + d_e - r$. We remark that this expression is the regularity of the $a^{\text{th}}$ power of the ideal of a complete intersection in $\mathbf{P}^r$ of hypersurfaces of degrees $d_1 \geq d_2 \geq \dots \geq d_e$. □

## 4.2 Castelnuovo-type bounds

In this section we consider the problem of bounding the regularity of a smooth complex projective variety $X \subseteq \mathbf{P}^r$ in terms of its degree. For curves a statement along these lines follows from a classical theorem of Castelnuovo, and so we speak in general of regularity bounds of Castelnuovo-type.

The first subsection gives an overview of the work in this direction, and proves a (variant of) a result of Mumford. Curves and surfaces are treated in Section 4.2.B. The third subsection presents some complements.

## 4.2.A Background and statements

By way of introduction, we start with some history. In 1893, Castelnuovo [36] proved

**Theorem 4.2.1.** *Let $C \subseteq \mathbf{P}^3$ be a smooth curve of degree $d$ that is not contained in a plane. Then hypersurfaces of degrees $k \geq d-2$ trace out a complete linear series on $C$, ie the natural homomrphisms*

$$\rho_k : H^0\big(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^r}(k)\big) \longrightarrow H^0\big(C, \mathcal{O}_C(k)\big)$$

*are surjective for $k \geq d-2$. Equivalently, $H^1\big(\mathbf{P}^3, \mathcal{I}_C(k)\big) \ = \ 0$ when $k \geq d-2$.*

Using the reasoning leading to his bound on the genus of a space curve, Castelnuovo's statement implies that $\mathrm{reg}(C) \leq d-1$.

As Joe Harris observed, the example of curves having a $(d-1)$-secant line shows that the result cannot be improved for curves in $\mathbf{P}^3$. However he proposed around 1980 that a stronger bound should hold for a curve $C \subseteq \mathbf{P}^r$ that is non-degenerate in the sense that it is not contained in hyperplane. Harris' question was answered affirmatively by Gruson, Peskine and the second author:

**Theorem 4.2.2.** [101] *Let $C \subseteq \mathbf{P}^r$ be a non-degenerate irreducible curve of degree $d$. Then*

$$H^1\big(\mathbf{P}^r, \mathcal{I}_C(k)\big) \ = \ 0 \ \ \textit{for} \ \ k \geq d+1-r,$$

*and* $\mathrm{reg}(C) \leq d+2-r$.

The statement is best-possible and in fact the borderline cases were classified in the cited paper. The proof in [101] was homological in nature: it revolves around an Eagon–Nortcott complex and the observation that one can read off regularity from a linear "resolution" that is exact off a curve. In the next subsection we give a simpler alternative proof of the Theorem, in the case that $C$ is non-singular, following the strategy of [125].

There is an obvious extrapolation of this statement to arbitrary dimension:

**Conjecture 4.2.3 (Castelnuovo-type Regularity Conjecture).** *Let $X \subseteq \mathbf{P}^r$ be a smooth non-degenerate variety of degree $d$ and dimension $n$. Then*

$$\mathrm{reg}(X) \ \leq \ d+n+1-r.$$

Again examples show that this is best-possible in general.

At first there wasn't much evidence for this statement, but somewhat later Pinkham [159] was able to adapt Castelnuovo's geometric approach to establish a near optimal statement for surfaces. Specifically, Pinkham showed that if $S \subseteq \mathbf{P}^r$ is a non-degenerate surface of degree $d$, then

$$\mathrm{reg}(S) \ \leq \ d+4-r.$$

Pinkham's argument was a tour de force that for the first time made it plausable to expect precise statements in higher dimensions.

Prior to [101], Gruson, Peskine and Szpiro had given a very quick proof of Castelnuovo's Theorem 4.2.1 using vector bundles on $\mathbf{P}^2$: the argument appears in Szpiro's notes [183]. The second author observed that one could modify their approach to reprove Theorem 4.2.2 and get the optimal statement for surfaces:

**Theorem 4.2.4** ([125]). *Let $S \subseteq \mathbf{P}^r$ be a smooth non-degenerate surface of degree $d$. Then*

$$\mathrm{reg}(S) \ \leq \ d + 3 - r.$$

The proof appears in §4.2.B. Interestingly the construction of Gruson, Peskine and Szpiro proving this actually shows that hypersurfaces of a very special shape cut out a complete linear series on $X$.

The hypothesis that $\dim S = 2$ comes into play only to control the singularities of a generic projection $S \longrightarrow \mathbf{P}^3$. By studying what singularities to expect in higher dimensons, Ran and Kwak among others were able to obtain statements only a little off from Conjecture 4.2.3 in somewhat larger dimensions:

**Theorem 4.2.5** ([164], [121]). *Let $X \subseteq \mathbf{P}^r$ be a smooth non-degenerate variety of degree $d$ and dimension $n$.*

*(i). If $\dim X = 3$, then $\mathrm{reg}(X) \leq d + 5 - r$.*

*(ii). If $\dim X = 4$, then $\mathrm{reg}(X) \leq d + 9 - r$.*

Successively weaker bounds are known for varieties of still larger dimension. We refer to §4.2.C for a fuller discussion (without proofs).

All these results are linear in the degree $d$ with coefficient = 1. Shortly after [101], Mumford [15] gave a linear bound valid in all dimensions. At the end of this subsection we use Theorem 4.1.1 to establish a slight variant:

**Theorem 4.2.6.** *Let $X \subseteq \mathbf{P}^r$ be a smooth complex projective variety of dimension $n$, degree $d$ and codimension $e = r - n$. Set $c = \min\{e, n+1\}$. Then*

$$\mathrm{reg}(X) \ \leq \ c \cdot d - n.$$

*In particular $\mathrm{reg}(X) \leq (n+1)d - n$.*

From a geometric viewpoint, the restriction that $X$ be non-singular is quite natural for these questions (although [101] holds for arbitrary irreducible curves). However algebraically the smoothness hypothesis appeared more artificial, and Eisenbud and Goto [63] proposed that that the bound $\mathrm{reg}(X) \leq d + n + 1 - r$ should hold for arbitrary reduced and irreducible varieties. The Eisenbud–Goto conjecture spawned a vast amount of work in the commutative algebra community devoted to establishing it in various special cases. However as an outgrowth of their construction of prime ideals with bad regularity, McCullough and Peeva showed that the bound can fail badly for arbitrary reduced and irreducible varieties. See §4.3.B.

We conclude by proving Mumford's bound. Besides 4.1.1 the main input to the proof is an earlier observation of Mumford [141]:

**Lemma 4.2.7.** *Let $X \subseteq \mathbf{P}^r$ be a smooth variety of dimension $n$ and degree $d$. Then $X$ is cut out scheme-theoretically by hypersurfaces of degree $d$.*

*Proof.* We may suppose that $\operatorname{codim} X \geq 2$. Let $\Lambda \subseteq \mathbf{P}^r$ be a linear space of dimension $(r-n-2)$ disjoint from $X$, and denote by $C_\Lambda(X) \subseteq \mathbf{P}^r$ the cone over $X$ centered along $\Lambda$. Thus $C_\Lambda(X)$ is a hypersurface of degree $d$, and we assert that

$$X \;=\; \bigcap_{\Lambda \cap X = \varnothing} C_\Lambda(X) \tag{*}$$

as schemes. We first check (*) as point-sets, to which end fix $P \notin X$. Then

$$X \,\cap\, \operatorname{Span}(\Lambda\, , P\,) \;=\; \varnothing$$

for general $\Lambda$, so $P \notin \cap\, C_\Lambda(X)$, as required. To show that (*) holds on the level of schemes, it remains to show that for fixed $x \in X$, the tangent spaces $T_x C_\Lambda(X)$ cut out $T_x X$. For this fix a tangent vector $v \in T_x\mathbf{P}^r$ not lying in $T_xX$. Then projection from sufficiently general $\Lambda$ defines

$$\pi_\Lambda \,:\, (\mathbf{P}^r - \Lambda) \longrightarrow \mathbf{P}^{n+1}$$

which maps $x$ to a smooth point $\overline{x} \in \pi_\Lambda(X)$ and $v$ to a non-zero vector not tangent to $\pi_\Lambda(X)$. For this $\Lambda$, $C_\Lambda(X)$ is smooth at $x$ and $v \notin T_x C_\Lambda(X)$. □

**Remark 4.2.8.** If $X \subseteq \mathbf{P}^r$ is a possibly singular irreducible variety of dimension $n$ and degree $d$, then the argument just completed shows that $X$ is set-theoretically cut out by hypersurfaces of degree $d$. But it seems not to be known whether the corresponding statement is true scheme-theoretically. □

*Proof of Theorem 4.2.6.* We assert to begin with that if $r > 2n+1$ then it suffices to prove the statement for the embedding $X \subseteq \mathbf{P}^{2n+1}$ obtained from a general linear projection. In fact, if hypersurfaces of degree $k$ in $\mathbf{P}^{2n+1}$ cut out a complete linear series on $X$, then so too do hypersurfaces of degree $k$ in $\mathbf{P}^r$. Therefore

$$H^1\big(\mathbf{P}^{2n+1}, \mathcal{I}_{X/\mathbf{P}^{2n+1}}(k)\big) \;=\; 0 \;\Longrightarrow\; H^1\big(\mathbf{P}^r, \mathcal{I}_{X/\mathbf{P}^r}(k)\big) \;=\; 0,$$

while the higher cohomology groups are the same for both embeddings. So we may assume that $X$ has codimension $c = \min\{e, n+1\}$. On the other hand, $X$ is cut out scheme-theoretically by hypersurfaces of degree $d$ thanks to the previous Lemma, and then 4.1.1 gives the stated regularity bound. □

**Remark 4.2.9** (**Curves in a product of projective spaces**). Lozovanu [130] and Cobb [41] have given extensions of Theorem 4.2.2 to curves lying in a product of projective spaces. □

## 4.2.B Regularity for non-singular curves and surfaces

This subsection is devoted to a proof of Theorems 4.2.2 and 4.2.4 following the approach of [183] and [125].

**Curves.** Suppose that $C \subseteq \mathbf{P}^r$ is a non-degenerate smooth curve of degree $d$. Fixing a general $(r-3)$-plane $\Lambda \subseteq \mathbf{P}^r$, we project from $\Lambda$ to define

$$\pi \,=\, \pi_\Lambda \;:\; C \longrightarrow \mathbf{P}^2.$$

For concreteness, write $z_0, z_1, \ldots, z_r$ for homogeneous coordinates on $\mathbf{P}^r$ and take $\Lambda$ to be defined by $z_0 = z_1 = z_2 = 0$, so that $\pi$ is given by

$$[z_0, \ldots, z_r] \mapsto [z_0, z_1, z_2].$$

We may and do assume that $\pi$ maps $C$ birationally onto a plane curve $\overline{C} \subseteq \mathbf{P}^2$ of degree $d$ having ordinary doubly points as its only singularities.

The next step is to construct a presentation of $\pi_*\mathcal{O}_C$ as an $\mathcal{O}_{\mathbf{P}^2}$-module that is adapted to the problem at hand. To begin with, there is a natural map $\mathcal{O}_{\mathbf{P}^2} \longrightarrow \pi_*\mathcal{O}_C$. In addition, the sections

$$z_3\,,\;\ldots\;,\;z_r \;\in\; \Gamma\big(C, \mathcal{O}_C(1)\big)$$

are realized by maps $\mathcal{O}_{\mathbf{P}^2} \longrightarrow \pi_*\mathcal{O}_C(1)$. We put these together to define a homomorphism

$$w \;:\; \mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(-1)^{r-2} \longrightarrow \pi_*\mathcal{O}_C. \tag{4.2.1}$$

Observe next that if $\Lambda$ is chosen generically, then *the homomorphism $w$ is surjective as a mapping of $\mathcal{O}_{\mathbf{P}^2}$-modules*. Indeed, this equivalent to the assertion that the coordinates $z_3, \ldots, z_r$ separate fibres of $\pi$ over $\mathbf{P}^2$, which is clear since these fibres consist of at worst two (reduced) points. Thus one arrives at an exact sequence

$$0 \longrightarrow F \longrightarrow \mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(-1)^{r-2} \stackrel{w}{\longrightarrow} \pi_*\mathcal{O}_C \longrightarrow 0, \tag{4.2.2}$$

defining a sheaf $F$ on $\mathbf{P}^2$. Since $\pi_*\mathcal{O}_C$ is Cohen–Macaulay as an $\mathcal{O}_{\mathbf{P}^2}$-module, it follows that $F$ is in fact locally free, of rank $r-1$. Noting that the map $F \longrightarrow \mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(-1)^{r-2}$ in (4.2.2) drops rank along a curve of degree $d$, we see that $\det\, F = \mathcal{O}_{\mathbf{P}^2}(-d-r-2)$.

Now consider the homomorphism

$$t_k \;:\; H^0\Big(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(k) \oplus \mathcal{O}_{\mathbf{P}^2}(k-1)^{r-2}\Big) \longrightarrow H^0\Big(C, \pi_*\mathcal{O}_C(k)\Big)$$

determined by $w$. By construction the image of $t_k$ consists of the restriction to $C$ of all homogeneous polynomials in $\mathbf{P}^r$ having the form

$$P_k(z_0, z_1, z_2) + z_3 \cdot Q_{k-1,3}(z_0, z_1, z_2) + \ldots + z_r \cdot Q_{k-1,r}(z_0, z_1, z_2)$$

where $P_k$ and the $Q_{k-1,j}$ are homogeneous polynomials of degrees $k$ and $k-1$ respectively on $\mathbf{P}^2$. Therefore

$$\mathrm{im}\big(t_k\big) \subseteq \mathrm{im}\Big(\rho_k : H^0\big(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(k)\big) \longrightarrow H^0\big(C, \mathcal{O}_C(k)\big)\Big).$$

It follows that the vanishing of $H^1\big(\mathbf{P}^2, F(k)\big)$ implies the vanishing of $H^1\big(\mathbf{P}^2, \mathcal{I}_C(k)\big)$. Since $H^1\big(C, \mathcal{O}_C(k)\big) = H^2\big(\mathbf{P}^2, F(k)\big)$ thanks to (4.2.2), we see that Theorem 4.2.2 will follow if we show that $F$ is $(d+2-r)$-regular.

To this end, observe to begin with that

$$H^1\big(\mathbf{P}^2, F\big) = 0 \quad , \quad H^0\big(\mathbf{P}^2, F(1)\big) = 0.$$

Indeed, referring to (4.2.2), the first vanishing follows from the fact that $w$ is an isomorphism on global sections, while the second expresses the non-degeneracy of $C \subseteq \mathbf{P}^r$. Serre duality then yields

$$H^1\big(\mathbf{P}^2, F^*(-3)\big) = H^2\big(\mathbf{P}^2, F^*(-4)\big) = 0,$$

and hence $F^*$ is $(-2)$-regular.

But now the required $(d+2-r)$-regularity of $F$ follows from the multiplicativity of regularity in tensor products (Corollary 3.1.17) in characteristic zero. In fact

$$\begin{aligned} F &= \Lambda^{\mathrm{rank}(F)-1} F^* \otimes \det F \\ &= \Lambda^{r-2} F^* \otimes \mathcal{O}_{\mathbf{P}^2}(-d-r+2), \end{aligned}$$

and hence

$$\mathrm{reg}(F) \leq (-2)(r-2) + (d+r-2) = d+2-r,$$

as required. □

**Remark 4.2.10** (**Work of Gruson, Peskine and Szpiro**)**.** The case $r=3$ of this argument appears in [183]. The resulting rank two vector bundle $F$ on $\mathbf{P}^2$ also plays a central role in the work [102] of Gruson and Peskine, where among other things it is established that $F$ is stable. These authors make the interesting observtation that the Bogomolov inequality $c_1(F)^2 < 4c_2(F)$ is equivalent to Castelnuovo's bound on the genus of a space curve. □

**Remark 4.2.11** (**Kernel bundles**)**.** The proof of Theorem 4.2.2 in [101] proceeded via a lemma that sometimes gives additional information; the argument can also be adapted to cover singular curves. Writing $V = H^0\big(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(1)\big)$, the given embedding $C \subseteq \mathbf{P}^r$ determines a surjective mapping

$$\mathrm{ev} : V \otimes_{\mathbf{C}} \mathcal{O}_C \longrightarrow \mathcal{O}_C(1)$$

of vector bundles on $C$. Let $M = \ker(\mathrm{ev})$, so that $M$ is a vector bundle on $C$ of rank $r$ and degree $= -d$. The key lemma in [101] is the following:

Let $A$ be a line bundle on $C$ with the property that

$$H^1\big(C, \Lambda^2 M \otimes A\big) \ = \ 0. \tag{*}$$

Then $C$ is $h^0(C, A)$-regular.

To see how this works, suppose that $C = \mathbf{P}^1$ is rational. Then by Grothendieck's theorem, $M$ decomposes as a direct sum of line bundles, say

$$M \ = \ \mathcal{O}_{\mathbf{P}^1}(-b_1) \oplus \ldots \oplus \mathcal{O}_{\mathbf{P}^1}(-b_r) \ \ , \ \ \text{with } b_1 \leq \ldots \leq b_r.$$

If $C$ is non-degenerate then $h^0(C, M) = 0$, and hence all $b_i \geq 1$. One has $\sum b_i = d$, and therefore (*) is satisfied with $A = \mathcal{O}_{\mathbf{P}^1}(d + 1 - r)$, the "worst" possibility being $b_1 = \ldots = b_{r-2} = 1$, $b_{r-1} + b_r = d - r + 2$. However if one happens to know that $M$ is more balanced, then (*) leads to a stronger regularity statement. □

**Surfaces.** We now indicate the modifications required to the argument just completed in order to establish the regularity Theorem 4.2.4 for smooth surfaces. The presentation follows [125], with substantial simplifications suggested by V. Greenberg.

Consider then a smooth non-degenerate surface $S \subseteq \mathbf{P}^r$ of degree $d$. We start by formalizing and generalizing slightly the construction of the mapping $w$ in (4.2.1). Fix a linear space $\Lambda \subseteq \mathbf{P}^r$ of dimension $r - 4$ disjoint from $X$, and denote by $p : M \ =_{\text{def}} \mathrm{Bl}_\Lambda(\mathbf{P}^r) \longrightarrow \mathbf{P}^r$ the blowing up along $\Lambda$. This admits a projection $q : M \longrightarrow \mathbf{P}^3$, and so one obtains for each $k \in \mathbf{N}$ a homomorphism

$$u_k : q_*\big(p^*\mathcal{O}_{\mathbf{P}^r}(k)\big) \longrightarrow q_*\big(p^*\mathcal{O}_S(k)\big) \tag{4.2.3}$$

of sheaves on $\mathbf{P}^3$. In fact, $M$ is identified with the projectivization $\mathbf{P}(U)$, where $U$ is the vector bundle $\mathcal{O}_{\mathbf{P}^3}(1) \oplus \mathcal{O}_{\mathbf{P}^3}^{r-3}$ on $\mathbf{P}^3$, and hence

$$q_*\big(p^*\mathcal{O}_{\mathbf{P}^r}(k)\big) \ = \ \mathrm{Sym}^k(U).$$

The map $u$ constructed in (4.2.1) is essentially (a twist of) the case $k = 1$ of (4.2.3).

We shall be particularly interested in $u_2$, which takes the form

$$u_2 : \mathcal{O}_{\mathbf{P}^3}^{N(r)} \oplus \mathcal{O}_{\mathbf{P}^3}(1)^{r-3} \oplus \mathcal{O}_{\mathbf{P}^3}(2) \longrightarrow \pi_*\mathcal{O}_S(2),$$

where $N(r) = \binom{r-2}{2}$. As in the argument for curves, we can make this more explicit by choosing coordinates on $\mathbf{P}^r$ in such a way that $\Lambda$ is given by $z_0 = \ldots = z_3 = 0$. Then the components of $w_2$ arise from multiplication by all monomials of degrees one and two in the remaining variables $z_4, \ldots, z_r$. Twisting by $\mathcal{O}_{\mathbf{P}^3}(-2)$ one arrives at the map

$$w_2 \ : \ \mathcal{O}_{\mathbf{P}^3}^{N(r)}(-2) \oplus \mathcal{O}_{\mathbf{P}^3}(-1)^{r-3} \oplus \mathcal{O}_{\mathbf{P}^3} \longrightarrow \pi_*\mathcal{O}_S, \tag{4.2.4}$$

that will be our focus.

So far we have not used in any essential way that $S$ has dimension two. This comes in to verify the critical

**Lemma 4.2.12.** *For a sufficiently general choice of $\Lambda$ the map $u_2$ – and hence also $w_2$ – is surjecive.*

*Proof.* It suffices to check this fibre by fibre over $\mathbf{P}^3$. Fix a point $y \in \mathbf{P}^3$, and consider the $(r-3)$-plane $L_y = p(q^{-1}(y))$ through $\Lambda$ corresponding to $y$. Denoting by $S_y$ the scheme-theoretic intersection of $S$ with $L_y$, one sees that $u_k \otimes \mathbf{C}(y)$ is identified with the restriction homomorphsm

$$\rho_k : H^0\big(L_y, \mathcal{O}_{L_y}(k)\big) \longrightarrow H^0\big(S_y, \mathcal{O}_{S_y}(k)\big).$$

Hence the surjectivity of $u_2$ it is equivalent to the assertion that $H^1\big(L_y, \mathcal{I}_{S_y/L_y}(2)\big) = 0$ for each $y \in \mathbf{P}^3$.

Assume now that $\Lambda$ is chosen so that $\pi : S \longrightarrow \overline{S} \subseteq \mathbf{P}^3$ has only the classical ordinary singularities: a curve of double points (along which $\overline{S}$ is given by the local analytic equation $uv = 0$); finitely many pinch points (corrresponding to the local equation $u^2 - wv^2 = 0$); and finitely many triple points (with local equation $uvw = 0$). (Cf [?].) Then $S_y$ is a scheme of length two in the first two cases, and length three in the third. In any event $\rho_2$ is surjective, and the Lemma is established. □

Now let $F = \ker(w_2)$, giving rise to an exact sequence

$$0 \longrightarrow F \longrightarrow \mathcal{O}_{\mathbf{P}^3}^{N(r)}(-2) \oplus \mathcal{O}_{\mathbf{P}^3}(-1)^{r-3} \oplus \mathcal{O}_{\mathbf{P}^3} \longrightarrow \pi_*\mathcal{O}_S \longrightarrow 0.$$

The smoothness of $S$ implies that $\pi_*\mathcal{O}_S$ is locally Cohen-Macaulay, and hence $F$ is locally free. One has

$$\operatorname{rank}(F) \ = \ N(r) + r - 2 \ \ , \ \ \det(F) \ = \ \mathcal{O}_{\mathbf{P}^3}\big( - 2N(r) - r + 3 - d \ \big),$$

and $F$ satisfies

$$H^1\big(\mathbf{P}^3, F\big) \ = \ 0 \ \ , \ \ H^0\big(\mathbf{P}^3, F(1)\big) = 0, \tag{4.2.5}$$

the second coming from the non-degeneracy of $S \subseteq \mathbf{P}^r$. Furthermore, Kodaira vanishing on $S$ implies:

$$H^2\big(\mathbf{P}^3, F(-1)\big) \ = \ H^1\big(\mathbf{P}^3, \pi_*\mathcal{O}_S(-1)\big) \ = \ H^1\big(S, \mathcal{O}_S(-1)\big) \ = \ 0. \tag{4.2.6}$$

As before, one can control the regularity of $S \subseteq \mathbf{P}^r$ via cohomological properties of $F$. Specifically, denote by

$$W_k \ \subseteq \ H^0\big(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(k)\big)$$

the subspace spanned by forms pulled back from $\mathbf{P}^3$ together with the monomials $\{z_i z_j\}$ and $\{z_i\}$ for $4 \le i, j \le r$ . Thus

$$W_k \ = \ H^0\Big( \ \mathbf{P}^3 \ , \mathcal{O}_{\mathbf{P}^3}^{N(r)}(k-2) \oplus \mathcal{O}_{\mathbf{P}^3}(k-1)^{r-3} \oplus \mathcal{O}_{\mathbf{P}^3}(k)\Big),$$

and hence $H^1\big(\mathbf{P}^3, F(k)\big) = 0$ if and only if $W_k$ maps onto $H^0\big(S, \mathcal{O}_S(k)\big)$. Therefore Theorem 4.2.4 follows from

**Proposition 4.2.13.** *The vector bundle $F$ is $(d+3-r)$–regular.*

*Proof.* This is proved just as in the case of curves . Specifically, applying Serre duality to the vanishings observed in (4.2.6) and (4.2.5), one finds that

$$H^1\big(\mathbf{P}^3, F^*(-3)\big) \ = \ H^2\big(\mathbf{P}^3, F^*(-4)\big) \ = \ H^3\big(\mathbf{P}^3, F^*(-5)\big) \ = \ 0.$$

Therefore $F^*$ is $(-2)$–regular. On the other hand,

$$\begin{aligned} F \ &= \ \Lambda^{\mathrm{rank}(F)-1}\left(F^*\right) \otimes \det(E) \\ &= \ \Lambda^{N(r)+r-3} F^* \otimes \mathcal{O}_{\mathbf{P}^3}\big(-2\cdot N(r) - r + 3 - d\big). \end{aligned}$$

Therefore, thanks to Corollary 3.1.17:

$$\begin{aligned} \mathrm{reg}(F) \ &\leq \ (-2)\cdot\big(N(r)+r-3\big) + \big(2\cdot N(r) + r - 3 + d\big) \\ &= d+3-r, \end{aligned}$$

as required. □

## 4.2.C Complements

We conclude by discussing without proof some extensions of these ideas.

**Singularities of projections in higher dimensions.** The hypothesis that $\dim S = 2$ comes into the argument just completed only to ensure that all the fibres of a general projection impose independent conditions on quadrics in their linear spans, which is what is required for Lemma 4.2.12. This in turn followed from the classical description of the singularities that arise for generic projections of smooth surfaces.

The appearance of [125] sparked renewed interest in the geometry of generic projections in higher dimensions, leading to some extensions of the regularity theorem for surfaces. We briefly summarize some of these developments here.

To begin with, following Greenberg and Kwak [121], one can axiomatize the computations appearing above. Specifically, consider a smooth non-degenerate projective variety $X \subseteq \mathbf{P}^r$ of dimension $n$. Fix a generic $(r-n-2)$-plane $\Lambda \subseteq \mathbf{P}^r$ and denote by

$$\pi \ = \ \pi_\Lambda : X \longrightarrow \mathbf{P}^{n+1}$$

the corresponding linear projection. The idea is that if one controls the number and degrees of generators of $\pi_*\mathcal{O}_X$ as an $\mathcal{O}_{\mathbf{P}^{n+1}}$- module, then one gets an effective regularity bound.

Writing $X_y \subseteq L_y$ for the fibre of $\pi$ over $y \in \mathbf{P}^{n+1}$, the question is equivalent as in Lemma 4.2.12 to giving generators for all the fibres $H^0\big(L_y, \mathcal{O}_{X_y}(k)\big)$. Kwak arrives at Theorem

4.2.5 and some extensions by appealing to results of Mather on the singularities of generic projections. We refer to his nice survey [121] for precise statements and proofs.

A construction of Flenner and Ran (cf [128]) shows then when $n$ is large a generic projection will have fibres whose length is exponential in $n$. However there are some general results and conjectures that are of interest even if they don't directly have applications to regularity questions. First, Ran [165] proves:

**Theorem 4.2.14 (The (dimension + 2)-Secant Lemma).** *Let $X \subseteq \mathbf{P}^r$ be a smooth projective variety of dimension $n$, and denote by*

$$\mathrm{Sec}^{n+2}(X) \ \subseteq \ \mathbf{P}^r$$

*the variety swept out by all the $(n+2)$-secant lines to $X$. Then*

$$\dim \mathrm{Sec}^{n+2}(X) \ \leq \ n+1.$$

This generalizes the classical fact that a smooth curve $C \subseteq \mathbf{P}^3$ has at most a one-dimensional family of trisecant lines. Ran's result implies that a generic projection $X \longrightarrow \mathbf{P}^{n+1}$ will not have any fibres consisting of $(n+2)$ collinear points. A quick proof of the Theorem appears in Appendix A of [14]. Further developments appear in the papers [103], [166] [167] of Gruson–Peskine and Ran.

In another direction, Beheshti and Eisenbud [20], [21] introduce and study a more subtle non-classical invariant of the fibres of a projection for which they obtain strong uniform bounds. We refer to their papers for precise statements and applications. We do however want to mention a very clean conjecture that they propose:

**Conjecture 4.2.15 (Conjecture of Beheshti and Eisenbud).** *Let $X \subseteq \mathbf{P}^r$ be a smooth projective variety of dimension $n$, and let $\pi : X \longrightarrow \mathbf{P}^{n+c}$ be a generic linear projection. Then for any point $y \in \mathbf{P}^{n+c}$, the fibre $X_y = \pi^{-1}(y)$ has Castelnuovo–Mumford regularity*

$$\mathrm{reg}(X_y) \ \leq \ \tfrac{n}{c}+1$$

*considered as a finite subset of $\mathbf{P}^r$.* □

**Double-point divisors and regularity of $\mathcal{O}_X$.** Mumford's original proof of 4.2.6 made use of double-point divisors of generic projections, and these have found other applications to this circle of ideas. Suppose as before that $X \subseteq \mathbf{P}^r$ is a non-degenerate smooth complex projective variety of dimension $n$ and degree $d$, and that $\pi_\Lambda : X \longrightarrow \mathbf{P}^{n+1}$ is a generic projection. The double points of $\pi_\Lambda$ determine an effective divisor

$$\Delta_\Lambda \ \equiv_{\mathrm{lin}} \ (d-n-2)H - K_X.$$

By varying $\Lambda$ one finds:

$$|\,(d-n-2)H - K_X\,| \text{ is a base-point free linear series.} \qquad (4.2.7)$$

This opens the door to using Kodaira vanishing to control the higher cohomology of line bundles of the form $\mathcal{O}_X(k)$.

To begin with, the exact sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbf{P}^r} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

shows that if $\mathcal{I}_X$ is $m$-regular, then $\mathcal{O}_X$ is $(m-1)$-regular. Therefore Conjecture 4.2.3 predicts that $\mathcal{O}_X$ should be $(d+n-r)$-regular. It was observed by Illic [113] that (4.2.7) quickly leads to a statement quite close to this:

**Proposition 4.2.16** ([113]). *The structure sheaf $\mathcal{O}_X$ is $(d-1)$-regular.*

*Proof.* One needs to show that $H^i\big(X, \mathcal{O}_X(d-i-1)\big) = 0$ for $1 \le i \le n$. Since

$$(d-i-1)H \ = \ K_X \ + \ \Big(\big(d-(n+2)\big)H - K_X\Big) \ + \ (n+1-i)H,$$

it follows from (4.2.7) that the divisor on the left is of the form $K_X$ + (ample). So the Proposition follows from Kodaira vanishing. □

Building on work of Kwak and Park [?], Noma considered also "inner projections," i.e. projecting from a linear space that meets $X$. By a detailed analysis along these lines, Noma [?] finally proved the very nice result that $\mathcal{O}_X$ is in fact $(d+n-r)$-regular.

Mumford's original proof of 4.2.6 also revolved around these double point divisors. In their interesting paper [122], Kwak and Park observe that one could greatly improve the numerics if one had more control over the linear series that these divisors generate. Specifically, with $X \subseteq \mathbf{P}^r$ as above, denote by

$$W \ \subseteq \ H^0\big(X, \mathcal{O}_X((d-n-2)H - K_X)\big)$$

the subspace spanned by (the defining sections of) all double-point divisors $\Delta_\Lambda$ of generic projections $\pi_\Lambda$. Kwak and Park show that if one can bound from above the codimension of $W$, then one can combine Mumford's argument with results of the authors from [51] to get a regularity bound that potentially comes quite close to the Castalnuovo-type Conjecture 4.2.3. We hope that these ideas will lead to further progress in the future.

**Singular varieties.** Although the construction of [135] discussed in the next section shows that a linear regularity bound cannot hold for arbitrarily singular varieties, it is interesting to ask what one can say for mildly singular varieties. Results along these lines for surfaces and threefolds have been obtained by Niu [144] and Niu–Park [146].

## 4.3 Constructions

The preceding results show that for ideals of smooth varieties, regularity is bounded linearly in input data such as the generating degree. By contrast, it has been known since at least the 1980s that this can fail badly for arbitrary ideals. In this section we briefly survey – largely without proofs – some of the examples that witness this phenomenon. Several of the constructions deal in the first instance with arithmetic regularity, but in view of Example 3.1.40 these lead also to geometric examples.

As we proceed, it is worthwhile to keep in mind a theorem of Galligo and Giusti that frames the question. Namely, suppose that $I \subseteq \mathbf{C}[z_0, \ldots, z_n]$ is an arbitrary homogeneous ideal whose generators have degrees $\leq d$. Then

$$\operatorname{arithreg}(I) \ \leq \ \big(2d\big)^{2^{n-1}}. \tag{4.3.1}$$

In their paper [15], Bayer and Mumford give a quick proof of the slightly weaker bound $\operatorname{arithreg}(I) \leq (2d)^{n!}$. This paper has been very influential in shaping the body of work around complexity of computation in algebraic geometry, and we strongly recommend it to the reader.

### 4.3.A Non-reduced schemes

Until recently, most "pathological" (i.e. interesting) examples involved ideals defining highly non-reduced schemes. The earliest constructions – which still exhibit the most extreme phenomena – had an ad hoc combinatorial flavor. Subsequently Ullery [184] found a more systematic and geometric approach to producing ideals with super-linear regularity. We will first explain Ullery's ideas in an illustrative case, and then say a few words (without proof) about some of the particular examples in the literature.

**Ullery's construction.** If $\mathcal{I} \subseteq \mathcal{O}_{\mathbf{P}}$ is an ideal with the property that $\mathcal{I}(1)$ is globally generated, then $\mathcal{I}$ cuts out a linear space and hence $\operatorname{reg}(\mathcal{I}) = 1$. On the other hand, it is not hard to exhibit $\mathcal{O}_{\mathbf{P}}$-modules $E$ generated in degree 1 with large regularity. Ullery's idea is that one can use examples of this sort systematically to build ideals whose regularity grows faster than linearly in the degrees of generators. We will illustrate her construction in a special case, producing for $\ell \gg 0$ a family of ideal sheaves $\mathcal{J}_\ell \subseteq \mathcal{O}_{\mathbf{P}^6}$ with the property that $\mathcal{J}_\ell(\ell+1)$ is globally generated, while

$$\operatorname{reg}(\mathcal{J}_\ell) \approx C \cdot \ell^{3/2}$$

for a suitable constant $C$. We refer to [184] for a more general description that includes parameters one can tune to exhibit various sorts of interesting behavior.

Turning to details, let $\Lambda = \mathbf{P}^2$ be a two-dimensional projective space, fix a large integer $k$, and consider on $\Lambda$ a general surjective homomorphism

$$u : \mathcal{O}_\Lambda^{k+2}(-1) \longrightarrow \mathcal{O}_\Lambda^{k}.$$

This determines an exact Eagon-Northcott (cf. [128, Appendix B]) complex having the shape

$$0 \longrightarrow \mathcal{O}_\Lambda(-k-2) \longrightarrow \Lambda^2\, \mathcal{O}_\Lambda^{k+2}(-1) \longrightarrow \mathcal{O}_\Lambda^{k+2}(-1) \otimes \mathcal{O}_\Lambda^{k} \longrightarrow \mathrm{Sym}^2 \mathcal{O}_\Lambda^{k} \longrightarrow 0. \tag{4.3.2}$$

Twisting by $\mathcal{O}_\Lambda(1)$, one arrives at an exact commutative diagram defining a vector bundle $E = E_k$ on $\Lambda$:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{O}_\Lambda(-k-1) & \longrightarrow & \mathcal{O}_\Lambda^{\binom{k+2}{2}}(-1) & & \longrightarrow & & \mathcal{O}_\Lambda^{k^2+2k} & \longrightarrow & \mathcal{O}_\Lambda^{\binom{k+1}{2}}(1) \longrightarrow 0. \\ & & & & & \searrow & & \nearrow & & & \\ & & & & & & E_k & & & & \end{array} \tag{4.3.3}$$

We see that

$$\mathrm{rank}(E_k) \;\approx\; \tfrac{k^2}{2} \;\;,\;\; \mathrm{reg}(E_k) \;=\; k,$$

and that $E_k(1)$ is globally generated.

The next step is to use $E_k$ to build an ideal on a larger projective space $\mathbf{P}^r$; to fix ideas we will (somewhat arbitrarily) take $r = 6$. For this, we start by choosing a linear embedding $\Lambda \subseteq \mathbf{P} = \mathbf{P}^6$. Then the conormal bundle $N^* = N^*_{\Lambda/\mathbf{P}}$ is a direct sum of four copies of $\mathcal{O}_\Lambda(-1)$: $N^* = \mathcal{O}_\Lambda^4(-1)$. Next, choose $\ell$ so that

$$\binom{\ell+3}{3} \;=\; \mathrm{rank}\,\mathrm{Sym}^\ell N^* \;\;\geq\;\; \mathrm{rank}\, E_k + 2 \;\approx\; \frac{k^2}{2}. \tag{*}$$

Since $E_k^*$ is globally generated by virtue of (4.3.3), we can then fix an embedding

$$\alpha : E_k(-\ell) \hookrightarrow \mathrm{Sym}^\ell\, N^*_{\Lambda/\mathbf{P}}.$$

Now when $\ell$ and $k$ are large, (*) requires that

$$\frac{\ell^3}{6} \;\gtrapprox\; \frac{k^2}{2}.$$

So we can suppose that

$$k \;\approx\; C \cdot \ell^{3/2}$$

for some constant $C$.

Finally consider on $\mathbf{P} = \mathbf{P}^6$ the conormal sequence

$$0 \longrightarrow \mathcal{I}^{\ell+1}_{\Lambda/\mathbf{P}} \longrightarrow \mathcal{I}^{\ell}_{\Lambda/\mathbf{P}} \longrightarrow \mathrm{Sym}^\ell\, N^*_{\Lambda/\mathbf{P}} \longrightarrow 0.$$

We can pull back the embedding $\alpha$ to define an ideal sheaf $\mathcal{J} = \mathcal{J}_\ell \subseteq \mathcal{I}^{\ell}_{\Lambda/\mathbf{P}}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}^{\ell+1}_{\lambda/\mathbf{P}} & \longrightarrow & \mathcal{J} & \longrightarrow & E_k(-\ell) & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow & & \Big\downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & \mathcal{I}^{\ell+1}_{\lambda/\mathbf{P}} & \longrightarrow & \mathcal{I}^{\ell}_{\Lambda/\mathbf{P}} & \longrightarrow & \mathrm{Sym}^\ell\, N^*_{\Lambda_{\mathbf{P}}} & \longrightarrow & 0. \end{array}$$

Thus the zeroes of $\mathcal{J}$ are supported on $\Lambda$, but with a non-trivial $\ell^{\text{th}}$ order scheme structure. Noting that $\operatorname{reg} \mathcal{I}_{\lambda/\mathbf{P}}^{\ell+1} = \ell + 1$, we see that $\mathcal{J}(\ell+1)$ is globally generated, and that

$$\operatorname{reg} \mathcal{J} \ = \ \operatorname{reg} E_k + \ell \ = \ k + \ell \ \approx \ C \cdot \ell^{3/2}.$$

Thus we have exhibited ideals whose regularity grows super-linearly in their generating degree.

We started with the Eagon–Northcott complex (4.3.2) only for the sake of concreteness. In her paper [184] Ullery begins with a more general pure complex as discussed in Lecture 2. By varyng this input, she settles several questions that had been raised in the literature, for example exhibitibg ideals whose regularity is revealed by degree jumps at different positions in the resolution.

**Caviglia's ideals.** As mentioned in Example 3.1.41, Caviglia discovered that the very simple-looking ideals

$$I_d \ = \ (x^d \, , \, y^d \, , \, xz^{d-1} - yw^{d-1}) \ \subseteq \ \mathbf{C}[x,y,z,w]$$

have surprisingly large arithmetic regularity:

$$\operatorname{arithreg}(I_d) \ = \ d^2 \, - \, 1.$$

He establishes this by studying the initial ideal of $I_d$ with respect to reverse lex order; it would be interesting to find a more geometric explanation. More recently, Choe [38] combined Caviglia's construction with the ideas of "unprojection" to produce some reduced varieties having large regularity.

**The examples of Mayer–Myer–Bayer–Stillman.** In their paper [17], Bayer and Stillman showed that constructions of Mayer and Myer, introduced to study problems in complexity theory, yield examples of ideals exhibiting doubly exponential regularity growth. Specifically, they produce ideals

$$I_n \ \subseteq \ \mathbf{C}[z_1, \ldots, z_n]$$

in the polynomial ring in $n = 10m + 1$ generators of degrees $\leq d$ with the property that

$$\operatorname{arithreg}(I_n) \ \geq \ \big(d-2\big)^{2^{m-1}} \ \approx \ d^{2^{n/10}}.$$

Thus the general shape of the upper bound (4.3.1) cannot be improved. Bayer and Stillman also observe that any example of extreme behavior for the ideal membership problem will lead to ideals with large regularity. Other families of highly irregular ideals appear in the paper [120] of Koh. The basic idea in all cases is to look at ideals generated by binomials whose exponent vectors have interesting combinatorial properties. See [180–182] for further analysis of these ideals. Once again, we believe that it would be nice to have a more "geometric" construction in the spirit of [184].

## 4.3.B The construction of McCullough and Peeva

For many years, it was unclear what to expect for the regularity of reduced and irreducible, but possibly singular, varieties $X \subseteq \mathbf{P}^r$: is their regularity bounded linearly in geometric data, or can it be much worse? In their breakthrough paper [135], McCullough and Peeva gave a construction that starts with a homogeneous ideal $I \subseteq S$ in a polynomial ring, and produces a prime ideal $P \subseteq T$ in a somewhat larger polynomial ring whose input parameters and arithmetic regularity are closely controlled by those of $I$. In particular, starting with one of the examples from the previous subsection, one arrives at a prime ideal with very large regularity. We present in this subsection a quick overview of their work, referring to [135] for detailed statements and proofs. McCullough's survey [134] is another valuable resource.

For simplicity, we start by explaining the idea in the affine setting. Let $A = \mathbf{C}[x_1, \ldots, x_n]$ be a polynomial ring in $n$ variables, and let $I \subseteq A$ be an ideal. Let $f_1, \ldots, f_m \in I$ be generators, and use these to fix a presentation

$$A^\ell \xrightarrow{\ \gamma\ } A^m \longrightarrow I \longrightarrow 0. \tag{4.3.4}$$

Thus $\gamma = (c_{i,j})$ is an $m \times \ell$ matrix of polynomials whose whose columns span the module of of syzygies among the $f_j$.

There are two familiar $A$-algebras associated to $I$. To begin with the symmetric algebra $\mathrm{Sym}(I)$ captures the "linear" relations among the generators. It can be realized as the quotient

$$\mathrm{Sym}(I) \ = \ A[y_1, \ldots, y_m] \, / \, L$$

of the polynomial ring $A[y_1, \ldots, y_m]$ modulo the ideal $L$ generated by the elements

$$r_i \ =_{\text{def}} \ c_{i,1} \cdot y_1 + \ldots + c_{i,m} \cdot y_m \ \in \ A[y_1, \ldots, y_m]. \tag{4.3.5}$$

Thus $\mathrm{Sym}(I)$ is easy to describe in terms of generators and relations, but it is typically not a domain or otherwise well-behaved.

One also has the Rees algebra $\mathrm{Rees}(I) = \oplus_{k \geq 0} I^k$ of $I$. It is customary to introduce a formal variable $t$ to keep track of the grading, so that

$$\mathrm{Rees}(I) \ = \ A[\, I \cdot t \,].$$

This algebra is a quotient of $\mathrm{Sym}(I)$ and it is always a domain (as $A[t]$ is). However the kernel $K$ of the natural map

$$A[y_1, \ldots, y_m] \longrightarrow \mathrm{Rees}(I) \ \ , \ \ y_j \mapsto f_j \cdot t,$$

can be difficult to pin down. Indeed, in addition to the elements $r_i$ from (4.3.5), $K$ often requires unpredictable generators of degrees $\geq 2$ in the $y_j$ arising from multiplicative relations among the $f_j$.

The beautiful idea of McCullough and Peeva is that by enlarging $\mathrm{Rees}(I)$, one arrives at an algebra that on the one hand is a domain while also admitting a uniform description in terms of generators, relations and higher syzygies.

**Definition 4.3.1** (McCullough–Peeva algebra)**.** Given an ideal $I \subseteq A$, define $\mathrm{MP}(I)$ to be the graded $A$-algebra

$$\mathrm{MP}(I) \ = \ A[I \cdot t\, ,\, t^2].$$

Explicitly,

$$\mathrm{MP}(I) \ = \ A \oplus I \cdot t \oplus A \cdot t^2 \oplus I \cdot t^3 \oplus A \cdot t^4 \oplus \dots . \qquad \square$$

Thus $\mathrm{MP}(I)$ is generated as an $A$-algebra by its components in degrees one and two.

**Example 4.3.2.** Take $A = \mathbf{C}[x]$ and $I = (x)$. Then

$$A[I \cdot t, t^2] \ = \ \mathbf{C}[x\, ,\, xt\, ,\, t^2]$$

is the coordinate ring of the Whitney umbrella $\{u^2 w - v^2 = 0\}$ in $\mathbf{C}^3$. In general, if $I$ is prime and $Z \subseteq \mathbf{A}^n$ is the variety defined by $I$, then $\mathrm{MP}(I)$ may be viewed as the coordinate ring of the variety obtained from $\mathbf{A}^n \times \mathbf{A}^1$ by identifying $(z, t)$ with $(z, -t)$ for $z \in Z$. $\square$

The algebra $\mathrm{MP}(I)$ admits a simple presentation as a quotient of a polynomial ring over $A$. Specifically, consider the homomoprphism

$$\begin{gathered} \phi \,:\, A[y_1, \dots, y_m, z] \longrightarrow A[I \cdot t, t^2] \\ y_j \mapsto f_j \cdot t \quad , \quad z \mapsto t^2, \end{gathered}$$

and denote by $Q \subseteq A[y_1, \dots, y_m, z]$ the kernel of $\phi$. McCullough and Peeva prove:

**Proposition 4.3.3.** *The prime ideal $Q$ is generated by the polynomials*

$$\begin{aligned} r_i \ &= \ \sum c_{i,j} y_j \quad (1 \leq i \leq \ell) \\ q_{i,j} \ &= \ y_i y_j - f_i f_j z \quad (1 \leq i \leq j \leq m). \qquad \square \end{aligned}$$

What about higher syzygies of $\mathrm{MP}(I)$? Not being in the local or graded situation, the question isn't completely well-posed. However as the Proposition hints, the idea is that a resolution can be constructed by suitably combining the higher syzygies of $I$ as an $A$-module with the resolution of the ideal $(y_1, \dots, y_m)^2$ over $\mathbf{C}[y_1, \dots, y_m]$.

Turning to the graded setting, the picture is essentially identical except that one is forced to deal initially with polynomial rings having non-standard gradings. Specifically, let $S = \mathbf{C}[x_0, \dots, x_n]$ be a (standard) polynomial ring, and let $I \subseteq S$ be a homogeneous ideal generated by forms $f_i$ with $\deg(f_i) = d_i$. Then

$$\mathrm{MP}(I) \ =_{\mathrm{def}} \ S[I \cdot t, t^2]$$

becomes a graded $S$-algebra (where $\deg(t) = 1$). Just as before we get a graded homomorphism

$$\begin{gathered} \phi \,:\, S[y_1, \dots, y_m, z] \longrightarrow A[I \cdot t, t^2] \\ y_j \mapsto f_j \cdot t \quad , \quad z \mapsto t^2, \end{gathered}$$

where now $S[y,z] = S[y_1,\ldots,y_m,z] = \mathbf{C}[x_0,\ldots,x_n,y_1,\ldots,y_m,z]$ is the polynomial ring with grading

$$\deg(x_k) \ = \ 1 \ \ , \ \ \deg(y_i) \ = \ d_i \ \ , \ \ \deg(z) \ = \ 2.$$

The prime ideal $Q = \ker(\phi) \subseteq S[y,t]$ has the same description as above (with the $c_{i,j}$ arising from the graded analogue of (4.3.4)), and it is homogeneous with respect to the indicated grading. McCullough and Peeva construct in [?, §3] a minimal graded free resolution of $Q$ over $S[y,z]$ in terms of the resolution of $I$ over $S$ and the known resolution of $(y_1,\ldots,y_m)^2$.

We now have a homogeneous prime ideal $Q \subseteq S[y,z]$ whose homological invariants are governed by those of $I$, but $S[y,z]$ is exotically graded. To remedy this, McCullough–Peeva introduce what they call "step-by-step homogenization." In brief, one makes the substitution $y_i \mapsto u_i v_i^{d_i - 1}$ and $z = u_0 v_0$ and pulls $Q$ back to a homogeneous ideal $P \subseteq T$ in the polynomial ring

$$T \ = \ S[u_0,\ldots,u_m,v_0,\ldots,v_m]$$

with the standard grading in which all variables have degree = 1. McCullough and Peeva prove [135, Theorem 1.6]:

**Theorem 4.3.4.** *The ideal $P \subseteq T$ is prime, and*

$$\text{arithreg}_T(P) \ = \ \text{arithreg}_S(I) + 2 + \sum_{i=1}^{m} \deg(f_i).$$

They also compute (among other invariants) the degree of Zeroes$(P)$, which leads to examples showing that the statement of Conjecture 4.2.3 can fail badly for arbitrary reduced and irreducible varieties.

## 4.4 Notes

As we noted in the text, the Eisenbud–Goto regularity conjecture – namely that Conjecture 4.2.3 should hold for arbitrarily singular reduced varieties – generated a vast amount of work in commutative algebra community before the counter-examples of McCullough and Peeva appeared. We refer to their paper [135] for a survey of some of this. In a different direction, the paper [86] discusses some interesting connections between Theorem 4.2.2 and combinatorics.

# Lecture 5

# Koszul Cohomology

In this Lecture we introduce the basic tool used to study individual terms in minimal free resolutions. The idea, which arose in passing in Lecture 1, is that these are computed as the cohomology of an explicit Koszul-type complex. One can then investigate for example the vanishing or non-vanishing of these groups degree by degree. Much of the material in this chapter originally appeared in Green's pioneering paper [88]. We have also drawn on the nice presentation in Chapters 1 - 3 of the notes [7] of Aprodu–Nagel.

Section 5.1 gives the definitions in both algebraic and geometric settings. In §5.2 we introduce vector bundles whose cohomology under favorable circumstances computes Koszul groups. This material is central to everything that follows. The third section is devoted to some general vanishing and non-vanishing theorems for Koszul groups of small weight, as well as a survey of other results concerning their geometry. In §5.4 we take up by way of examples two topics that show the general machinery in action: we discuss at length a theorem of Green on the syzygies of curves of large degree, and we present a theorem of Hochster that computes the syzygies of squaree-free monomial ideals in terms of simplicial homology.

## 5.1 Definitions and first properties

This section introduces Koszul cohomology groups, and establishes their connection with syzygies. We start in a purely algebraic setting before turning to the geometric situation that will be our main concern.

### 5.1.A Algebra

Let $V$ be a vector space over $\mathbf{C}$ of dimension $r+1$, and denote by $S = \mathrm{Sym}(V)$ the symmetric algebra on $V$.[1] Fix a finitely generated graded $S$-module $E = \oplus E_q$. Since $V = S_1$ there are

[1] Everything in this section works without change starting with vector spaces over an arbitrary field.

multiplication maps $V \otimes E_q \longrightarrow E_{q+1}$. Using the embedding $\Lambda^p V \longrightarrow \Lambda^{p-1} V \otimes V$, these in turn give rise to homomorphisms

$$\delta = \delta_{p,q} \, : \, \Lambda^p V \, \otimes \, E_q \longrightarrow \Lambda^{p-1} V \, \otimes_{\mathbf{k}} E_{q+1}.$$

Explicitly,

$$\delta\big( (v_1 \wedge \ldots \wedge v_p) \otimes g \big) \; = \; \sum_{k=1}^{p} (-1)^k \cdot (v_1 \wedge \ldots \wedge \widehat{v_k} \wedge \ldots \wedge v_p) \otimes (v_k \cdot g) \, .$$

One checks as usual that $\delta_{p,q} \circ \delta_{p+1,q-1} = 0$, and so one arrives at the *Koszul complex*:

$$\ldots \longrightarrow \Lambda^{p+1} V \otimes E_{q-1} \longrightarrow \Lambda^p \otimes E_q \longrightarrow \Lambda^{p-1} V \otimes E_{q+1} \longrightarrow \ldots . \tag{5.1.1}$$

**Definition 5.1.1** (**Koszul cohomology**). The cohomology groups of (5.1.1) are the *Koszul cohomology* groups of $E$ (with respect to $V$):

$$K_{p,q}(E) \; = \; K_{p,q}(E; V) \; =_{\text{def}} \; \frac{\ker \, \delta_{p,q}}{\operatorname{Im} \delta_{p+1,q-1}} \, . \qquad \square$$

In the sequel we write

$$Z_{p,q}(E) \; = \; \ker \, \delta_{p,q} \quad , \quad B_{p,q}(E) \; = \; \operatorname{Im} \delta_{p+1,q-1} \tag{5.1.2}$$

for the spaces of Koszul cycles and boundaries respectively.

**Example 5.1.2.** Denote by $\mathfrak{m} = S_+ \subseteq S$ the irrelevant maximal ideal of $S$. Then

$$K_{0,q}(E) \; = \; E_q / \mathfrak{m} E_{q-1}$$

is the vector space of homogeneous minimal generators of $M$ in degree $q$. $\square$

The importance of these vector spaces is that they compute the graded pieces of the minimal resolution of $E$. The following result formalizes an observation made in Lecture 1.

**Theorem 5.1.3** (**Koszul cohomology as syzygies**). *Denote by $\mathbf{k} = S/S_+$ the residue field of $S$ modulo the irrelevant ideal.*[2] *Then there is a canonical isomorphism*

$$K_{p,q}(E) \; = \; \operatorname{Tor}_p^S(\, E \, , \, \mathbf{k} \,)_{p+q} \tag{5.1.3}$$

*of the $(p, q)$ Koszul cohomology group of $E$ with the degree $(p+q)$ component of the indicated* Tor *module. In particular, if*

$$P_\bullet : \quad 0 \longrightarrow P_{r+1} \xrightarrow{\delta_{r+1}} P_r \xrightarrow{\delta_n} \ldots \longrightarrow P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\varepsilon} E \longrightarrow 0$$

*is the minimal free resolution of $M$, then*

$$P_p \; \cong \; \bigoplus_q K_{p,q}(E) \otimes S(-p-q). \tag{5.1.4}$$

[2]Thus $\mathbf{k} \cong \mathbf{C}$, but as before we use this notation to emphasize that we view $\mathbf{k}$ as an $S$-module.

*Proof of Theorem 5.1.3.* We flesh out the argument sketched in Remark 1.3.2. To begin with, tensoring $P_\bullet$ by $\mathbf{k} = S/S_+$ results in a complex with zero differentials. It follows that generators of $P_p$ in each degree are given by the graded pieces of $\mathrm{Tor}_p^S(E, \mathbf{k})$. So it remains to verify (5.1.3). For this consider the graded Koszul resolution $K_\bullet$ of $\mathbf{k}$:

$$0 \longrightarrow \Lambda^{r+1}V \otimes_{\mathbf{C}} S(-r-1) \longrightarrow \Lambda^r V \otimes_{\mathbf{C}} S(-r) \longrightarrow \ldots \longrightarrow V \otimes_{\mathbf{C}} S(-1) \longrightarrow S \longrightarrow \mathbf{k} \longrightarrow 0. \tag{5.1.5}$$

Then

$$\mathrm{Tor}_p^S(E, \mathbf{k}) \ = \ H_p\big(K_\bullet \otimes_S E\big)$$

thanks to the symmetry of Tor. But

$$\big(K_p \otimes_S E\big)_{p+q} \ = \ \big(\Lambda^p V \otimes E(-p)\big)_{p+q} \ = \ \Lambda^p V \otimes E_q.$$

Thus

$$H_p\big(K_\bullet \otimes_S E\big)_{p+q} \ = \ K_{p,q}(E)$$

by definition of the Koszul cohomology groups, as required. □

Here are some examples outlining various properties of these groups.

**Example 5.1.4 (Maps in the resolution).** Under the identification (5.1.4), the boundary maps $\delta_p : P_p \longrightarrow P_{p-1}$ are given by homorphisms

$$\mu_{p,q}^e : \ K_{p,q}(E) \longrightarrow K_{p-1,q-e}(E) \otimes \mathrm{Sym}^{e+1}(V)$$

for each $e \geq 0$, but these are only partially intrinsic. The linear piece (corresponding to $e = 1$) comes from the mapping

$$\Lambda^p V \otimes E_q \longrightarrow \Lambda^{p-1}V \otimes V \otimes E_q,$$

determining a natural homomorphism $Z_{p,q}(E) \longrightarrow Z_{p-1,q}(E) \otimes V$. In general, the reader may check that $\mu_{p,q}^e$ is canonically defined on the kernel of $d_{p,q}^{e-1}$. □

**Example 5.1.5 (Exact sequences).** Let

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

be a short exact sequence of finitely generated graded $S$-modules. Then the corresponding Koszul groups sit in a long exact sequence

$$\ldots \longrightarrow K_{p+1,q-1}(E'') \longrightarrow K_{p,q}(E') \longrightarrow K_{p,q}(E) \longrightarrow K_{p,q}(E'') \longrightarrow K_{p-1,q+1}(E') \longrightarrow \ldots.$$

(This is the long exact sequence of Tor.) □

**Example 5.1.6** (**Multiplication**)**.** Let $E$ be a finitely generated graded $S$-module, and consider a linear form $h \in S_1$. Then the homomorphism

$$K_{p,q-1}\big(E\big) = K_{p,q}\big(E(-1)\big) \longrightarrow K_{p,q}\big(E\big).$$

arising from multiplication by $h$ is the zero mapping. The same holds for multiplication by $f \in S_m$. (Suppose

$$z \;=\; \sum\nolimits_I \alpha_I \otimes g_I \;\in\; \Lambda^p V \otimes E_{q-1}$$

is a $(p, q-1)$-Koszul cycle for $E$. Then $h \cdot z = \sum \alpha_I \otimes (h \cdot g_I) = \delta(w)$, where

$$w \;=\; \sum\nolimits_I (h \wedge \alpha_I) \otimes g_I \;\in\; \Lambda^{p+1} V \otimes E_{q-1}.$$

The statement for multiplication by $f$ follows upon applying the degree one case term by term.) □

**Example 5.1.7** (**Restrictions, I**)**.** Let $E$ be a finitely generated $S$-module, and let $h \in V = S_1$ be a linear form. Assume that $h$ is a non-zerodivisor for $E$, and put $E' = E/\big(h \cdot E(-1)\big)$. Write $S' = \mathrm{Sym}(V/\mathbf{C} \cdot h)$, and denote by $K'_{p,q}(E')$ the Koszul-cohomology groups of $E'$ considered as an $S'$-module. Then

$$K_{p,q}(E) \;\cong\; K'_{p,q}(E') \quad \text{for every } \; p \text{ and } q.$$

(Examples 5.1.5 and 5.1.6 imply that the Koszul groups $K_{p,q}(E')$ of $E'$ as a module over $S$ sit in exact sequences

$$0 \longrightarrow K_{p,q}(E) \longrightarrow K_{p,q}(E') \longrightarrow K_{p-1,q+1}(E) \longrightarrow 0.$$

On the other hand write $V = U \oplus W$ where $U = \mathbf{C} \cdot h$ and $W$ maps isomorphically to $V/\mathbf{C} \cdot h$. Since $U$ acts trivially on $E'$, one sees that this splitting of $V$ determines a splitting of the Koszul complex computing the Koszul cohomology of $E'$ over $S$. This gives rise to a splitting

$$K_{p,q}(E') \;\cong\; K'_{p,q}(E') \,\oplus\, K'_{p-1,q+1}(E'),$$

and the assertion follows with an induction.) □

**Example 5.1.8** (**Products**)**.** If $E_1$ and $E_2$ are graded $S$-modules, then there are product maps

$$K_{p_1,q_1}(E_1) \,\otimes\, K_{p_2,q_2}(E_2) \longrightarrow K_{p_1+p_2,q_1+q_2}(E_1 \otimes E_2).$$

These and other cup products are studied in [88, §1.c]. □

**Example 5.1.9** (**First syzygies as Koszul classes**)**.** It is sometimes useful to make explicit the identification of Theorem 5.1.3. We outline here how this goes in the first non-trivial case. Consider then homogeneous minimal generators

$$m_1 \,, \,\dots\,, m_k \,\in\, E,$$

say with $m_i \in E_{a_i}$. Suppose that $g = (g_1, \ldots, g_k)$ is the coefficient vector of a minimal syzygy of degree $b$ among the $m_i$, so that $g_i$ is a homogeneous polynomial of degree $b - a_i$ and

$$\sum g_i \cdot m_i \ = \ 0. \tag{*}$$

We wish to represent this syzygy as a class in $K_{1,b-1}(E)$. To this end fix a basis $z_0, \ldots, z_r \in V$, and choose polynomials $h_{j,i}$ of degree $b - a_i - 1$ with the property that

$$g_i \ = \ z_0 \cdot h_{0,i} \ + \ \ldots \ + \ z_r \cdot h_{r,i}. \tag{**}$$

We may view the vector

$$\gamma \ = \ \begin{pmatrix} h_{0,1} \cdot m_1 + \ldots + h_{0,k} \cdot m_k \\ \vdots \\ h_{r,1} \cdot m_1 + \ldots + h_{r,k} \cdot m_k \end{pmatrix}$$

as an element of $V \otimes M_{b-1}$. It follows from (*) and (**) that $\gamma$ is a cycle for the Koszul complex, and the reader may check that it represents the syzygy in question. □

**Remark 5.1.10** (**Higher syzygies as Koszul classes**)**.** In principle one can compute Koszul representatives of coefficient vectors of higher syzygies by considering the double complex $P_\bullet \otimes K_\bullet$. Cohomology classes of the associated total complex are represented by "zig-zags" whose outer terms lead to the required identification. We refer to [129, §3] for an application of this procedure in the local setting. Herzog finds some explicit formulae in his paper [109]. □

## 5.1.B Geometry

We now turn to a geometric setting. Let $X$ be an irreducible complex projective variety of dimension $n$. Let $L$ be an ample and globally generated line bundle on $X$, and fix a subspace

$$V \ \subseteq \ H^0\big(X, L\big)$$

of dimension $r + 1$ that generates $L$. (In practice we will often take $V = H^0\big(X, \mathcal{O}_X(L)\big)$. These data determine a finite morphism

$$\phi \ = \ \phi_V \ : \ X \longrightarrow \mathbf{P}(V).$$

As above we write $S = \mathrm{Sym}(V)$ for the homogeneous coordinate ring of $\mathbf{P}(V)$.

Now fix an arbitrary coherent sheaf $\mathcal{F}$ on $X$, and – keeping the convention of §1.3.C – set

$$E \ = \ E_{\mathcal{F}} \ = \ \bigoplus_{m \gg \infty} H^0\big(X, \mathcal{F} \otimes L^{\otimes m}\big).^3$$

[3] Recall that if $\mathcal{F}$ has no zero-dimensional embedded primes, then the sum ranges over all $m \in \mathbf{Z}$. In the contraray case one fixes $m_0$ so that $h^0\big(X, \mathcal{F}(mL)\big)$ is constant for $m \leq m_0$ and then as in (1.3.4) one takes the sum over $m \geq m_0$.

This is a finitely generated graded $S$-module via the maps

$$V \otimes H^0\big(\mathcal{F} \otimes L^{\otimes m}\big) \longrightarrow H^0\big(\mathcal{F} \otimes L^{\otimes m+1}\big).$$

If $P_\bullet$ is the minimal graded free resolution of $E_{\mathcal{F}}$, then then the sheafification $\mathcal{P}_\bullet$ of $P_\bullet$ is a locally free resolution of the sheaf $\phi_*(\mathcal{F})$ on $\mathbf{P}(V)$.

**Definition 5.1.11** (**Koszul cohomology of a coherent sheaf**). The Koszul cohomology groups of $\mathcal{F}$ with respect to $V$ are defined to be

$$K_{p,q}(X, \mathcal{F}; V) \ = \ K_{p,q}(E_{\mathcal{F}}; V). \tag{5.1.6}$$

When $V = H^0\big(X, L\big)$ we write $K_{p,q}(X, \mathcal{F}; L)$, or – when $X$ and $L$ are understood – simply $K_{p,q}(\mathcal{F})$. We often take $\mathcal{F} = \mathcal{O}_X$, in which case we write $K_{p,q}(X; L)$ or $K_{p,q}(X; V)$. □

Explicitly, then, $K_{p,q}(X, \mathcal{F}; V)$ is the cohomology of the complex

$$\Lambda^{p+1}V \otimes H^0\big(\mathcal{F} \otimes L^{\otimes(q-1)}\big) \longrightarrow \Lambda^p V \otimes H^0\big(\mathcal{F} \otimes L^{\otimes q}\big) \longrightarrow \Lambda^{p-1}V \otimes H^0\big(\mathcal{F} \otimes L^{\otimes(q+1)}\big) \tag{5.1.7}$$

(at least when $q \gg -\infty$). As above, we denote by

$$Z_{p,q}(X, \mathcal{F}; V) \quad \text{and} \quad B_{p,q}(X, \mathcal{F}; V)$$

the subgroups of $\Lambda^p V \otimes H^0\big(\mathcal{F} \otimes L^{\otimes q}\big)$ consisting of Koszul cycles and boundaries respectively.

**Remark 5.1.12** (**Notation**). Note that the groups $K_{p,q}(X, \mathcal{F}; V)$ depend critically on the line bundle $L$ with $V \subseteq H^0\big(X, L\big)$. However in the interests of lighter notation, we allow it to remain implicit in (5.1.6).

**Example 5.1.13** (**Index shifting**). It follows from the definitions that

$$K_{p,q}(X, \mathcal{F}; V) \ = \ K_{p,q-1}(X, \mathcal{F} \otimes L; V). \qquad \square$$

**Example 5.1.14** (**Linearly normal embeddings**). Let $L$ be a very ample line bundle on $X$, and consider the embedding

$$X \ \subseteq \ \mathbf{P}H^0(L) \ = \ \mathbf{P}^{r(L)}$$

defined by the complete linear series $|L|$, so that $r = r(L) = h^0(L) - 1$. We will often be interested in the groups $K_{p,q}(X; L)$ governing the resolution of $\mathcal{O}_X$ as an $\mathcal{O}_{\mathbf{P}^r}$-module. Here $K_{0,1}(X; L) = 0$ since we are dealing with a linearly normal embedding, and

$$K_{0,q}\big(X; L\big) \ = \ 0 \ \text{ for all } q \geq 2 \quad \Longleftrightarrow \quad L \ \text{ is normally generated,}$$

meaning that the maps

$$\mathrm{Sym}^m H^0\big(X, L\big) \longrightarrow H^0\big(X, L^{\otimes m}\big)$$

are surjective for all $m \geq 1$. When this is satisfied, the remaining groups $K_{p,q}(X; L)$ control the resolution of the homogeneous ideal $I_X$ of $X$: for instance, $K_{1,q}(X; L)$ is the space of minimal generators of $I_X$ in degree $q + 1$, etc. □

**Example 5.1.15** (**Very positive embeddings**). In the situation of the previous example, suppose that $\dim X = n$ and that $L$ is sufficiently positive so that

$$H^i\big(X, L^{\otimes m}\big) \ = 0 \quad \text{for all} \ \ i\,,\, m \ > \ 0. \tag{5.1.8}$$

Then $K_{p,q}\big(X; L\big) = 0$ for all $q \geq n+2$ and all $p$. If in addition $H^n\big(X, \mathcal{O}_X\big) = 0$, then $K_{p,n+1}(X; L) = 0$ for every $p$. (The hypothesis (5.1.8) implies that $\mathcal{O}_X$ is $(n+1)$-regular viewed as a sheaf on $\mathbf{P}H^0(L)$. Hence the assertion follows from Theorem 3.1.8. If moreover $H^n\big(X, \mathcal{O}_X\big) = 0$, then $\mathcal{O}_X$ is $n$-regular. Alternatively (but equivalently), one argue directly by studying twists of the Koszul complex determined by $H^0(L) \otimes_{\mathbf{C}} \mathcal{O}_X \longrightarrow L$.)

**Example 5.1.16** (**A twisted quartic curve**). One can sometimes compute dimensions of Koszul cohomology groups by directly finding the shape of a resolution. For example, let $X = \mathbf{P}^1$, $L = \mathcal{O}_{\mathbf{P}^1}(4 \cdot \text{point})$ the line bundle of degree 4 on $X$, and consider the subspace

$$V \ = \ \big\langle \ s^4\,,\, s^3t\,,\, st^3\,,\, t^4 \,\big\rangle \ \subseteq \ H^0\big(\mathbf{P}^1, L\big),$$

defining an embedding of $\mathbf{P}^1$ as a twisted quartic $C \subseteq \mathbf{P}^3$. One convinces oneself that the resolution of $\mathcal{O}_C$ over $\mathcal{O}_{\mathbf{P}^3}$ has the shape

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^3}(-3)^3 \longrightarrow \mathcal{O}_{\mathbf{P}^3}(-2)^5 \longrightarrow \mathcal{O}_{\mathbf{P}^3}(-1) \oplus \mathcal{O}_{\mathbf{P}^3} \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

So $K_{0,0}(\mathbf{P}^1; V) = K_{0,1}(\mathbf{P}_1; V) = \mathbf{C}$, whereas $K_{1,1}(\mathbf{P}^1; V)$ and $K_{2,1}(\mathbf{P}^1; V)$ have dimension 5 and 3 respectively. We will recompute these groups more mechanically in Example 5.2.6 below. Note that the failure of $C \subseteq \mathbf{P}^3$ to be projectively normal means that these Koszul cohomology groups do not give the resolution of the homogeneous ideal of $C$. □

**Example 5.1.17** (**Restrictions, II**). Keeping notation as before, let $B$ be a line (or vector) bundle on the irreducible variety $X$, and assume that

$$H^1\big(X, B \otimes L^{\otimes m}\big) \ = \ 0 \quad \text{for all } m \in \mathbf{Z}. \tag{*}$$

Fix a section $h \in V \subseteq H^0\big(X, L\big)$ cutting out a divisor $Y \subseteq X$, and write

$$L_Y = L \mid Y \quad , \quad B_Y = B \mid Y \quad , \quad W \ = \ V/ < h >$$

for the restrictions of the given data to $Y$. Then

$$K_{p,q}\big(X, B; V\big) \ = \ K_{p,q}\big(Y, B_Y; V_Y\big)$$

for every $p$ and $q$. (Write $R(X, B) = \oplus H^0\big(X, L^{\otimes m}\big)$ for the graded module over $S = \mathrm{Sym} V$ determined by $B$, and similarly for $R(Y, B_Y)$. The hypothesis (*) yields an exact sequence

$$0 \longrightarrow R(X, B)(-1) \xrightarrow{\cdot s} R(X, B) \longrightarrow R(Y, B_Y) \longrightarrow 0$$

of graded $S$-modules, and then the assertion follows from Example 5.1.7. □

Finally, let us recall some facts about duality. Assume that $X$ is smooth of dimension $n$, and as before let $L$ be an ample line bundle on $X$, and $V \subseteq H^0(L)$ a generating subspace of dimension $r+1$ defining a finite map $\phi : X \longrightarrow \mathbf{P}(V)$. Now consider a locally free sheaf $B$ on $X$ with the property that

$$H^i\big(X, B \otimes L^{\otimes m}\big) \;=\; 0 \tag{5.1.9}$$

for all $0 < i < n$ and all $m \in \mathbf{Z}$. In this case $E_B = \oplus H^0\big(B \otimes L^{\otimes m}\big)$ is Cohen-Macaulay as a module over $S = \operatorname{Sym} V$, and hence has a graded free resolution

$$0 \longrightarrow P_{r-n} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow E_B \longrightarrow 0.$$

Moreover, $\operatorname{Ext}^i_S(E_B, S(-r-1)) = 0$ for $0 \leq i < r-n$, while

$$\operatorname{Ext}^{r-n}_S\big(E_B, S(-r-1)\big) \;=\; E_{\omega_X \otimes B^*}$$

is the graded module corresponding to $\omega_X \otimes B^*$, where $\omega_X$ is the canonical bundle of $X$ (cf [87]). In other words, $E_{\omega_X \otimes B^*}$ has the minimal graded free resolution

$$0 \longrightarrow P_0^\vee(-r-1) \longrightarrow P_1^\vee(-r-1) \longrightarrow \ldots\ldots P_{r-n}^\vee(-r-1) \longrightarrow E_{\omega_X \otimes B^*} \longrightarrow 0,$$

where we write $P^\vee = \operatorname{Hom}(P, S)$ for a free module $P$. Applying Theorem 5.1.3 to both $E_B$ and $E_{\omega_X \otimes B^*}$, this yields the following result, which was established by Green [88]:

**Theorem 5.1.18** (**Duality, I**). *Let $B$ be a locally free sheaf on the smooth $n$-dimensional variety $X$ that satisfies the vanishings* (5.1.9). *Then*

$$K_{p,q}(X, B; V) \;\cong\; K_{r-n-p,n+1-q}(X, \omega_X \otimes B^*; V)^*. \qquad \square$$

We will derive a slightly sharper version of this statement (Theorem 5.2.11) in the next section using classical Serre duality for vector bundles.

**Remark 5.1.19** (**Explicit duality**). Green [88] makes explicit the duality in the previous theorem. Specifically, continuing to assume the vanishings (5.1.9), and assuming for simplicity that one is working over $\mathbf{C}$, Green observes that

$$K_{r-n,n+1}\big(X, \omega_X; V\big) \;=\; \mathbf{C},$$

and he writes down an explicit generator for this group. Then the duality in (5.1.18) arises from the cup product noted in Example 5.1.8. $\square$

## 5.2 Kernel bundles

In this section we discuss a construction that allows one, under mild hypotheses, to compute Koszul groups as the cohomology of a vector bundle on a variety $X$. This opens the door to using the geometry of this bundle to study syzygies, which has turned out to be quite useful.

As in the previous section, let $X$ be an irreducible complex projective variety, let $L$ be an ample line bundle on $X$, and let $V \subseteq H^0\big(X, L\big)$ be an $(r+1)$-dimensional subspace of sections that generates $L$. Writing

$$V_X \;=\; V \otimes_{\mathbf{C}} L$$

for the trivial vector bundle on $X$ modeled on $V$, there is a natural surjective map

$$\mathrm{ev} \;=\; \mathrm{ev}_{V,L} \,:\, V_X \longrightarrow L$$

of vector bundles given by evaluation.

**Definition 5.2.1** (**Kernel bundle**)**.** The *kernel bundle* $M = M_V$ of $L$ with respect to $V$ is the kernel of the evaluation map $\mathrm{ev}_{V,L}$. Thus $M_V$ sits in the exact sequence:

$$0 \longrightarrow M_V \longrightarrow V_X \longrightarrow L \longrightarrow 0. \tag{5.2.1}$$

When $V = H^0\big(X, L\big)$ is the full space of sections of $L$, we denote this bundle by $M_L$. □

Thus $M_V$ is a vector bundle on $X$ with

$$\mathrm{rank}(M_V) \;=\; r \quad , \quad \det M_V \;=\; L^*.$$

One can also view $M_V$ as a twisted pullback of the cotangent bundle of projective space under the morphism

$$\phi = \phi_V : X \longrightarrow \mathbf{P}(V)$$

defined by $V$. Specifically, $M_V = \phi^*\big(\Omega^1_{\mathbf{P}(V)}(1)\big)$, and (5.2.1) pulls back from (a twist of) the Euler sequence on $\mathbf{P}(V)$. When the context is clear, we often simply write $M$ for the kernel bundle in question.

**Example 5.2.2** (**Alternate description**)**.** Let $\Delta \subseteq X \times X$ be the diagonal, with ideal sheaf $\mathcal{I}_\Delta \subseteq \mathcal{O}_{X\times X}$. Then

$$M_L \;=\; \mathrm{pr}_{1,*}\left(\mathcal{I}_\Delta \otimes \mathrm{pr}_2^* L\right),$$

and (5.2.1) arises as the pushforward under the first projection of the short exact sequence

$$0 \longrightarrow \mathcal{I}_\Delta \otimes \mathrm{pr}_2^* L \longrightarrow \mathrm{pr}_2^* L \longrightarrow \mathrm{pr}_2^* L \otimes \mathcal{O}_\Delta \longrightarrow 0$$

of sheaves on $X \times X$. □

**Example 5.2.3** (**Kernel bundles on $\mathbf{P}^1$**)**.** By a well-known theorem of Grothendieck, any vector bundle on $\mathbf{P}^1$ splits as a direct sum of line bundles. It is often quite easy to determine the splitting type of a kernel bundle.

(i). If $L = \mathcal{O}_{\mathbf{P}^1}(d)$, then $M_L \cong \mathcal{O}_{\mathbf{P}^1}^d(-1)$.

(ii). If $V = \langle s^4, s^3t, st^3, t^4\rangle \subseteq H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(4)\big)$ is the subspace defining the twisted quartic curve $C \subseteq \mathbf{P}^3$ from Example 5.1.16, then

$$M_V \;\cong\; \mathcal{O}_{\mathbf{P}^1}^2(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-2).$$

(iii). Let $T = \mathbf{C}[s,t]$ be the homogeneous coordinate ring of $\mathbf{P}^1$. A subspace

$$V \subseteq H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d)\big)$$

gives rise to a homomorphism $f : V \otimes_{\mathbf{C}} T \longrightarrow T(d)$, and $M_V$ is the sheafification of $\ker f$. In other words, the splitting type of $M_V$ encodes the number of minimal relations in each degree among the polynomials appearing in $V$. □

**Example 5.2.4** (**Canonical kernel bundle on a curve**)**.** Let $C$ be a smooth projective curve of genus $g \geq 2$, and denote by $\omega_C$ the canonical bundle on $C$. Then the corresponding kernel bundle $M_{\omega_C}$ is identified with the conormal bundle to $C$ in its Jacobian under the Abel–Jacobi embedding $C \subseteq \mathrm{Jac}(C)$. These are very interesting bundles: see [126, §1.6.1], as well as the proof of the theorems of Noether and Petri in §7.1.D below. □

Under suitable hypotheses, kernel bundles compute the Koszul cohomology groups associated to $L$.

**Theorem 5.2.5** (**Koszul cohomology via kernel bundles**)**.** *Let $\mathcal{F}$ be a coherent sheaf on $X$, and $V \subseteq H^0\big(X, L\big)$ a generating subspace. Assume that*

$$H^1\big(X, \mathcal{F} \otimes L^{\otimes(q-1)})\big) \;=\; 0.$$

*Then*

$$K_{p,q}\big(X, \mathcal{F}; V\big) \;=\; H^1\Big(\, X\,,\, \Lambda^{p+1} M_V \otimes \mathcal{F} \otimes L^{\otimes(q-1)}\,\Big).$$

**Example 5.2.6** (**The twisted quartic revisited**)**.** As an illustration, let us return to the twisted quartic $C \subseteq \mathbf{P}^3$ from Examples 5.1.16 and 5.2.3 (ii). Here $L = \mathcal{O}_{\mathbf{P}^1}(4)$ and $\mathcal{F} = \mathcal{O}_{\mathbf{P}^1}$, so the hypothesis of the theorem is satisfied as soon as $q \geq 1$. Since $M = \mathcal{O}_{\mathbf{P}^1}^2(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-2)$, we see that

$$h^1\big(\mathbf{P}^1, M_V\big) \;=\; 1 \quad , \quad h^1\big(\mathbf{P}^1, \Lambda^2 M_V\big) \;=\; 5 \quad , \quad h^1\big(\mathbf{P}^1, \Lambda^3 M_V\big) \;=\; 3.$$

Thus we recover the dimensions of the groups $K_{p,1}$ appearing in 5.1.16. □

*Proof of Theorem 5.2.5.* Consider to begin with the Koszul complex arising from the surjection $V_X \otimes L^* \longrightarrow \mathcal{O}_X$. This takes the form of a long exact sequence

$$\ldots \longrightarrow \Lambda^2 V_X \otimes L^{\otimes -2} \longrightarrow V_X \otimes L^* \longrightarrow \mathcal{O}_X \longrightarrow 0$$

of bundles on $X$. Tensoring through by $\mathcal{F} \otimes L^{\otimes(p+q)}$, we arrive at a long exact sequence of sheaves that includes:

$$\ldots \longrightarrow \Lambda^{p+1} V_X \otimes \mathcal{F} \otimes L^{\otimes(q+1)} \longrightarrow \Lambda^p V_X \otimes \mathcal{F} \otimes L^{\otimes q} \longrightarrow \Lambda^{p-1} V_X \otimes \mathcal{F} \otimes L^{\otimes(q-1)} \longrightarrow \ldots \;. \quad (5.2.2)$$

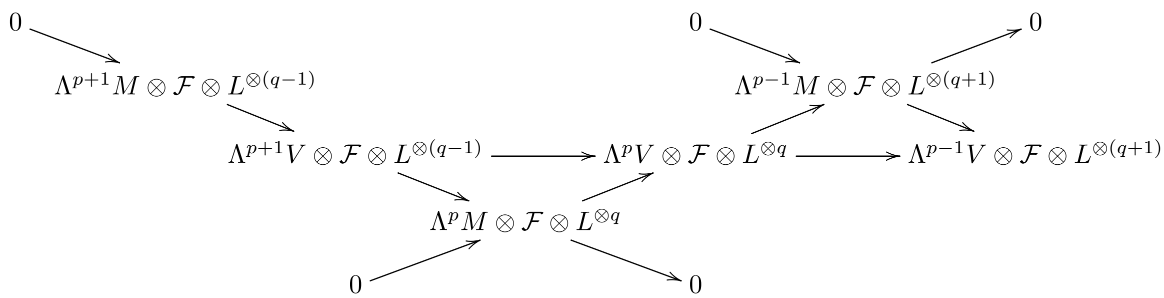



Figure 5.1: Diagram for Proof of Theorem 5.2.5

The complex (5.1.7) of vector spaces computing $K_{p,q}(X, \mathcal{F}; V)$ is obtained by taking global sections of each term in (5.2.2). On the other hand, we can split (5.2.2) into short exact sequences of sheaves arising from the sequence (5.2.1) defining $M_V$. Specifically, taking exterior products in (5.2.1) gives rise to

$$0 \longrightarrow \Lambda^p M_V \longrightarrow \Lambda^p V_X \longrightarrow \Lambda^{p-1} M_V \otimes L \longrightarrow 0. \tag{5.2.3}$$

Twists of these then fit together in an exact commutative diagram of sheaves shown in Figure 5.1. Looking at the upward-pointing diagonal sequence in that Figure, one finds first of all that the group $Z_{p,q}(X, \mathcal{F}; V) \subseteq \Lambda^p V \otimes H^0\big(\mathcal{F} \otimes L^{\otimes q}x\big)$ of Koszul cycles is computed as

$$Z_{p,q}(X, \mathcal{F}; V) \;=\; H^0\big(X, \Lambda^p M_V \otimes \mathcal{F} \otimes L^{\otimes q}\big).$$

On the other hand, assuming that $H^1\big(\mathcal{F} \otimes L^{\otimes(q-1)}\big) = 0$, the downward-pointing diagonal on the left of the diagram leads to the exact sequence

$$\Lambda^{p+1} V \otimes H^0\big(\mathcal{F} \otimes L^{\otimes(q-1)}\big) \longrightarrow H^0\big(\Lambda^p M_V \otimes \mathcal{F} \otimes L^{\otimes q}\big) \longrightarrow H^1\big(\Lambda^{p+1} M_V \otimes \mathcal{F} \otimes L^{\otimes(q-1)}\big) \longrightarrow 0 \tag{5.2.4}$$

on cohomology. The Theorem follows. □

**Remark 5.2.7** (**Computation without vanishing**)**.** Even in the absence of any vanishing, this computation shows that $K_{p,q}(X, \mathcal{F}; V)$ is the cokernel of the map

$$\Lambda^{p+1} V \otimes H^0\big(\mathcal{F} \otimes L^{\otimes(q-1)}\big) \longrightarrow H^0\big(\Lambda^p M_V \otimes \mathcal{F} \otimes L^{\otimes q}\big)$$

appearing in (5.2.4). □

**Remark 5.2.8** (**Koszul cycles**)**.** For later reference, we repeat from the proof just completed that, independent of any vanishing hypotheses, the group of $(p, q)$-Koszul cycles is given by

$$Z_{p,q}(X, \mathcal{F}; V) \;=\; H^0\big(X, \Lambda^p M_V \otimes \mathcal{F} \otimes L^{\otimes q}\big). \qquad \square$$

Assuming some additional vanishings, one can alternatively compute $K_{p,q}$'s via higher cohomology:

**Proposition 5.2.9.** *In the situation of Theorem 5.2.5, assume in addition that*

$$H^1\big(\mathcal{F}\otimes L^{\otimes(q-2)}\big) \;=\; H^2\big(\mathcal{F}\otimes L^{\otimes(q-3)}\big) \;=\; \dots \;=\; H^{k-1}\big(\mathcal{F}\otimes L^{\otimes(q-k)}\big) \;=\; 0$$

$$H^2\big(\mathcal{F}\otimes L^{\otimes(q-2)}\big) \;=\; H^3\big(\mathcal{F}\otimes L^{\otimes(q-3)}\big) \;=\; \dots \;=\; H^{k}\big(\mathcal{F}\otimes L^{\otimes(q-k)}\big) \;=\; 0.$$

*Then*

$$K_{p,q}\big(X,\mathcal{F};V\big) \;=\; H^k\big(X,\Lambda^{p+k}M_V\otimes\mathcal{F}\otimes L^{\otimes(q-k)}\big).$$

*Proof.* In fact, using the exact sequences (5.2.3), the stated vanishings imply that

$$H^1\big(\Lambda^{p+1}M_V\otimes\mathcal{F}\otimes L^{\otimes(q-1)}\big)=H^2\big(\Lambda^{p+2}M_V\otimes\mathcal{F}\otimes L^{\otimes(q-2)}\big)=\dots=H^k\big(\Lambda^{p+k}M_V\otimes\mathcal{F}\otimes L^{\otimes(q-k)}\big).$$

Thus the assertion follows from 5.2.5. □

**Example 5.2.10 (Koszul cohomology on $\mathbf{P}^1$).** Let $X=\mathbf{P}^1$, let $L=\mathcal{O}_{\mathbf{P}^1}(d)$ be the line bundle of degree $d$, and let $B=\mathcal{O}_{\mathbf{P}^1}(b)$ with $0\le b\le d-1$. Then

$$\begin{aligned} K_{p,0}\big(\mathbf{P}^1,B;L\big) &\neq 0 \quad \text{when} \;\; 0\le p\le b,\\ K_{p,1}\big(\mathbf{P}^1,B;L\big) &\neq 0 \quad \text{when} \;\; b+1\le p\le d-1,\end{aligned}$$

and all other Koszul groups vanish. □

As an application, we use classical Serre duality to give a slight strengthening of Theorem 5.1.18. This statement appears as Theorem 2.c.6 in [88].

**Theorem 5.2.11 (Duality, II).** *Let $X$ be a smooth projective variety of dimension $n\ge 2$, and let $B$ be a locally free sheaf on $X$. As above let $L$ be an ample line bundle on $X$, and $V\subseteq H^0\big(X,L\big)$ a generating subspace of dimension $r+1$. Assume that*

$$\begin{aligned} H^1\big(B\otimes L^{\otimes(q-1)}\big) \;&=\; \dots \;=\; H^{n-1}\big(B\otimes L^{\otimes(q-n+1)}\big) \;=\; 0\\ H^1\big(B\otimes L^{\otimes(q-2)}\big) \;&=\; \dots \;=\; H^{n-1}\big(B\otimes L^{\otimes(q-n)}\big) \;=\; 0.\end{aligned}$$

*Then there is an isomorphism*

$$K_{p,q}\big(X,B;V\big) \;\cong\; K_{r-n-p,n+1-q}\big(X,\omega_X\otimes B^*\big)^*,$$

*where $\omega_X$ is the canonical bundle on $X$.*

**Remark 5.2.12 (Duality on curves).** Keeping in mind Remark 5.2.7, the argument that follows will show that if $n=1$, then the conclusion of the Theorem holds without assuming any vanishings. (Compare also Theorem 5.1.18.) □

*Proof of Theorem 5.2.11.* For compactness, write $M_V = M$. To begin with Proposition 5.2.9 applies to yield

$$K_{p,q}(X, B; V) \;=\; H^{n-1}\Big(X, \Lambda^{p+n-1}M \otimes B \otimes L^{\otimes(q-n+1)}\Big).$$

This $H^{n-1}$ is Serre dual to

$$H^1\Big(X, \Lambda^{p+n-1}M^* \otimes \omega_X \otimes B^* \otimes L^{\otimes(n-q-1)}\Big).$$

On the other hand, recalling that $\text{rank} M = r$ and $\det M = L^*$, one has that

$$\Lambda^{p+n-1}M^* \;=\; \Lambda^{r+1-p-n}M \otimes L$$

Thus $K_{p,q}(X, B; V)$ is dual to

$$H^1\Big(X, \Lambda^{r+1-p-n}M \otimes \omega_X \otimes B^* \otimes L^{\otimes(n-q)}\Big). \tag{*}$$

But $H^1\big(\omega_X \otimes B^* \otimes L^{\otimes(n-q)}\big) = 0$ thanks to the vanishing of $H^{n-1}\big(B \otimes L^{\otimes(q-n)}\big)$ in the hypothesis (which we haven't used up to now), so Theorem 5.2.5 shows that the group in (*) computes

$$K_{r-p-n,n+1-q}\big(X, \omega_X \otimes B^*; V\big),$$

as asserted. □

# 5.3 Syzygies of weights zero and one

We present here some additional information about the Koszul groups $K_{p,0}$ and $K_{p,1}$. The first subsection sketches some useful vanishing and non-vanishing theorems. In the second, we survey without proof some further constructions and results involving these syzygies.

## 5.3.A Some vanishing and non-vanishing theorems

Consider an irreducible projective variety $X$, and as above let $V \subseteq H^0\big(X, L\big)$ be a generating subspace. The following result of Green gives a useful criterion for the vanishing of some Koszul groups on $X$.

**Theorem 5.3.1** (**Green's vanishing theorem**). *Let $\mathcal{F}$ a torsion-free sheaf on $X$, and let $p$ be an integer with*

$$p \;\geq\; h^0\big(X, \mathcal{F}\big).$$

*Then $K_{p,0}\big(X, \mathcal{F}; V\big) = 0$.*

**Remark 5.3.2.** Recall (Example 5.1.13) that $K_{p,q}\big(X,\mathcal{F};V\big) = K_{p,0}\big(X,\mathcal{F}\otimes L^{\otimes q};V\big)$. Hence the theorem also leads to a vanishing criterion for $K_{p,q}$: if $p \geq h^0\big(X,\mathcal{F}\otimes L^{\otimes q}\big)$, then $K_{p,q}(X,\mathcal{F};V) = 0$. □

*Proof of Theorem 5.3.1.* We will show that in fact $Z_{p,0}\big(X,\mathcal{F};V\big) = 0$, for which we use a simple instance of construction that will come up on several occasions.

Specifically, fix a point $x \in X$. The image of $x$ in the projective space $\mathbf{P}(V)$ under the map $\phi : X \longrightarrow \mathbf{P}(V)$ corresponds to a one-dimensional quotient $V \twoheadrightarrow W_x$ of $V$. It gives rise to a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_V & \longrightarrow & V\otimes_{\mathbf{C}}\mathcal{O}_X & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \rho_x} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_x & \longrightarrow & W_x\otimes_{\mathbf{C}}\mathcal{O}_X & \longrightarrow & L\otimes\mathcal{O}_{\{x\}} & \longrightarrow & 0, \end{array}$$

defining $\rho = \rho_x : M_V \longrightarrow \mathcal{I}_x$, where $\mathcal{I}_x = \mathcal{I}_{x/X}$ is the ideal sheaf of $\{x\}$ in $X$. Supposing that $\dim V = r+1$, choose general smooth points $x_0,\ldots,x_r \in X$ whose images span $\mathbf{P}(V)$. Then the maps $\rho_{x_i}$ assemble to give an injective homomorphism

$$\rho : M_V \longrightarrow \mathcal{I}_{x_0}\oplus\ldots\oplus\mathcal{I}_{x_r}$$

of sheaves on $X$. This in turn determines for every $p > 0$ an injection

$$0 \longrightarrow \Lambda^p M_V \longrightarrow \bigoplus_{\#J=p}\mathcal{I}_{Z_J},$$

where $Z_J \subseteq X$ denotes the $p$-element subset of $X$ consisting of the points $x_j$ with $j \in J$ for $J \subseteq [0,r]$ and $\#J = p$. Assuming that we've chosen the $x_j$ generally enough so that $\mathcal{F}$ is locally free near each of them, tensoring by $\mathcal{F}$ yields an inclusion

$$0 \longrightarrow \Lambda^p M_V\otimes\mathcal{F} \longrightarrow \bigoplus_{\#J=p}\mathcal{F}\otimes\mathcal{I}_{Z_J}.$$

In particular, recalling that $Z_{p,0}\big(X,\mathcal{F};V\big) = H^0\big(X,M_V\otimes\mathcal{F}\big)$, it follows that

$$Z_{p,0}\big(X,\mathcal{F};V\big) \ \subseteq\ \bigoplus_{\#J=p} H^0\big(X,\mathcal{F}\otimes\mathcal{I}_{Z_J}\big). \tag{*}$$

But now note that if $\mathcal{F}$ is a torsion-free sheaf with $h^0\big(X,\mathcal{F}\big) > 0$, and if $x \in X$ is a general point, then

$$h^0\big(X,\mathcal{F}\otimes\mathcal{I}_x\big) \ <\ h^0\big(X,\mathcal{F}\big).$$

In particular if $h^0\big(X,\mathcal{F}\big) \leq p$, and if $Z \subseteq X$ consists of $p$ general points, then

$$H^0\big(X,\mathcal{F}\otimes\mathcal{I}_Z\big) \ =\ 0.$$

Thus each of the groups on the right-hand side of (*) vanishes, and hence $Z_{p,0}\big(X,\mathcal{F};V\big) = 0$. □

The next result is a partial converse to Theorem 5.3.1. It appears in [52], and was inspired by a construction of Otavianni and Paoletti [148]:

**Proposition 5.3.3.** *Let $B$ be a line bundle on $X$, and assume that $L$ is sufficiently positive so that*

$$H^0\big(X, B \otimes L^{\otimes -m}\big) = 0 \quad \textit{for} \;\; m \geq 1 \tag{5.3.1}$$

$$H^0\big(X, L \otimes B^*\big) \neq 0. \tag{5.3.2}$$

*Then $K_{p,0}\big(X, B; L\big) \neq 0$ when $p < h^0\big(X, B\big)$.*

*Proof.* Thanks to (5.3.1) it suffices to show that $Z_{p,0}\big(X, B; V\big) \neq 0$ for $p < h^0\big(X, B\big)$. For this use (5.3.2) to fix a non-zero element $s \in H^0\big(X, L \otimes B^*\big)$ and choose linearly independent sections

$$f_0, \ldots, f_p \in H^0\big(X, B\big).$$

Thus $f_i s \in H^0\big(X, L\big)$. Now consider $\alpha \in \Lambda^p H^0(L) \otimes H^0(B)$ given by

$$\alpha = \sum_{j=0}^{p} (-1)^j \big(f_0 s \wedge \ldots \wedge \widehat{f_j s} \wedge \ldots \wedge f_p s\big) \otimes f_j.$$

Then $\alpha$ is killed by the Koszul differential and hence exhibits the required cycle. □

The next result, due to Green and the second author, gives a useful criterion for the non-vanishing of certain Koszul groups of weight one.

**Theorem 5.3.4.** *Let $X$ be an irreducible projective variety of dimension $n$, and let $L$ be a globally generated ample line bundle on $X$. Assume that*

$$L \cong L_1 \otimes L_2,$$

*where $h^0\big(X, L_i\big) = r_i + 1$ for some $r_i \geq 1$. Then*

$$K_{r_1+r_2-1,1}\big(X; L\big) \neq 0.$$

*Sketch of Proof.* We will assume for simplicity that $L_1$ and $L_2$ are themselves globally generated. Choose divisors $E_1 \in |L_1|, E_2 \in |L_2|$, giving rise to embeddings

$$L_2 \cong L(-E_1) \subseteq L \;\; , \;\; L_1 \cong L(-E_2) \subseteq L.$$

We choose a basis $s_0, \ldots, s_r \in \Gamma\big(X, L\big)$ in such a way that $E_1 + E_2 = \{s_0 = 0\}$ and the two subspaces of $\Gamma\big(X, L\big)$ in question are given by the spans

$$\Gamma\big(X, L(-E_1)\big) = < s_0, s_1, \ldots, s_{r_2} > \;\; , \;\; \Gamma\big(X, L(-E_2)\big) = < s_0, s_{r-r_1+1}, \ldots, s_r > .$$

We denote by $u_0, u_1, \dots, u_{r_2} \in \Gamma(L_2)$ and $v_0, v_{r-r_1+1}, \dots, v_r \in \Gamma(L_1)$ the corresponding sections. The goal is to produce $\alpha \in \Gamma\big(X, \Lambda^{r_2+r_1-1} M_L \otimes L\big)$ representing a non-trivial Koszul class.

To this end, note that there is natural inclusion of sheaves

$$M_{L_2} \oplus M_{L_1} \ \subseteq \ M_L$$

deduced from the homomorphism $L_2 \oplus L_1 \longrightarrow L$ determined by the $E_i$. Taking wedge products, one gets further inclusions

$$\begin{aligned}(\Lambda^{r_2} M_{L_2} \otimes L_2) \otimes (\Lambda^{r_1-1} M_{L_1} \otimes L_1) \ &\subseteq \ \Lambda^{r_2+r_1-1}\big(M_{L_2} \oplus M_{L_1}\big) \otimes (L_2 \otimes L_1) \\ &\subseteq \ \Lambda^{r_2+r_1-1} M_L \otimes L.\end{aligned} \tag{5.3.3}$$

We will specify sections

$$\sigma_2 \in \Gamma\big(\Lambda^{r_2} M_{L_2} \otimes L_2\big) \ \ , \ \ \sigma_1 \in \Gamma\big(\Lambda^{r_1-1} M_{L_1} \otimes L_1\big),$$

and then take $\alpha$ to be the image of $\sigma_2 \otimes \sigma_1$ under the map on global sections determined by (5.3.3).

To this end note that

$$\Gamma\big(\Lambda^{r_2} M_{L_2} \otimes L_2\big) \ = \ \Lambda^{r_2+1}\Gamma(L_2) \ \ , \ \ \Gamma\big(\Lambda^{r_1-1} M_{L_1} \otimes L_1\big) \ \supseteq \ \Lambda^{r_1}\Gamma(L_1).$$

We take

$$\sigma_2 \ = \ u_0 \wedge \ldots \wedge u_{r_2} \ \ , \ \ \sigma_1 \ = \ v_{r-r_1+1} \wedge \ldots \wedge v_r.$$

Thus we have produced a Koszul cycle

$$\alpha \ = \ \text{image}(\sigma_2 \otimes \sigma_1) \ \in \ \Gamma\big(X, \Lambda^{r_2+r_1-1} M_L \otimes L\big).$$

Under the inclusion

$$\Gamma(\Lambda^{r_2+r_1-1} M_L \otimes L) \ \subseteq \ \Lambda^{r_2+r_1-1} H^0(L) \otimes H^0(L),$$

one sees that $\alpha$ maps to

$$\widetilde{\alpha} \ = \ \sum_{\substack{0 \le i \le r_2 \\ r-r_1+1 \le j \le r}} \varepsilon_{i,j} \cdot \big(s_0 \wedge \ldots \wedge \widehat{s_i} \wedge \ldots \wedge s_{r_2} \wedge s_{r-r_1+1} \wedge \ldots \wedge \widehat{s_j} \wedge \ldots \wedge s_r\big) \otimes u_i \cdot v_j,$$

where $\varepsilon_{i,j} = \pm 1$. For the verification that $\alpha$ represents a non-zero cohomology class, we refer to [88]. □

## 5.3.B Additional geometry of $K_{p,0}$ and $K_{p,1}$

There are a number of additional results and constructions in the literature concerning the special properties of syzygies of weights zero and one. We survey some of these here without proof.

**Green's linear syzygy theorem.** Returning to the algebraic setting of Subsection 5.1.A, consider a positively graded finitely generated $S$-module

$$E = \bigoplus_{q \geq 0} E_q .$$

Since $E$ is positively graded, one has $K_{p,q}(E) = 0$ if $q < 0$. The *linear strand* of the graded free resolution $P_\bullet = P_\bullet(E)$ of $E$ is the subcomplex consisting of terms of lowest weight $q = 0$, i.e.

$$\ldots \longrightarrow K_{2,0}(E) \otimes S(-2) \longrightarrow K_{1,0}(E) \otimes S(-1) \longrightarrow K_{0,0}(E) \otimes S \longrightarrow 0.$$

Note that this only depends on the map $V \otimes E_0 \longrightarrow E_1$. Define the variety of rank one linear relations of $E$ to be the algebraic subset

$$R(E) =_{\text{def}} \big\{ v \otimes e \in V \otimes E_0 \mid v \cdot e = 0 \in E_1 \big\}.$$

Green [94] proved:

**Theorem 5.3.5.** *The length $\ell$ of the linear strand of $P_\bullet(E)$ satisfies*

$$\ell \leq \max \big( \dim E_0 - 1 \, , \, \dim R(E) \, \big).$$

*In other words, $K_{p,0}(E) = 0$ whenever $p$ is larger than the expression on the right.*

This had been conjectured by Eisenbud and Koh [65], motivated in part by the argument in [88] leading to Theorem 5.3.1. Green's proof of Theorem 5.3.5 uses an interesting construction involving the so-called exterior minors of a matrix of linear forms. Eisenbud subsequently put the argument into a wider context via the BGG-correspondence: this is an association between linear complexes over the symmetric algebra of vector space $V$ and modules over the exterior algebra of $V^*$. We recommend Chapter 7 of [60] for an informative and detailed presentation.

**Green's generalization of Castelnuovo's lemma and the $K_{p,1}$-Theorem.** Consider a (reduced) finite set

$$X \subseteq \mathbf{P}^r = \mathbf{P}(V)$$

consisting of $d \geq r+1$ points in linear general position. A classical lemma of Castelnuovo asserts that if $d \geq 2r+3$ then $X$ imposes only $2r+1$ conditions on quadrics if and only if the points in question lie on a rational normal curve. Green [88] proved a very nice generalization of this involving higher syzygies.

Specifically, let $I_X \subseteq S$ be the homogeneous ideal of $X$, and denote by $R_X = S/I_X$ the homogeneous coordinate ring of $X$, viewed as a graded $S$-module.

**Theorem 5.3.6.** *The points of $X$ lie on a rational normal curve if and only if*

$$K_{r-1,1}(R_X) \neq 0.$$

The non-vanishing of the group in question for subsets of a rational normal curve is elementary, and if $d \geq 2r+3$ then the classical statement applies. However when $r+4 \leq d \leq 2r+2$ this is an interesting and delicate result. Green uses it to prove the following "$K_{p,1}$-theorem."

**Theorem 5.3.7.** *Let $X \subseteq \mathbf{P}^r = \mathbf{P}H^0(L)$ be a smooth projective $n$-fold embedded by a complete linear series.*

(*i*). *If $K_{r-n,1}(X; L) \neq 0$, then $X$ is a variety of minimal degree.*

(*ii*). *If $K_{r-n-1,1}(X; L) \neq 0$ and $\deg X \geq r-n-3$, then $X$ sits as a subvariety of codimension one in a variety of minimal degree.*

Green establishes these results in §3.c of [88]. We refer also to [7, §3.3] for a nice account.

**Syzygy schemes and $K_{p,1}$-classes of small rank.** Consider as above a linearly normal embedding

$$X \subseteq \mathbf{P}^r = \mathbf{P}H^0(L)$$

of a smooth $n$-fold $X$. It sometimes happens that $X$ lies on a rational scroll $Z \subseteq \mathbf{P}^r$ of codimension $p$ in $\mathbf{P}^r$, and in this case $K_{p,1}(X; L) \neq 0$. In fact, the resolution of $I_Z$ is linear, and the linear strand of the resolution of $Z$ injects for reasons of weight into the linear strand of the resolution of $X$. The question arises whether one can recognize intrinsically classes $\gamma \in K_{p,1}(X; L)$ that arise in this fashion. This involves a circle of ideas around syzygy schemes nicely summarized in Chapter 3 of [7].

In brief, write $V = H^0(L)$. One starts by showing that there is a unique smallest linear subspace $W \subseteq V$ such that $\gamma \in K_{p,1}(X; L)$ is represented by an element in $\Lambda^p W \otimes V$: one defines the *rank* of $\gamma$ to be the dimension of $W$. Classes $\gamma$ of rank $p$ don't arise in our setting, so the first interesting case occurs when $\mathrm{rank}(\gamma) = p+1$.

**Proposition 5.3.8.** *If rank$(\gamma) = p+1$, then $\gamma$ is supported on a rational normal scroll $Z \subseteq \mathbf{P}^r$ of codimension $p$ that contains $X$.*

The proof involves an analysis of the *syzygy scheme* defined by a class $\gamma \in K_{p,1}(X; L)$. Specifically, if we represent $\gamma$ by an element in $\Lambda^p V \otimes V$, the condition that $\delta(\gamma) = 0 \in \Lambda^{p-1} V \otimes H^0(L^{\otimes 2})$ means that we can view $\gamma$ as defining an element $\delta'(\gamma) \in \Lambda^{p-1}(V) \otimes I_2(X)$, where $I_2(X)$ denotes the space of quadrics through $X$. The image of the resulting map $\Lambda^{p-1} V^* \longrightarrow I_2(X)$ defines a linear system of quadrics though $X$ whose common zeroes are defined to be the syzygy scheme

$$\mathrm{syz}(\gamma) \subseteq \mathbf{P}^r.$$

The Proposition is established by proving that the syzygy scheme of a class of rank $p+1$ is a rational normal scroll of codimension $p$. We again refer to [7, Chapter 3] for details and more information.

## 5.4 Examples

In this section, we present by way of illustration some applications and examples of the material developed above. In the first subsection we discuss an influential theorem of Green concerning the syzygies of curves of large degree. We then give a quick sketch of the connection between syzygies of square-free monomial ideals and the homology of simplicial complexes.

### 5.4.A Green's theorem on curves of large degree

This subsection focuses on a theorem of Green concerning the syzygies of curves of large degree. Lecture 7 is devoted to a more detailed study of syzygies of curves, and the result also motivates the content of the next Lecture, so the present discussion is perhaps a little out of sequence. However we felt that the reader might find it useful to see early on some concrete geometric applications of the material developed in the preceding sections.

Let $C$ be a smooth projective curve of genus $g$, and let $L$ be a line bundle of degree $d$ on $C$. It is elementary that if $d \geq 2g+1$ then $L$ is very ample, and it was established by Castelnuovo [36], Mattuck [133] and Mumford [141] that $L$ is normally generated, i.e. that it defines an embedding

$$C \ \subseteq \ \mathbf{P}H^0(L) \ = \ \mathbf{P}^{d-g}$$

in which $C$ is projectively normal. Later Fujita [79] and Saint-Donat [171] showed that if $d \geq 2g+2$, then $C$ is cut out by quadrics. Classically this seemed to be the end of the story, but Green realized in [88] that these are just the first cases of a more general theorem for higher syzygies. Green's result ultimately inspired much of the work discussed in the chapters that follow.

By way of motivation, let us recast the classical statements in terms of Koszul groups. Given a linearly normal embedding $C \subseteq \mathbf{P}H^0(L)$, we've seen (Example 5.1.14) that

$$C \ \text{ is projectively normal } \iff K_{0,q}(C;L) \ = \ 0 \ \text{ for all } \ q \ \geq \ 2.$$

When this is satisfied, the homogeneous ideal $I_C$ is generated by quadrics if and only if $K_{1,q}(C;L) = 0$ for all $q \geq 2$. So the classical results assert the vanishing of $K_{0,q}(C;L)$ and $K_{1,q}(C;L)$ for $q \geq 2$ provided that $L$ has sufficiently large degree.

This suggests the following

**Definition 5.4.1 (Property ($\mathbf{N_k}$)).** The line bundle $L$ defining $C \subseteq \mathbf{P}H^0(L)$ satisfies *Property* ($N_k$) provided that

$$K_{p,q}(C;L) \ = \ 0$$

for all $q \geq 2$ and $0 \leq p \leq k$.

In other words, the first $k$ steps of the resolution of $R(C;L) = \oplus H^0\big(C, L^{\otimes m}\big)$ as a module over $S = \operatorname{Sym} H^0(L)$ should be linear.

Thus, very concretely, $(N_0)$ asks that $C$ be projectively normal, while $(N_1)$ means that in addition the homogeneous ideal $I_C$ is generated in degree 2. The first non-classical condition is $(N_2)$, which requires that if $q_\alpha \in I_{C,2}$ are quadratic generators, then the module of syzygies among the $q_\alpha$ should be generated by relations of the form

$$\sum \ell_\alpha \cdot q_\alpha \ = \ 0$$

where the $L_\alpha$ are *linear* polynomials. (The terminology was motivated by Mumford's use of "normal generation" and "normal presentation" for $(N_0)$ and $(N_1)$ in [141].)

**Example 5.4.2.** A rational normal curve $C \subseteq \mathbf{P}^3$ of degree three has a resolution of the form

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^3}^2(-3) \longrightarrow \mathcal{O}_{\mathbf{P}^3}^3(-2) \longrightarrow \mathcal{O}_{\mathbf{P}^3} \longrightarrow \mathcal{O}_C \longrightarrow 0,$$

and hence satisfies $(N_2)$. An elliptic curve $E \subseteq \mathbf{P}^3$ of degree four is a complete intersection of two quadrics, so is resolved by a Koszul complex

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^3}(-4) \longrightarrow \mathcal{O}_{\mathbf{P}^3}^2(-2) \longrightarrow \mathcal{O}_{\mathbf{P}^3} \longrightarrow \mathcal{O}_E \longrightarrow 0.$$

Thus $E$ satisfies $(N_1)$ but not $(N_2)$. □

Green's result generalizes the classical statements in a very pleasing way:

**Theorem 5.4.3 (Green).** *Assume that*

$$d \ = \ \deg(L) \ \geq \ 2g + 1 + k.$$

*Then $L$ satisfies Property $(N_k)$.*

We will give three proofs of the result. The first, via duality and Green's vanishing theorem, is the one appearing in [88]. A second argument proceeds directly with kernel bundles, while a third reduces the question to the syzygies of finite sets.

*First Proof of Theorem 5.4.3.* In view of Example 5.1.15, the issue is to verify that

$$K_{p,2}\big(C; L\big) \ = \ 0$$

provided that $p \leq d - 2g - 1$. Since $H^1\big(C, L\big) = 0$, it is equivalent by duality (Remark 5.2.12) to show that

$$K_{r-1-p,0}(C, \omega_C; L) \ = \ 0, \tag{*}$$

where $r = r(L) = d - g$. But note that

$$r - 1 - p \ = \ d - g - 1 - p \ \geq \ g \ = \ h^0\big(C, \omega_C\big)$$

by hypothesis, so (*) follows from Theorem 5.3.1. □

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_{L(-D)} & \longrightarrow & H^0\big(L(-D)\big)\otimes\mathcal{O}_C & \longrightarrow & L(-D) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_L & \longrightarrow & H^0\big(L\big)\otimes\mathcal{O}_C & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Sigma_D & \longrightarrow & W_D\otimes\mathcal{O}_C & \longrightarrow & L\otimes\mathcal{O}_D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Figure 5.2: Diagram of bundles on $C$

**Secant bundles.** In preparation for the second proof of Theorem 5.4.3, we start with some remarks concerning secant constructions with kernel bundles. Given $C \subseteq \mathbf{P}H^0(L)$ as above, let $D$ be an effective divisor on $C$ with the property that $L(-D)$ is globally generated: this means that if $\overline{D} \subseteq \mathbf{P}^{r(L)}$ is the linear span of $D$, then $C \cap \overline{D} = D$. Write

$$W_D \;=\; \frac{H^0\big(C, L\big)}{H^0\big(C, L(-D)\big)},$$

so that $\mathbf{P}(W_D) = \overline{D} \subseteq \mathbf{P}H^0(L)$. These data fit together in the exact commutative diagram shown in Figure 5.2, which defines a vector bundle $\Sigma_D$ on $C$ with $\operatorname{rank}\Sigma_D = \dim W_D$.

For the proof of 5.4.3 we take $D = x_1 + \ldots + x_{r-1}$ to be the sum of $r-1$ general points of $C$, so that $D$ spans a linear space $\overline{D} = \mathbf{P}^{r-2}$ of codimension two. In this case $h^0\big(C, L(-D)\big) = 2$,

$$M_{L(-D)} \;=\; L^*(D) \quad , \quad \Sigma_D \;=\; \oplus\, \mathcal{O}_C(-x_i).$$

and the left-hand column of Figure 5.2 becomes the exact sequence

$$0 \longrightarrow L^*(D) \longrightarrow M_L \longrightarrow \oplus\, \mathcal{O}_C(-x_i) \longrightarrow 0. \tag{5.4.1}$$

Green's theorem will follow quickly from this sequence.

*Second Proof of Theorem 5.4.3.* We need to show that $H^1\big(C, \Lambda^{p+1}M_L \otimes L\big) = 0$ when $p \leq d - 2g - 1$. To this end, take wedge products in (5.4.1) to get an exact sequence

$$0 \longrightarrow \mathcal{O}(D) \otimes \Lambda^p\big(\oplus\, \mathcal{O}_C(-x_i)\big) \longrightarrow \Lambda^{p+1}M_L \otimes L \longrightarrow L \otimes \Lambda^{p+1}\big(\oplus\, \mathcal{O}_C(-x_i)\big) \longrightarrow 0.$$

The term on the right is a direct sum of line bundles of degree $d - p - 1 \geq 2g$ and hence has vanishing $H^1$. As for the term on the left, it is the direct sum of line bundles corresponding to the sum of

$$r - 1 - p \;=\; d - g - 1 - p$$

general points. But $d - g - 1 - p \geq g$ by assumption, so these summands also have vanishing $H^1$. □

**Syzygies of finite sets.** Yet another approach to Theorem 5.4.3 is to reduce the question to a statement about the syzygies of finite sets. Starting with a projectively normal embedding $C \subseteq \mathbf{P}^{d-g}$ defined by a line bundle $L$ of degree $d = 2g+1+k$, take a general hyperplane section of $C$. Writing $n = d-g-1$, this is a finite set

$$X \subseteq \mathbf{P}^n \quad \text{with} \quad \#X \;=\; 2n+1-k, \tag{*}$$

and it is well known (at least in characteristic zero) that the points of $X$ are in linearly general position. Moreover thanks to Proposition 1.3.8 the syzygies of $C$ restrict to those of $X$.

Now consider any finite set $X \subseteq \mathbf{P}^n = \mathbf{P}(V)$, with homogeneous ideal $I_X \subseteq S = \mathrm{Sym}(V)$, and denote by $R_X = S/I_X$ the homogeneous coordinate ring $X$. Then Green's result follows from a vanishing statement for the syzygies of $R_X$ or $I_X$:

> Theorem: If $X \subseteq \mathbf{P}^n$ is a finite set of $(2n+1-k)$ points in linearly general position, then
>
> $$K_{p,2}(R_X;V) \;=\; K_{p-1,3}(I_X;V) \;=\; 0 \;\text{ for } 0 \le p \le k.$$

(The equality of the two Koszul groups results from Example 5.1.5.) This was established by Green and the second author in [98] via an explicit calculation. Alternatively, one can argue with vector bundles along the lines of the second proof of 5.4.3, as follows.

Consider the blowing up

$$\mu : \mathbf{P}' \;=\; \mathrm{Bl}_X(\mathbf{P}) \longrightarrow \mathbf{P}$$

of $\mathbf{P} = \mathbf{P}^n$ along $X$. Let $E = \sum E_i$ be the exceptional divisor, where $E_i$ lies over the $i^{\text{th}}$ point $x_i \in X$, and note that $\mu_*\mathcal{O}_{\mathbf{P}'}(-E) = \mathcal{I}_{X/\mathbf{P}}$. Writing $L = \mu^*\mathcal{O}_{\mathbf{P}}(1)$, it follows from 5.2.5 that the issue is to prove the vanishing

$$H^1\big(\mathbf{P}', \Lambda^p M_L \otimes L^{\otimes 2}(-E)\big) = 0 \quad \text{for} \;\; p \le k. \tag{*}$$

For this, define a vector bundle $\Sigma$ of rank $n+1$ on $\mathbf{P}'$ by an exact sequence analogous to the bottom row in Figure 5.2:

$$0 \longrightarrow \Sigma \longrightarrow V \otimes \mathcal{O}_{\mathbf{P}'} \longrightarrow L \otimes \mathcal{O}_E \longrightarrow 0.$$

Assuming that the points of $X$ span $\mathbf{P}^n$, this bundle fits into an exact sequence

$$0 \longrightarrow M_L \longrightarrow \Sigma \longrightarrow L \otimes \mathcal{O}_{\mathbf{P}'}(-E) \longrightarrow 0,$$

from which one shows that (*) is implied by the vanishing

$$H^1\big(\mathbf{P}', \Lambda^{p+1}\Sigma \otimes L\big) \;=\; 0 \tag{**}$$

in the same range of $p$. On the other hand, by decomposing $X$ into suitable subsets one sees that $\Sigma$ sits in a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}'}(-E') \longrightarrow \Sigma \longrightarrow \oplus_{i=1}^n \mathcal{O}_{\mathbf{P}'}(-E_i) \longrightarrow 0,$$

where $E' = E_{n+1} + \ldots + E_{2n+1-k}$ is the sum of $(n+1-k)$ exceptional components. Taking wedge products as in the second proof of 5.4.3, the assertion (**) follows using the hypothesis that the points of $X$ are in linearly general position. □

| | 0 | 1 | $\cdots$ | $(d-2g-1)$ | $(d-2g)$ | $\cdots$ | $(d-g-1)$ |
|---|---|---|---|---|---|---|---|
| 0 | 1 | – | $\cdots$ | – | – | $\cdots$ | – |
| 1 | – | $*$ | $\cdots$ | $*$ | $\diamond$ | $\cdots$ | $\diamond$ |
| 2 | – | – | $\cdots$ | – | $*$ | $\cdots$ | $*$ |

Figure 5.3: Betti table of curve of large degree

**Boundary examples and overview.** The conclusion of Theorem 5.4.3 is the best possible in the sense that for any curve $C$ of genus $g \geq 2$, there exist very ample line bundles $L$ of degree $d = 2g + k$ for which Property $(N_k)$ fails. Specifically, take $L = \omega_C(D)$ where

$$D = x_1 + \ldots + x_{k+2}$$

is a general effective divisor of degree $k+2$: thus $D$ spans a $(k+2)$-secant $k$-plane in $\mathbf{P}H^0(L)$. Then the bundle $\Sigma_D$ in Figure 5.2 has rank $= k+1$ and

$$\det \Sigma_D = \mathcal{O}_C(-D).$$

Therefore $\omega_C$ is a quotient of $\Lambda^{k+1} M_L \otimes L$, and consequently $K_{k,2}(C; L) \neq 0$ thanks to 5.2.5. It turns out that these examples, along with hyperelliptic curves, are precisely the borderline cases in Green's theorem: see [98].

It is interesting to put together what we know so far about the syzygies of a linearly normal embedding $C \subseteq \mathbf{P}H^0(L)$ defined by a line bundle of large degree $d \gg 0$. Since $C$ is projectively Cohen–Macaulay, the length of its resolution is $r - 1 = d - g - 1$. Considerations of Castelnuovo–Mumford regularity (Example 5.1.15) show that all syzygies of weight $q \geq 3$ vanish, and Theorem 5.4.3 asserts that

$$K_{p,2}(C; L) = 0 \quad \text{for} \ \ p \leq d - 2g - 1.$$

On the other hand, $K_{p,2}(C; L)$ is dual to $K_{d-g-1-p,0}(C, \omega_C; L)$ and by Proposition 5.3.3 these groups are non-zero (for $d \gg 0$) when

$$d - g - 1 - p < g = h^0\big(C, \omega_C\big),$$

i.e. when $p \geq d - 2g$. Thus the only groups whose vanishing or non-vanishing remain in question are the $K_{p,1}(C; L)$ in the range $d - 2g \leq p \leq d - g - 1$. The situation is summarized in Figure 5.3, which shows the Betti table for $C$. Entries marked with a dash are zero, those with an asterisk are non-zero, while a diamond indicates the groups that aren't determined by these considerations. (The "missing" $K_{p,1}$ groups are the subject of the Gonality Theorem discussed in Lecture 7. In brief, roughly the first half turn out to be non-zero for relatively elementary reasons. The others depend in an interesting (but understood) way on the intrinsic geometry of $C$.)

### 5.4.B Hochster's theorem

As a second example, we briefly discuss a circle of ideas going back to Hochster expressing the syzygies of square-free monomial ideals in terms of simplicial homology. The following paragraphs only scratch the surface of a very interesting body of work. For fuller presentations we refer for instance to [137, Chapters 1 and 5] or [110, Chapter 5].

Consider a simplicial complex $\Delta$ on the index set $[n] = \{1, \ldots, n\}$. As explained in Section 3.2.C, this is the same as giving a square-free monomial ideal

$$I_\Delta \subseteq \mathbf{C}[x_1, \ldots, x_n] :$$

by definition, $I_\Delta$ is generated by the monomials $x^\tau$ where $\tau \subseteq [n]$ is a non-face of $\Delta$. We wish to study the Koszul cohomology groups $K_{p,q}(I_\Delta)$. These are naturally $\mathbf{N}^n$ graded, and it turns out that the only non-zero components correspond to weight vectors having entries 0 or 1, which in turn are parametrized by square-free monomials $x^\sigma$. It is then easiest to work directly with Tor's, so the goal is to describe $\mathrm{Tor}_p(I_\Delta, \mathbf{k})_\sigma$, where as usual $\mathbf{k} = \mathbf{C} = S/S_+$.

Given such an exponent vector $\sigma$, define its support $|\sigma| \subseteq [n]$ to be the subset of $[n]$ it determines. We then form the simplicial complex

$$\Delta^\sigma \subseteq |\sigma|$$

whose faces are all $\tau \subseteq \sigma$ with the property that $\sigma - \tau$ is *not* a face of $\Delta$.[4] For example, if $\Delta$ is the simplicial complex shown in Figure 3.1, and $|\sigma| = \{1, 2, 3\}$, then $\Delta^\sigma$ consists of the isolated vertex $\{2\}$. If $|\sigma| = [4]$, then $\Delta^\sigma = \Delta^\vee$.

Hochster's result is the following:

**Theorem 5.4.4.** *For $p \geq 1$ one has*

$$\mathrm{Tor}_p\big(I_\Delta, \mathbf{k}\big)_\sigma \;=\; \widetilde{H}_{p-1}\big(\Delta^\sigma; \mathbf{C}\big),$$

*where the group on the right is the reduced simplicial homology of $\Delta^\sigma$.*

For example, returning to the ideal $I_\Delta \subseteq \mathbf{C}[x_1, \ldots, x_4]$ arising from Figure 3.1, we saw in the resolution (3.2.4) that $\mathrm{Tor}_1(I_\Delta, \mathbf{k}) = \mathbf{C}$, with the generator having weight $\sigma = [4]$. It corresponds to the one-dimensional reduced homology group $\widetilde{H}^0(\Delta^\vee)$ of the Alexander dual $\Delta^\vee$.

*Sketch of Proof of Theorem 5.4.4.* Fix $\sigma \subseteq [n]$, and write $V$ for the vector space of linear forms in $\mathbf{C}[x_1, \ldots, x_n]$. Then $\mathrm{Tor}_i(I_\Delta, \mathbf{k})_\sigma$ is computed as the homology of the weight $\sigma$

[4] This notation is non-standard: $\Delta^\sigma$ is more properly described as the link of $|\sigma|$ in the Alexander dual $\Delta^\vee$ of $\Delta$.

subcomplex $(\Lambda^\bullet V \otimes I_\Delta)_\sigma$ of the Koszul complex. Now $(\Lambda^p V \otimes I_\Delta)_\sigma$ has dimension 1 or 0, depending on whether or not there is a degree $p$ square-free monomial $x^\tau$ such that

$$x^{\sigma-\tau} \in I_\Delta. \tag{*}$$

In other words, $\mathrm{Tor}_p(I_\Delta, \mathbf{k})_\sigma$ is the homology of the complex $K_{\bullet,\sigma}(I_\Delta)$ having

$$K_{p,\sigma}(I_\Delta) = \{x^\tau \mid \deg(x^\tau) = p\,,\, x^{\sigma-\tau} \in I_\Delta\},$$

with the usual Koszul differential.

On the other hand, consider any simplicial complex $\Pi$ on the vertex set $|\sigma|$. Then the $\mathbf{C}$-valued reduced simplicial $(p-1)$-chains $C_{p-1}(\Pi)$ of $\Pi$ are identified with monomials $x^\tau$ of degree $p$ such that $x^\tau \in \Pi$. Applying this to $\Pi = \Delta^\sigma$, we see that

$$C_{p-1}(\Delta^\sigma) = K_{p,\sigma}(I_\Delta),$$

as required. □

**Remark 5.4.5.** As we intimated above, there are many further results in the same direction. For example, given $\sigma$, let $\Delta|\sigma$ denote the subcomplex formed by those faces of $\Delta$ contained in $\sigma$. Then Theorem 5.4.4 is Alexander dual to an isomorphism:

$$\mathrm{Tor}_p(I_\Delta, \mathbf{k}) = \widetilde{H}^{|\sigma|-p-2}(\Delta|\sigma; \mathbf{C}).$$

One can also compute local cohomology, regularity, projective dimension, etc. We again refer the interested reader to [137, Chapters 1 and 5] or [110, Chapter 5] for details and references.

## 5.5 Notes ◇

# Lecture 6

# Linearity of Syzygies: Property $(\mathbf{N_k})$

In Section 5.4.A we discussed Green's theorem on the syzygies of curves of large degree. It asserts that if $L$ is a line bundle of degree $\geq 2g+1+k$ on a smooth projective curve $C$ of genus $g$, then $L$ satisfies Property $(N_k)$: the first $k$ steps of the resolution of $C$ in the embedding defined by $L$ are linear. Green's result suggested that analogous linearity statements should hold for increasingly positive embeddings of other varieties. The present lecture is devoted to a body of work in this direction.

We start in Section 6.1 with a rather soft result that makes precise the slogan that on an arbitrary projective variety $X$, Property $(N_k)$ holds "linearly in the positivity" of a very ample line bundle $L$. The remaining sections focus on the more interesting problem of giving effective statements in various natural settings. Section 6.2 is devoted to the Koszul cohomology of projective space: we prove a linearity theorem due to Green [89] for the syzygies of Veronese varieties, and give also some other applications. In Section 6.3 we survey without proof some results about the syzygies of abelian varieties. Finally, Section 6.4 considers a smooth projective variety $X$ embedded by line bundles of the type $L_d = \mathcal{O}_X(K_X + dB)$ where $B$ is a divisor satisfying various strong positivity hypotheses.

## 6.1 A linearity theorem for very positive embeddings

Let $X$ be an irreducible projective variety, and let $L$ be a very ample line bundle on $X$. We will be interested in the embedding

$$X \ \subseteq \ \mathbf{P}^r \ = \ \mathbf{P}H^0(L)$$

defined by the complete linear series $|L|$. Thus $r = r(L) = h^0\big(X, L\big) - 1$, and $K_{0,1}\big(X; L\big) = 0$ (Example 5.1.14).

We start with a definition that already appeared (in the case of curves) in Section 5.4.A.

**Definition 6.1.1 (Property $(\mathbf{N_k})$).** We say that $L$ satisfies *Property* $(N_k)$ if

$$K_{p,q}(X;L) \ = \ 0 \quad \text{for all} \quad q \geq 2$$

and every $0 \leq p \leq k$. □

(When $L$ is understood, one sometimes speaks of $(N_k)$ holding for $X \subseteq \mathbf{P}^r$.) Thus, very concretely:

$$(N_0) \text{ holds for } L \quad \Longleftrightarrow \quad \begin{cases} \text{For every } m \geq 0 \text{, the natural maps} \\ \qquad S^m H^0\big(X,L\big) \longrightarrow H^0\big(X,L^{\otimes m}\big) \\ \text{are surjective;} \end{cases}$$

$$(N_1) \text{ holds for } L \quad \Longleftrightarrow \quad \begin{cases} (N_0) \text{ is satisfied, and the homogeneous ideal } I = I_{X/\mathbf{P}^r} \text{ of} \\ X \text{ in } \mathbf{P}^r \text{ is generated by quadrics;} \end{cases}$$

$$(N_2) \text{ holds for } L \quad \Longleftrightarrow \quad \begin{cases} (N_1) \text{ is satisfied, and the first module of syzygies among} \\ \text{quadratic generators } q_\alpha \in I \text{ is spanned by relations of} \\ \text{the form} \\ \qquad \sum \ell_\alpha \cdot q_\alpha \ = \ 0, \\ \text{where the } \ell_\alpha \text{ are linear forms;} \end{cases}$$

and so on. In other words, the first $k$ steps of the resolution of $R(X;L) = \oplus H^0\big(X, L^{\otimes m}\big)$ should be linear. As noted in Section 5.4.A, the name "$(N_k)$" is intended to suggest the connection with the Mumford's terminology "normal generation" and "normal presentation" for $(N_0)$ and $(N_1)$ in [141].

Green [89, §3] proved that given $k$, any sufficiently positive line bundle on $X$ satisfies $(N_k)$: we will outline his approach later in the section. The following slightly more precise statement shows that in fact the required positivity for $L$ grows linearly in $k$.

**Theorem 6.1.2.** *Let $X$ be an irreducible projective variety of dimension $n$. Then there exists a very ample line bundle $A$ on $X$ with the property that $(N_k)$ holds for $L = A^{\otimes (k+1)}$. More generally, if*

$$L \ = \ P \otimes A^{\otimes (k+1)}$$

*for any ample (or nef) bundle $P$, then $L$ satisfies Property $(N_k)$.*

We will prove the theorem shortly. However it is good to keep in mind that if $\dim X \geq 2$, then $(N_k)$ also *fails* linearly in the positivity of $L = A^{\otimes (k+1)}$ (at least when $X$ is smooth). Specifically, in the situation of the Theorem there is a positive number $c > 0$ depending on $X$ and $A$ such that

$$K_{p,2}(X;L) \neq 0 \ \text{ for } p \geq c \cdot k.$$

Non-vanishings of this sort are discussed in Lecture 8. For the case $X = \mathbf{P}^2$, where best-possible statements are known, see Proposition 6.2.8 in the next section. We will also outline in Lecture 8 a very nice result (Theorem 8.1.3) of Jinhyung Park giving a substantial strengthening of Theorem 6.1.2 for syzygies of higher weight.

*Proof of Theorem 6.1.2.* To begin with, fix a very ample line bundle $N$ with the property that $N \otimes P$ is very ample with vanishing higher cohomology for any nef $P$. The existence of $N$, which follows from a vanishing theorem of Fujita, is sketched in Example 6.1.3.

The essential point is the following

> CLAIM: There exist integers $0 < d_0 < \ldots < d_n$ and another integer $e > d_n$ (all depending only on $N$ and $X$) with the following property. Suppose that
>
> $$L = L_d = N^{\otimes d} \otimes P$$
>
> where $d \geq e$ and $P$ is nef. Then $M_L$ is resolved by a (possibly infinite) exact sequence whose first $n+1$ terms are of the form:
>
> $$\ldots \longrightarrow V_1 \otimes N^{\otimes(-d_1)} \longrightarrow V_0 \otimes N^{\otimes(-d_0)} \longrightarrow M_L \longrightarrow 0, \qquad (*)$$
>
> where $V_0, \ldots, V_n$ are vector spaces depending on $d$ and $P$.

Granting this for the time being, fix an integer $0 \leq p \leq k$. Then the $(p+1)$-fold tensor power $T^{p+1}M_L = M_L \otimes \ldots \otimes M_L$ of $M_L$ is resolved by the $(p+1)$-fold tensor power of the complex (*) resolving $M_L$. Thus $T^{p+1}M_L$ has a resolution

$$\ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow T^{p+1}M_L \longrightarrow 0 \qquad (**)$$

whose first $n$ terms $F_0, \ldots, F_n$ have the shape

$$F_j = \bigoplus_i N^{\otimes(-b_{i,j})} \quad \text{where} \quad b_{i,j} \leq (p+1)d_j,$$

and in particular $b_{i,j} < (p+1)e$. Keeping in mind that by assumption $H^i\big(X, N^{\otimes a} \otimes P\big) = 0$ for any $i, a > 0$ and nef $P$, we read off from (**) that if $d \geq (k+1)e$ then

$$H^i\Big( X\,, T^{p+1}M_{L_d} \otimes L_d^{\otimes(q-1)} \Big) = 0$$

for $i > 0$ and $q \geq 2$. On the other hand, as we are in characteristic zero, $T^{p+1}M$ contains $\Lambda^{p+1}M$ as a summand, and therefore also $H^1\big(X, \Lambda^{p+1}M \otimes L^{\otimes(q-1)}\big) = 0$. Thanks to Theorem 5.2.5, we conclude that $K_{p,q}\big(X; L_d\big) = 0$ when $q \geq 2$. Thus the statement of the Theorem holds with $A = N^{\otimes(e+1)}$.

It remains to prove the Claim. For this we pass to $X \times X$ and consider the diagonal $\Delta \subseteq X \times X$ with ideal sheaf $\mathcal{I}_\Delta$. As usual, write $B_1 \boxtimes B_2$ for the exterior product of line bundles $B_1, B_2$ on $X$. Since $N \boxtimes N$ is ample, we can use Serre vanishing to construct a (possibly infinite) resolution of $\mathcal{I}_\Delta$ having the shape

$$\ldots \longrightarrow W_1 \otimes \big(N^{\otimes(-d_1)} \boxtimes N^{\otimes(-d_1)}\big) \longrightarrow W_0 \otimes \big(N^{\otimes(-d_0)} \boxtimes N^{\otimes(-d_0)}\big) \longrightarrow \mathcal{I}_\Delta \longrightarrow 0, \quad (***)$$

where $0 < d_0 < d_1 < \ldots$ and the $W_i$ are finite dimensional vector spaces. (See [128, 1.2.21] for details.) Set $e = d_{2n} + 1$, and for $d \geq e$ tensor the exact sequence (***) by $\mathrm{pr}_2^*\big(N^{\otimes d} \otimes P\big)$. This gives a resolution of $\mathcal{I}_\Delta \otimes \mathrm{pr}_2^*(N^{\otimes d} \otimes P)$ for which the higher direct images $R^i\mathrm{pr}_{1,*}$ of the first $2n$ terms vanish. On the other hand,

$$M_{L_d} \ = \ \mathrm{pr}_{1,*}\big(\mathcal{I}_\Delta \otimes \mathrm{pr}_2^*(N^{\otimes d} \otimes P)\big)$$

(Example 5.2.2). Applying $\mathrm{pr}_{1,*}$ to this resolution, we arrive at the complex asserted in the Claim, with $V_i = W_i \otimes H^0\big(X, N^{\otimes(d-d_i)} \otimes P\big)$. □

**Example 6.1.3** (**An application of Fujita's vanishing theorem**). We outline the existence of a line bundle $N$ with the properties asserted at the beginning of the previous proof. The key tool is a theorem of Fujita asserting that Serre vanishing can be made to work uniformly with respect to twists by nef line bundles. More precisely, fix a very ample line bundle $B$ and a coherent sheaf $\mathcal{F}$ on an irreducible projective variety $X$ of dimension $n$. Fujita's result is that there exists an integer

$$m_0 \ = \ m_0(X, B, \mathcal{F})$$

having the property that if $P$ is any nef line bundle and $m \geq m_0$ then

$$H^i\big(X, \mathcal{F} \otimes B^{\otimes m} \otimes P\big) \ = \ 0 \ \text{ for all } \ i > 0. \qquad (*)$$

The crucial point is that $m_0$ is independent of $P$. We refer to [128, §1.4.D] for a fuller discussion and the proof. In the setting of the previous proof, we apply this with $\mathcal{F} = B^{\otimes -(n+2)}$ and take $N = B^{\otimes m_0}$. Then (*) implies that $N \otimes P$ is $(-1)$-regular with respect to $B$ for any any nef $P$. If follows (Theorem 3.2.5) that $N \otimes P$ is very ample and has vanishing higher cohomology, as required. □

**Example 6.1.4** (**Vanishing for Koszul cohomology of a coherent sheaf**). An analogous statement holds for the Koszul cohomology of any coherent sheaf $\mathcal{F}$ on $X$. Specifically, a small modification of the proof Theorem 6.1.2 shows that one can take the line bundle $A$ in to have the property that if $L = P \otimes A^{\otimes(k+1)}$ with $P$ nef, then

$$K_{p,q}\big(X, \mathcal{F}; L\big) \ = \ 0 \quad \text{for} \quad q \ \geq \ 2 \ \text{ and } \ 0 \leq p \leq k. \qquad \square$$

Finally, we say a word about the "multi-diagonal" method introduced by Green in [89] to show that any sufficiently positive line bundle on an irreducible variety $X$ satisfies $(N_k)$. Fix an integer $p \geq 0$, and consider the $(p+2)$-fold product

$$Y \ = \ Y_p \ = \ X \times \cdots \times X$$

of $X$ with itself: write $y = (x_0, x_1, \ldots, x_{p+1}) \in Y_p$ for the typical element of $Y$. Now consider the (reduced) algebraic subset

$$\Sigma \;=\; \Sigma_p \;\subseteq\; Y_p$$

defined by:

$$\Sigma \;=\; \big\{ (x_0, \ldots, x_{p+1}) \;|\; x_0 = x_i \text{ for some } 1 \leq i \leq p+1 \big\}.$$

So for instance if $p = 0$ then $Y_0 = X \times X$, and $\Sigma = \Delta_X$ is the diagonal; when $p = 1$, $\Sigma \subseteq X \times X \times X$ is the union (in the hopefully evident notation) of the two (big) diagonals $\Delta_{01}$ and $\Delta_{02}$; and so on.

It is classical that one can use the diagonal $\Delta \subseteq X \times X$ to study multiplication maps

$$H^0\big(X, L\big) \otimes H^0\big(X, L^{\otimes(q-1)}\big) \longrightarrow H^0\big(X, L^{\otimes q}\big).$$

Green's nice idea is that the $\Sigma_p$ play a similar role for higher syzygies. As a matter of notation, given line bundles $B_0, \ldots, B_{p+1}$ on $X$, write

$$B_0 \boxtimes \ldots \boxtimes B_{p+1} \;=_{\text{def}}\; \otimes \operatorname{pr}_i^* B_i$$

for their exterior product on $Y_p$.

**Proposition 6.1.5** (**Green's multi-diagonal criterion**). *Let $L$ be a line bundle on $X$ satisfying $H^j\big(X, L^{\otimes m}\big) = 0$ for $j, m > 0$ and assume that*

$$H^1\Big( Y_p\,,\, \mathcal{I}_\Sigma \otimes L^{\otimes(q-1)} \boxtimes L \cdots \boxtimes L \Big) \;=\; 0 \tag{6.1.1}$$

*for some $q \geq 2$. Then $K_{p,q}(X; L) = 0$.*

Granting the Proposition, it follows immediately from Serre vanishing that for fixed $p$, any sufficiently positive $L$ satisfies $K_{p,q}(X; L) = 0$ for all $q \geq 2$.

Green's proof of 6.1.5 was completed (and corrected) by Inamdar in [?]. Their argument goes by induction on $p$. Alternatively, one can argue via Künneth that the sheaf appearing in (6.1.1) pushes down with vanishing higher direct images to $T^{p+1}(M_L) \otimes L^{\otimes(q-1)}$ under projection to the first copy of $X$ (corresponding to the index 0 in the definition of $\Sigma$). It then follows from (6.1.1) that $H^1\big(X, T^{p+1}(M_L) \otimes L^{\otimes(q-1)}\big) = 0$, which as above yields the stated vanishing of $K_{p,q}$.

## 6.2 Projective space

This section is devoted to computations involving Koszul cohomology on projective space. The first subsection presents some vanishing theorems due to Green, implying in particular that the $d^{\text{th}}$ Veronese of projective space satisfies $(N_d)$. One of Green's motivations for developing the theory was to study algebraic questions that come up when one makes infinitesimal computations in Hodge theory, and in Section 6.2.B we give a very small sample of some of the many applications in this direction. Finally, Section 6.2.C briefly takes up Koszul modules, recently introduced and applied to great affect by Aprodu et. al.[4, 5].

## 6.2.A Koszul cohomology of projective space

Let $V$ be a complex vector space of dimension $n+1$, and let $\mathbf{P} = \mathbf{P}(V)$ be the corresponding $n$-dimensional projective space. We start with a result of Green [89, §2] concerning the Koszul cohomology of line bundles on $\mathbf{P}$:

**Theorem 6.2.1 (Green).** *Given $b \geq 0$ write $B = \mathcal{O}_{\mathbf{P}}(b)$, and fix $d \geq 1$. Then for $q \geq 1$ the Koszul cohomology groups of $B$ with respect to $\mathcal{O}_{\mathbf{P}}(d)$ satisfy the vanishing*

$$K_{p,q}\big(\,\mathbf{P}, B\,;\,\mathcal{O}_{\mathbf{P}}(d)\,\big) \;=\; 0$$

*provided that $p \leq b + (q-1)d$.*

**Corollary 6.2.2.** *The line bundle $\mathcal{O}_{\mathbf{P}}(d)$ on $\mathbf{P}$ satisfies property $(N_d)$.* □

In other words, the first $d$ steps of the resolution of the ideal of the $d^{\text{th}}$ Veronese variety are linear.

**Remark 6.2.3.** In view of index shifting (Example 5.1.13), the hypothesis that $b \geq 0$ in 6.2.1 involves no loss in generality. □

The quickest approach to the theorem is via Castelnuovo–Mumford regularity.

**Lemma 6.2.4.** *Let $M_d$ be the kernel bundle on $\mathbf{P}$ associated to $\mathcal{O}_{\mathbf{P}}(d)$, defined by the exact sequence*

$$0 \longrightarrow M_d \longrightarrow S^dV \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}} \longrightarrow \mathcal{O}_{\mathbf{P}}(d) \longrightarrow 0. \qquad (*)$$

*Then $M_d$ is $(-1)$-regular.*

*Proof.* We have to show that $H^i\big(\mathbf{P}, M_d(1-i)\big) = 0$ for $i > 0$. When $i = 1$ this follows from the fact that $M_d$ is the kernel bundle associated to the complete linear series $|\mathcal{O}_{\mathbf{P}}(d)|$. For $i \geq 2$ is it read off from twists of (*). □

*Proof of Theorem 6.2.1.* The theorem follows from the lemma thanks to the good behavior of regularity with respect to wedge products of vector bundles on projective space in characteristic zero. Specifically, the Lemma and Corollary 3.1.17 show that $\Lambda^{p+1}M_d$ is $(p+1)$-regular, and hence $\operatorname{reg}\big(\Lambda^{p+1}M_d \otimes B\big) \leq (p+1)-b$. In particular, this means that if $(q-1)d \;\geq\; (p+1-b)-1$, then

$$H^1\Big(\,\mathbf{P}\,,\Lambda^{p+1}M_d \otimes B \otimes \mathcal{O}_{\mathbf{P}}((q-1)d)\,\Big) \;=\; 0.$$

The stated vanishing of Koszul cohomology groups follows from Theorem 5.2.5. □

**Example 6.2.5 (Syzygies of weight 0).** Assuming (as we may without loss of generality) that $0 \leq b \leq d-1$, the groups $K_{p,0}\big(\mathbf{P}^n, B; \mathcal{O}_{\mathbf{P}^n}(d)\big)$ computing syzygies of weight $q = 0$ are governed by Theorem 5.3.1 and Proposition 5.3.3. Specifically

$$K_{p,0}\big(\mathbf{P}^n, B; \mathcal{O}_{\mathbf{P}^n}(d)\big) \;\neq\; 0 \quad\Longleftrightarrow\quad p \;<\; h^0\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(b)\big). \qquad □$$

**Example 6.2.6.** The conclusion and proof of Theorem 6.2.1 remain valid if $B$ is any vector bundle (or even a coherent sheaf) on $\mathbf{P}$ that is $(-b)$-regular for $b \geq 0$. □

**Remark 6.2.7 (Syzygies of flag manifolds).** Manivel [132] gives a very pleasing extension of Corollary 6.2.2 to homogeneous line bundles on arbitrary flag manifolds. □

The statements of Theorem 6.2.1 and Corollary 6.2.2 are not (expected to be) sharp. In the case of the projective plane, the best-possible bounds were established by Ciliberto and Ottaviani–Paoletti [148]:

**Proposition 6.2.8 (Syzygies of $\mathbf{P}^2$).** *Propterty $(N_k)$ holds for $\mathcal{O}_{\mathbf{P}^2}(d)$ if and only if*

$$k \ \leq \ 3d-3.$$

*Proof.* The essential point is to show that

$$K_{p,2}\big(\mathbf{P}^2; \mathcal{O}_{\mathbf{P}^2}(d)\big) \ = \ 0 \iff p \ \leq \ 3d-3.$$

Writing $r_d = h^0\big(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(d)\big) - 1$, this group is dual to $K_{r_d-2-p,1}\big(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(-3); \mathcal{O}_{\mathbf{P}^2}(d)\big)$, which in turn is isomorphic to

$$K_{r_d-3-p,0}\big(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(d-3); \mathcal{O}_{\mathbf{P}^2}(d)\big).$$

By Theorem 5.3.1 and Proposition 5.3.3 this is non-vanishing if and only if $r_d - 3 - p \leq r_{d-3}$, i.e. if and only if

$$p \ \leq \ \binom{d+2}{2} - \binom{d-1}{2} - 3 \ = \ 3d-3,$$

as claimed. □

Ottaviani and Paoletti conjecture that $(N_k)$ is satisfied in the same range on projective space of any dimension, but as of this writing only partial results are known [?], [?]. Veronese syzygies are discussed at much greater length in Lecture 8.

**Remark 6.2.9 (Nonvanishing for $T^{d+2} M_d$).** The proof of Theorem 6.2.1 actually establishes the vanishing

$$H^1\big(\mathbf{P}, T^{p+1}M_d \otimes \mathcal{O}_{\mathbf{P}}(d)\big) \ = \ 0$$

for the tensor powers of the kernel bundle associated to $\mathcal{O}_{\mathbf{P}}(d)$ when $p \leq d$. In contrast to the statement for wedge powers implicit in the result and conjecture of Ottaviani–Paoletti, this cannot be improved: one has

$$H^1\big(\mathbf{P}, T^{d+2}M_d \otimes \mathcal{O}_{\mathbf{P}}(d)\big) \ \neq \ 0.$$

(It suffices to prove the same non-vanishing for the symmetric power $S^{d+2}M_d$. For this, note first that the restriction of $M_d$ to a line $\mathbf{P}^1 \subseteq \mathbf{P}$ maps onto $\mathcal{O}_{\mathbf{P}^1}(-1)$, from which it follows that $S^{d+2}M_d \otimes \mathcal{O}_{\mathbf{P}}(d+1)$ is not globally generated. On the other hand, the short exact sequence

$$0 \longrightarrow S^{d+2}M_d \longrightarrow S^{d+2}H^0\big(\mathcal{O}_{\mathbf{P}}(d)\big) \otimes \mathcal{O}_{\mathbf{P}} \longrightarrow S^{d+1}H^0\big(\mathcal{O}_{\mathbf{P}}(d)\big) \otimes \mathcal{O}_{\mathbf{P}}(d) \longrightarrow 0$$

shows that the vanishing $H^1\big(\mathbf{P}, S^{d+2}M_d \otimes \mathcal{O}_{\mathbf{P}}(d)\big) = 0$ would imply the $(d+1)$-regularity of $S^{d+2}M_d$.)

In the next subsection we will discuss some applications of these computations to Hodge theoretic questions. For this one needs a strengthening of Theorem 6.2.1, also due to Green [89, Theorem 2.16], for incomplete linear series.

**Theorem 6.2.10.** *Let $W \subseteq H^0\big(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(d)\big)$ be a subspace of codimension $c$ that defines a basepoint-free linear series, and as before put $B = \mathcal{O}_{\mathbf{P}}(b)$ with $b \geq 0$. Then*

$$K_{p,q}\big(\mathbf{P}, B\,;\,W\big) \;=\; 0$$

*when $p \leq b + (q-1)d - c$.*

*Sketch of Proof.* Write $M_W$ for the kernel bundle associated to the linear series $W$. Since $W$ is basepoint-free, $M_W$ sits in a short exact sequence

$$0 \longrightarrow M_W \longrightarrow M_d \longrightarrow U \longrightarrow 0$$

where $U$ is the rank $c$ trivial bundle on $\mathbf{P}$ modeled on the vector space $H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(d))/W$. By splitting the Eagon–Northcott complex ($\mathrm{EN}_{p+1}$) in [128, Appendix B2] into two pieces, one arrives at a long exact sequence

$$\ldots \longrightarrow \Lambda^{p+3+c} M_d \otimes S^2 U^* \longrightarrow \Lambda^{p+2+c} M_d \otimes U^* \longrightarrow \Lambda^{p+1+c} M_d \longrightarrow \Lambda^{p+1} M_W \otimes \det U \longrightarrow 0$$

of bundles on $\mathbf{P}$. Since $\det U = \mathcal{O}_{\mathbf{P}}$ it follows using Lemma 6.2.4 that $\Lambda^{p+1} M_W$ is $(p+1+c)$-regular, and then one concludes as in the proof of Theorem 6.2.1. (Alternatively, by choosing a flag of linear spaces between $W$ and $H^0(\mathcal{O}_{\mathbf{P}}(d))$ each having codimension one in the next, one can argue by a double induction on $c$ and $p$ using only classical Koszul complexes: see [90, §4].) □

## 6.2.B Applications to Hodge theory

One of the motivations underlying Green's papers [88, 89] was to study systematically algebraic questions arising from infinitesimal computations in Hodge theory. This circle of ideas led to many beautiful applications by Green, Voisin, Nori and others; the surveys [90, 92, 189] give nice overviews, and detailed accounts appear for example in the books [35, 191]. To convey the flavor, we sketch here one of the earliest results in this direction, an improvement of the classical Noether–Lefschetz theorem from [89].

Working on $\mathbf{P} = \mathbf{P}^3$, let $U_d \subseteq |\mathcal{O}_{\mathbf{P}}(d)|$ denote the open set parameterizing all smooth surfaces of degree $d \geq 4$. A classical theorem of Noether and Lefschetz states that there are countably many proper subvarieties $Z_{\alpha,d} \subseteq U_d$ parametrizing surfaces $X \subseteq \mathbf{P}^3$ whose Picard groups are not generated by the hyperplane class $h_X$ of $X$:

$$\mathrm{Pic}(X) \supsetneq \mathbf{Z} \cdot h_X.$$

The question arises: how big can these Noether–Lefschetz $Z_{\alpha,d}$ loci be? Green's result is:

**Theorem 6.2.11** ([89], Theorem 4.1)**.** *Assume that $d \geq 4$. If $Z \subseteq U_d$ is an irreducible component of the Noether–Lefschetz locus, then*

$$\operatorname{codim}_{U_d} Z \ \geq \ d-3.$$

The statement is optimal: the locus of surfaces containing a line has codimension $= d-3$. In fact, Green and Voisin show elsewhere [188] that these are precisely the components of the Noether–Lefschetz locus having maximal dimension.

The theorem is established by combining infinitesimal line of reasoning pioneered in [34] with Theorem 6.2.10. Referring to [191, Chapters 5.3, 6.2] or [35, Chapter 7.5] for the precise argument, we give an informal explanation of the idea. Let $X = X_0 \subseteq \mathbf{P}^3$ be a smooth surface of degree $d$ that carries a line bundle $L = L_0$ that is not the restriction of a divisor on $\mathbf{P}^3$. Then $L$ defines a non-zero primitive class

$$\gamma \ = \ c_1(L) \ \in \ H^{1,1}_{\text{prim}}(X; \mathbf{Z}).$$

Now imagine moving $X$ in a family $\pi : \mathcal{X} \longrightarrow S$ of smooth surfaces in $\mathbf{P}^3$: so $X = \pi^{-1}(0)$ for some $0 \in S$. Writing $X_t = \pi^{-1}(t)$, the integral cohomology groups $H^2(X_t, \mathbf{Z})$ are locally identified via the Gauss–Manin connection, and by the Lefschetz $(1,1)$-theorem, $L$ deforms in a family $L_t$ of holomorphic line bundles if and only if $\gamma$ remains of type $(1,1)$ under the deformation of $X$, i.e.

$$\gamma \ \in \ H^{1,1}_{\text{prim}}(X_t, \mathbf{Z}) \ = \ H^{1,1}_{\text{prim}}(X_t, \mathbf{C}) \cap H^2(X, \mathbf{Z}). \tag{6.2.1}$$

The idea of [34] is that there is a first-order obstruction to this holding.

Specifically, denote by $\Theta_X$ the tangent bundle of $X$, so that $H^1\big(X, \Theta_X\big)$ parametrizes first-order deformations of the complex structure of $X$. Cup product and trace defines a homomorphism

$$H^1\big(X, \Theta_X\big) \otimes H^1\big(X, \Omega^1_X\big) \longrightarrow H^2\big(X, \mathcal{O}_X\big) \tag{6.2.2}$$

that gives the first-order obstruction to the class $\gamma \in H^{1,1}_{\text{prim}}(X, \mathbf{C})$ remaining of type $(1,1)$ under a deformation of $X$ corresponding to $\theta \in H^1(X, \Theta_X)$. Using the duality $H^{0,2}(X) = H^{2,0}(X)^*$, (6.2.2) gives rise to a mapping

$$H^1(X, \Theta_X) \otimes H^{2,0}(X) \longrightarrow H^{1,1}_{\text{prim}}(X)^*, \tag{6.2.3}$$

and if (6.2.1) holds to first order, then $\gamma : H^{1,1}_{\text{prim}}(X)^* \longrightarrow \mathbf{C}$ vanishes on the image of $\theta \otimes H^{2,0}(X)$.

So far we have not used in any essential way that $X$ is a surface of degree $d$ in $\mathbf{P}^3$; this comes into the picture via a basic computation going back to Griffiths that interprets the Hodge groups of $X$ via its Jacobian algebra. Write $S$ for the homogeneous coordinate ring of $\mathbf{P}^3$, let $J_X \subseteq S$ be the ideal generated by the four partials of the defining equation of $X$, and put

$$R \ = \ R_X \ = \ S/J_X.$$

Then $R$ is a finite-dimensional Gorenstein algebra with socle in degree $4d-8$, i.e. $R_{4d-8} = \mathbf{C}$ and multiplication $R_a \otimes R_{4d-8-a} \longrightarrow R_{4d-8}$ gives a duality $R_a = R^*_{4d-8-a}$.

**Lemma 6.2.12.** *One has*

$$H^1(X,\Theta_X) \cong R_d \quad , \quad H^{2,0}(X) \cong R_{d-4}$$
$$H^{1,1}_{\text{prim}}(X) \cong H^{1,1}_{\text{prim}}(X)^* \cong R_{2d-4}.$$

*Moreover under these isomorphisms, the map* (6.2.3) *corresponds to multiplication*

$$R_d \otimes R_{d-4} \longrightarrow R_{2d-4}.$$

For the proof see for instance [191, §6.1.3] or [35, §3.2].

We now indicate the proof of Theorem 6.2.11. Let $Z \subseteq U_d$ be an irreducible component of the Noether–Lefschetz locus having codimension $c$, and fix a smooth point $0 \in Z$ corresponding to a surface $X = X_0 \subseteq \mathbf{P}^3$. By the discussion above, the tangent space $T_0Z$ determines a subspace

$$\overline{W} \subseteq H^1(X,\Theta_X) = R_d$$

of codimension $c$ having the property that

$$\overline{W} \otimes R_{d-4} \longrightarrow R_{2d-4}$$

is not surjective. Pulling back to the polynomial ring, we find a subspace $W \subseteq S_d$ of codimension $c$ for which $W \otimes S_{d-4} \longrightarrow S_{2d-4}$ fails to surject. In other words,

$$K_{0,1}\big(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}}(d-4); W\big) \neq 0.$$

But $W$ is basepoint-free since it contains the Jacobian ideal, and hence it follows from Theorem 6.2.10 that $c > d-4$.

## 6.2.C Koszul modules

In an important series of papers [4,5], Aprodu, Farkas, Papadima, Raicu and Weyman have introduced and studied an interesting construction starting from a subspace of alternating 2-tensors on a vector space. They show that these *Koszul modules* have surprisingly many applications, ranging from statements in geometric group theory to the solution of a long-standing problem on the syzygies of the tangent developable surface of a rational normal curve leading to a new proof of Voisin's theorem on canonical curves (see Section 7.4). We show here how the sort of argument appearing in Section 6.2.A leads to a quick proof of their basic vanishing theorem. We recommend [6] for a nice survey of some of the applications of this theory.

Let $V$ be a complex vector space of dimension $n+1$, and let $K \subseteq \Lambda^2 V$ be a subspace. The Koszul complex gives an exact sequence $\Lambda^2 V \otimes S^q V \longrightarrow V \otimes S^{q+1} V \longrightarrow S^{q+2} V$. Splicing in $K$ yields a complex

$$K \otimes S^q V \longrightarrow V \otimes S^{q+1} V \longrightarrow S^{q+2} V. \tag{6.2.4}$$

We denote by

$$W_q \;=\; W_q(V,K)$$

the cohomology of (6.2.4): this is the (degree $q$ piece of) the Koszul module determined by $K$. One says that $K \subseteq \Lambda^2 V$ has *vanishing resonance* if

$$K^{\perp} \cap \mathbf{G}(V,2) \;=\; 0,$$

where $\mathbf{G}(V,2)$ is the Grassmanian of decomposable two-forms in $\Lambda^2 V^{\vee}$.

One of the key technical results of the theory is a vanishing theorem for non-resonant Koszul modules:

**Theorem 6.2.13** ([5], Theorem 3.1)**.** *Assume that $K \subseteq \Lambda^2 V$ has vanishing resonance. Then*

$$W_q(V,K) \;=\; 0 \;\; \textit{for} \;\; q \;\geq\; n-2.$$

*Proof.* Write $\mathbf{P} = \mathbf{P}(V)$, and denote by $M = \Omega^1_{\mathbf{P}}(1)$ the rank $n$ kernel bundle of $\mathcal{O}_{\mathbf{P}}(1)$. Recalling that $H^0\big(\mathbf{P}, M(1)\big) = \Lambda^2 V$, we get a natural map

$$\alpha : K \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}} \longrightarrow M(1),$$

and the hypothesis that $K$ is non-resonant means exactly that $\alpha$ is surjective. Let $J = \ker(\alpha)$, so that $J$ sits in a short exact sequence

$$0 \longrightarrow J \longrightarrow K \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}} \longrightarrow M(1) \longrightarrow 0. \tag{*}$$

Then $W_q = H^1\big(\mathbf{P}, J(q)\big)$, so the issue is to prove that this group vanishes when $q \geq n-2$.

The next step is to use the Eagon–Northcott complex $(\mathrm{EN})_1$ [128, Appendix B] to construct a resolution of $J(-1)$ from (*). Write $Q = M^*$, and note that $S^{\ell}Q = (S^{\ell}M)^*$ since we are in characteristic zero. The resulting resolution takes the form

$$0 \longleftarrow J(-1) \longleftarrow \Lambda^{n+1}K \otimes \mathcal{O}_{\mathbf{P}}(-n-1) \otimes \det Q \longleftarrow \Lambda^{n+2}K \otimes \mathcal{O}_{\mathbf{P}}(-n-2) \otimes Q \otimes \det Q \longleftarrow \ldots,$$

the $\ell^{\text{th}}$ term being

$$P_{\ell} \;=\; \Lambda^{n+\ell}K \otimes \mathcal{O}_{\mathbf{P}}(-n-\ell) \otimes \det Q \otimes S^{\ell-1}Q.$$

Now $\det Q = \mathcal{O}_{\mathbf{P}}(1)$ and $Q$ is 0-regular (from Euler sequence), so $P_{\ell}$ is $(n+\ell-1)$ regular. Therefore $J(-1)$ is $n$-regular, and the required vanishing follows. □

## 6.3 Abelian varieties ⋄

This section will survey a body of work [150], [151], [?] by Pareschi, Popa and others concerning syzygies of ablian varieties. It has not yet been written.

## 6.4 Hyper-adjoint bundles

The natural generalization to arbitrary dimensions of large degree line bundles on curves came into focus only in the 1980s. The observation is that if $C$ is a curve of genus $g$, then a divisor $D$ on $C$ has degree $\geq 2g-2+m$ if and only if $D \equiv_{\text{lin}} K_C + mA$ for some ample divisor $A$ on $C$. So on an arbitrary smooth projective variety $X$ of dimension $n$, one is led to consider *adjoint divisors*, i.e. those of the form $K_X + P$ for suitably positive divisors $P$. In particular, Mukai suggested that one should study the syzygetic properties of such bundles.

Except in dimension two – where a theorem of Reider [170] applies (see also [127]) – it is not known as of this writing that $K_X + P$ is very ample if we require simply that $P$ be a small multiple of an ample divisor. So it is unrealistic at the moment to ask refined algebraiic questions. On the other hand, we saw already in Proposition 3.2.10 that divisors of this shape become very easy to analyze if we ask that $P$ be a multiple of a very ample, or an ample and globally generated, divisor. While the terminology is non-standard, we will refer to these as *hyper-adjoint* bundles to emphasize the very strong positivity conditions we impose. The present section is devoted to some results about their syzygies. In the first subsection we discuss the main results. The second is devoted to complements.

### 6.4.A Syzygies of hyper-adjoint embeddings

Throughout this subsection, $X$ denotes a smooth complex projective variety of dimension $n$. The first result is due to the authors [51]:

**Theorem 6.4.1.** *Let $B$ be a very ample divisor on $X$, and write*

$$L_d \;=\; \mathcal{O}_X(K_X + dB + P) \quad , \quad N_f \;=\; \mathcal{O}_X(K_X + fB + Q)$$

*where $P$ and $Q$ are nef.*[1] *Then $L_d$ satisfies Property $(N_k)$ when $d \geq n+1+k$. More generally, if $d \geq n+1$ then*

$$K_{p,1}(X, N_f; L_d) \;=\; 0 \quad \textit{when} \quad f \;\geq\; (n+1)+p.$$

Note that when $X = \mathbf{P}^n$ and $B$ is the hyperplane divisor, these statements reduce to Theorem 6.2.1.

Recall (Proposition 3.2.10 and Example 3.2.11) that a divisor of the form

$$K_X + (n+1)B + (\text{nef})$$

is free. So by dimension shifting and absorbing terms into $N_f$, the result for $K_{p,1}$ gives information also for other Koszul groups. For instance, one finds that

$$K_{p,q}(X; L_d) \;=\; 0 \;\text{ for }\; (q-1)d \;\geq\; (q-1)(n+1)+p.$$

[1] Recall that a divisor is nef its intersection with every effective curve is $\geq 0$. As explained in Section 4.1.A, such divisors should be seen as "non-negative:" they are limits of ample ones. See [128, Chapter 1.4] for a detailed discussion.

It is natural to ask whether one can weaken the requirement that $B$ be very ample. Lacini and Purnaprajna [13] have recently established a nice generalization of the main case of 6.4.1:

**Theorem 6.4.2.** *Assuming only that $B$ is ample and globally generated, $\mathcal{O}_X(K_X + dB)$ satisfies $(N_k)$ when $d \geq n + 1 + k$.*

The proofs of the theorems are somewhat involved, so instead of discussing the general statements we'll focus here on the normal generation of $\mathcal{O}_X(K_X + (n+1)B + P)$ i.e. the case $k = 0$. As we'll see momentarily, there is a very quick proof of this instance of Theorem 6.4.1, going back to [24]. However we'll also present two longer arguments to illustrate the approaches of [51] and [13]. One preliminary remark is in order before proceeding. Namely, if $(X, B) = (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$, then $K_X + (n+1)B \equiv_{\text{lin}} 0$, in which case $(N_0)$ has to be understood as surjectivity of trivial multiplication maps. However it's known that in every other case $K_X + (n+1)B$ is very ample when $B$ is.

**Normal generation via Kawamata–Viehweg vanishing.** Assuming that $B$ is very ample, the normal generation $\mathcal{O}_X(K_X + dB + P)$ for $d \geq n+1$ follows quickly from Kawamata–Viehweg vanishing (Theorem 4.1.5), much as in the proof of Theorem 4.1.1. The starting point is:

**Lemma 6.4.3.** *Denote by $\mathcal{I}_\Delta \subseteq \mathcal{O}_{X\times X}$ the ideal sheaf of the diagonal $\Delta \subseteq X \times X$. Then*

$$\big(\mathcal{O}_X(B) \boxtimes \mathcal{O}_X(B)\big) \otimes \mathcal{I}_\Delta$$

*is globally generated.*

*Proof.* Since $B$ is very ample, it suffices to prove this for the hyperplane bundle on $\mathbf{P}^r$, where it is clear. □

With $L_d$ as in 6.4.1, we focus on the vanishing $K_{0,2}(X; L_d) = 0$, which is equivalent to showing that $H^1\big(X \times X, (L_d \boxtimes L_d) \otimes \mathcal{I}_\Delta\big) = 0$. For this, let

$$\mu \,:\, Y \,=_{\text{def}}\, \text{Bl}_\Delta(X \times X) \longrightarrow X \times X$$

be the blowing up of the diagonal, and denote by $E$ the exceptional divisor. The issue is to prove that

$$H^1\big(Y, \mu^*(L_d \boxtimes L_d)(-E)\big) \;=\; 0. \tag{*}$$

Now recall that $K_Y = \mu^*(K_X \dotplus K_X) + (n-1)E$, and note that $\mu^*(B \dotplus B)(-E)$ is free thanks to the Lemma.[2] Since $d \geq n+1$ we see as in the proof of 4.1.1 that

$$\mu^*(L_d \boxtimes L_d)(-E) \;=\; \mathcal{O}_Y\big(K_Y + \mu^*(B \dotplus B) + P'\big),$$

where $P'$ is nef. So (*) follows from vanishing for big and nef divisors.

[2]Recall that we write $D_1 \dotplus D_2$ for the "exterior sum" $\text{pr}_1^* D_1 + \text{pr}_2^* D_2$ of two divisors.

**Normal generation via Le Potier vanishing.** Although one can get a weaker statement using Green's multi-diagonal criterion (Proposition 6.1.5), Theorem 6.4.1 does not as far as we know follow directly from vanishing theorems for divisors. Instead, the proof in [51] draws on vanishing theorems originally due to Le Potier for vector bundles. We explain the approach by reproving the normal generation of $L_d$, always assuming that $B$ is very ample and that $d \geq n+1$. Writing $M_d$ for the kernel of the evaluation map $H^0(L_d) \otimes_{\mathbf{C}} \mathcal{O}_X \longrightarrow L_d$, we focus as above on showing that $H^1\big(X, M_d \otimes L_d\big) = 0$.

Turning to details, consider the projective embedding

$$X \subseteq \mathbf{P} = \mathbf{P}^r$$

defined by $B$, so that $\mathcal{O}_X(1) = \mathcal{O}_X(B)$. Via extension by zero, we view $L_d$ as coherent sheaf on $\mathbf{P}$. Since $d \geq n+1$, it follows from Kodaira vanishing (Theorem 3.2.9) that

$$H^i\big(\mathbf{P}, L_d \otimes \mathcal{O}_{\mathbf{P}}(-i)\big) \ = \ 0 \quad \text{for } i \ \geq \ 0.$$

In other words, $L_d$ is 0-regular as an $\mathcal{O}_{\mathbf{P}}$-module. Therefore (Proposition 3.1.14) $L_d$ admits a locally free resolution $P_\bullet$

$$\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\ \varepsilon\ } L_d \longrightarrow 0,$$

where $P_i = V_i \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}}(-i)$ for suitable vector spaces $V_i$ with $V_0 = H^0\big(X, L_d\big)$. Note that the restriction $\varepsilon|X$ of $\varepsilon$ to $X$ is the evaluation map, so $M_d = \ker(\varepsilon|X)$.

Now consider the complex $Q_\bullet = P_\bullet \otimes_{\mathcal{O}_{\mathbf{P}}} L_d$. This is a complex of vector bundles on $X$, but it is not acyclic: its homology are the sheaves

$$\mathcal{H}_i(Q_\bullet) \ = \ \mathcal{T}or_i^{\mathcal{O}_{\mathbf{P}}}\big(L_d, L_d\big) \ = \ \Lambda^i \mathcal{N}^* \otimes L_d^{\otimes 2},$$

where $\mathcal{N}$ is the normal bundle of $X$ in $\mathbf{P}^r$. (Here we are using the fact (cf. [112, Chapter 11.1]) that if $L$ and $L'$ are any line bundles on $X$, then $\mathcal{T}or_i^{\mathcal{O}_{\mathbf{P}}}\big(L, L'\big) = \Lambda^i \mathcal{N}^* \otimes L \otimes L'$.) On the other hand, thanks to the right-exactness of tensor product, we see that $P_1 \otimes L_d$ maps onto $M_d \otimes L_d$. So we arrive at a complex $Q'_\bullet$ of bundles on $X$ having the shape:

$$\ldots \longrightarrow Q_2 \longrightarrow Q_1 \xrightarrow{\ \delta\ } M_d \otimes L_d \longrightarrow 0,$$

where $Q_i = V_i \otimes_{\mathbf{C}} L_{d-i}$, $\delta$ is surjective, and

$$\mathcal{H}_i \ = \ \mathcal{H}_i(Q'_\bullet) \ = \ \Lambda^i \mathcal{N}^* \otimes L_d^{\otimes 2}$$

for $i \geq 1$. A diagram chase reveals that to conclude the desired vanishing $H^1(M_d \otimes L_d) = 0$ it suffices to show:

$$H^i\big(X, Q_i\big) \ = \ 0 \ \text{ for } 1 \ \leq i \ \leq n \tag{6.4.1}$$

$$H^{i+1}\big(X, \mathcal{H}_i\big) \ = \ 0 \ \text{ for } 1 \ \leq i \ \leq n-1. \tag{6.4.2}$$

Since $d \geq n+1$, (6.4.1) follows directly from Kodaira. For (6.4.2), the plan is to invoke vanishing theorems for vector bundles.

Specifically, observe (from the adjunction formula) that

$$\det\big(\mathcal{N} \otimes B^*\big) \;=\; \mathcal{O}\big(K_X + (n+1)B\big).$$

Therefore, writing $e = \operatorname{rank}(\mathcal{N})$ and recalling that $d \geq n+1$, we find:

$$\begin{aligned} \Lambda^i \mathcal{N}^* \otimes L_d^{\otimes 2} \;&=\; \Lambda^i(\mathcal{N}^* \otimes B) \otimes L_{d-i} \otimes L_d \\ &=\; \Lambda^{e-i}(\mathcal{N} \otimes B^*) \otimes L_{d-i} \otimes U, \end{aligned}$$

where $U$ is a nef line bundle. But $\mathcal{N} \otimes B^*$ is globally generated (since $T\mathbf{P}(-1)$ is), and therefore (6.4.2) follows from the following version of Le Potier vanishing:

**Theorem 6.4.4.** *Let $E$ be a globally generated vector bundle of rank $e$ on $X$, and let $A$ be an ample divisor. Then*

$$H^j\big(X, \Lambda^p E \otimes \mathcal{O}_X(K_X + A)\big) \;=\; 0 \quad \textit{for} \;\; j \,>\, e \,-\, p.$$

See [128, Chapter 7.3, especially Example 7.3.16] for a discussion and proof.

**Remark 6.4.5.** Keeping notation as above, the main technical result of [51] is the vanishing

$$H^j\big(X, (M_d)^{\otimes s} \otimes N_f\big) \;=\; 0$$

for $j \geq 1$ provided that $d \geq n+1$ and $f \geq n+1+s-j$. Theorem 6.4.1 follows from the case $j = 1$ and $s = p+1$. For the proof, one considers the $s$-fold tensor product of the restriction $P_\bullet \otimes \mathcal{O}_X$ of $P_\bullet$ to $X$: the homology of this is computed using the Künneth formula. After twisting by $N_f$ one is reduced to a variant of Le Potier's theorem involving tensor products of the $\Lambda^i \mathcal{N}^*$.

**Normal generation via linkage.** The argument just completed makes crucial use of the hypothesis that $B$ is very ample. In their extension (Theorem 6.4.2) assuming only that $B$ is ample and globally generated, Lacini and Purnaprajna [13] start instead by using $B$ to define a branched covering

$$\phi : X \longrightarrow \mathbf{P}^n \quad \text{with} \quad \mathcal{O}_X(B) \;=\; \phi^*\mathcal{O}_{\mathbf{P}^n}(1).$$

They then study the subvariety of $X \times X$ linked to $\Delta_X$ by the pullback of the diagonal $\Delta_{\mathbf{P}^n}$ under $\phi \times \phi : X \times X \longrightarrow \mathbf{P}^n \times \mathbf{P}^n$. To illustrate the idea, we briefly sketch a third proof of normal generation of $\mathcal{O}_X(K_X + (n+1)B)$ along these lines. The argument will involve an additional simplifying hypothesis, which however can always be arranged if one starts with very ample $B$.

We begin by recalling and generalizing from Section 4.1.B some of the constructions around linkage. Let $M$ be a smooth variety of dimension $m$ and let $U$ be a vector bundle of rank $e \leq m$ on $M$. Suppose that we are given a section $s \in \Gamma\big(M, U\big)$ whose zero-locus $Z = \text{Zeroes}(s)$ contains a smooth subvariety $X \subseteq M$ of codimension $e$. In happy situations $Z = X \cup Y$ will consist of the union of $X$ and another subvariety $Y \subseteq M$ also of codimension $e$: we say then that $s$ *links* $X$ to $Y$.

As in Lecture 4, there is a natural notion of what it means for this linkage to be generic. Namely, consider the blowing-up

$$\mu : M' \,=\, \text{Bl}_X(M) \longrightarrow M$$

of $M$ along $X$, with exceptional divisor $E \subseteq M'$. Then $s$ gives rise to a section

$$s' \,\in\, \Gamma\big(M', (\mu^*U)(-E)\big),$$

and we ask that $Y' =_{\text{def}} \text{Zeroes}(s') \subseteq M'$ be a smooth subvariety of codimension $e$ mapping birationally to $Y$. Assuming that this holds, the analogue of Theorem 4.1.15 is that the Koszul complex of $s$ gives rise to a long exact sequence

$$\begin{aligned} 0 \longrightarrow \omega_M \longrightarrow U \otimes \omega_M \longrightarrow \Lambda^2 U \otimes \omega_M \longrightarrow \dots \\ \dots \longrightarrow \Lambda^{e-1} U \otimes \omega_M \longrightarrow \mathcal{I}_{X/M} \otimes \det U \otimes \omega_M \longrightarrow \mu_* \omega_{Y'} \longrightarrow 0 \end{aligned} \tag{6.4.3}$$

of sheaves on $M$. One uses this to study the cohomology of $\mathcal{I}_{X/M}$.

Returning to our basepoint-free ample divisor $B$ on $X$, choose a subspace

$$W \,\subseteq\, H^0\big(X, \mathcal{O}_X(B)\big)$$

of dimension $n+1$ defining a branched covering $\phi : X \longrightarrow \mathbf{P}^n$. We propose to apply the previous discussion on $M = X \times X$, using $X \times_{\mathbf{P}^n} X$ to link the diagonal $\Delta_X$ to a subvariety $Y \subseteq M$. To this end, note that according to a well-known construction of Beilinson (cf. [147, Proof of Theorem 3.1.3]) the diagonal $\Delta_{\mathbf{P}} \subseteq \mathbf{P}^n \times \mathbf{P}^n$ is the zero-locus of a section of $T\mathbf{P}(-1) \boxtimes \mathcal{O}_{\mathbf{P}}(1)$. Writing $Q = \phi^* T\mathbf{P}(-1)$ and

$$U \,=\, Q \boxtimes B,$$

we arrive at a section $s \in \Gamma\big(X \times X, U\big)$ vanishing on the diagonal $\Delta \subseteq X \times X$. We suppose that $s$ defines a generic linkage: if for instance $B$ is very ample, we can arrange for this by choosing $W$ generically. Now $\det U = \mathcal{O}_X(B) \boxtimes \mathcal{O}_X(nB)$, so we twist (6.4.3) by $T =_{\text{def}} \mathcal{O}_X(nB) \otimes \mathcal{O}_X(B)$. A computation with Koszul complexes shows that

$$H^i\big(X \times X, \Lambda^{n-i} U \otimes \omega_{X \times X} \otimes T\big) \,=\, 0 \;\text{ for }\; i > 0,$$

and the higher cohomology of $\omega_{Y'} \otimes \mu^* T$ vanishes thanks to Kawamata–Viehweg vanishing. Thus one reads off from (6.4.3) the required vanishing

$$H^1\big(X \times X, \mathcal{I}_{\Delta/X \times X} \otimes \mathcal{O}_X(K_X + (n+1)B) \boxtimes \mathcal{O}_X(K_X + (n+1)B)\big) \,=\, 0.$$

**Remark 6.4.6.** Lacini and Purnaprajna cannot assume in [13] that $s$ defines a generic linkage. Instead the consider first the case when $\phi : X \longrightarrow \mathbf{P}^n$ is Galois, and then study what happens when one passes to the Galois closure. This involves some delicate analysis of the singularities that can arise.

## 6.4.B Complements.

We summarize briefly some related results and conjectures.

**Multiplication maps on curves.** Let $X$ be a smooth projective curve of genus $g$, and let $L_1$, $L_2$ be line bundles on $X$ Mumford [141] showed many years ago that if $L_1$ and $L_2$ have degrees $\geq 2g+1$ and $\geq 2g$ respectively, then the multiplication map

$$H^0(L_1) \otimes H^0(L_2) \longrightarrow H^0(L_1 \otimes L_2)$$

is surjective. This can be established by the methods of Section 5.4.A, but as Butler observed one can also proceed by proving the stability of the kernel bundle $M_{L_1}$. Then $M_{L_1} \otimes L_2$ is stable of slope $> 2g-2$, and hence has vanishing $H^1$. The advantage of this approach is that it opens the door to studying multiplication maps

$$H^0(E_1) \otimes H^0(E_2) \longrightarrow H^0(E_1 \otimes E_2) \tag{*}$$

for vector bundles of higher rank on $X$. For example, Butler shows in [33] that if $E_1$ is semistable of slope $\mu(E_1) > 2g$ then its kernel bundle $M_{E_1}$ is semistable, and he deduces that (*) is surjective provided that $E_2$ is also semistable of slope $\geq 2g$. This leads to surjectivity statements for multiplications involving adjoint-type line bundles on ruled varieties $\mathbf{P}(E)$ over a curve.

In a related direction, Butler conjectured that if $L$ is a general very ample line bundle on a Brill–Noether general curve, then $M_L$ is stable with one exception. This was recently established by Farkas and Larson [75] after partial results by several other authors. We refer to their paper for references and an account of the history of Butler's conjecture.

**Adjoint syzygies on surfaces.** Let $X$ be a smooth projective surface, and $L$ an ample divisor on $X$. Reider's theorem [170] implies the following:

> Assume that $\big(L^2\big) > 9$ and that $\big(L \cdot C\big) \geq 3$ for every irreducible curve $C \subseteq X$. Then
>
> $$K_X + L \text{ is very ample.}$$
>
> In particular, if $A$ is any ample divisor, then $K_X + 4A$ is very ample.

Mukai asked whether $K_X + L$ is then normally generated, and more generally whether there are analogous statements for higher syzygies on surfaces.

Mukai's problem remains open as of this writing, but in an interesting series of papers, Gallego–Purnaprajna and others have made substantial progress under additional assumptions. For example, it is established in [84, Theorem 1.23] that if $X$ is anticanonical (i.e. $-K_X$ is effective) and $A$ is ample, then $K_X + nA$ satisfies $(N_k)$ if $n \geq k+4$. Other results appear in [12, 81, 83, 104, 145]. We recommend [82] for a survey. In this paper, Gallego and Purnaprajna make the appealing

**Conjecture 6.4.7.** *Assume that $X$ is regular (i.e. $H^1\big(X, \mathcal{O}_X\big) = 0$), and let $L$ be an ample line bundle on $X$. Assume that*

$$\big(L^2\big) \,>\, (k+3)^2 \quad , \quad \textit{and} \quad \big(L \cdot C\big) \,\geq\, k+3$$

*for every effective curve $C \subseteq X$. Then $\mathcal{O}_X(K_X + L)$ satisfies $(N_k)$,*

## 6.5 Notes

Many of the results of this Lecture are accompanied by statements to the effect that various natural coordinate rings are Koszul. Let $R$ be a graded commutative algebra, and put $k = R/R_+$. One says that $R$ is said to be *Koszul* if $\mathrm{Tor}^i_R(k,k)$ has pure degree $k$. For coordinate rings this is quite close to asking that condition $(N_1)$ hold, and the Koszul condition has been established in most of the settings where $(N_1)$ holds. We refer for example to [169], [160], [42] for some statements along these lines.

Concerning syzygies of Veronese varieties, we point to the papers [31] and [30] of Bruns–Conca–Römer for an interesting discussion from a more algebraic perspective.

# Lecture 7

# Syzygies of Curves

This lecture is devoted a circle of results and conjectures concerning the syzygies of smooth projective curves. The theme is that one can expect to see precise and delicate reflections of the intrinsic geometry of a curve in the shape of its resolution under appropriate embeddings.

## 7.1 Overview

We start with an introductory survey of the questions at hand. Throughout this Lecture, $C$ denotes a smooth projective curve of genus $g = g(C) \geq 2$. We denote by $\omega_C$ the canonical bundle of $C$, and by $K_C$ a canonical divisor.

### 7.1.A Canonical curves and their defining equations

One of the most interesting constructions involving $C$ starts with the fact that $\omega_C$ is globally generated, and hence defines a mapping

$$\phi_C : C \longrightarrow \mathbf{P}H^0(\omega_C) = \mathbf{P}^{g-1}.$$

Unless $C$ is hyperelliptic (i.e. it is expressed as a double covering $C \longrightarrow \mathbf{P}^1$), $\phi_C$ is an embedding and realizes $C$ as a *canonical curve*

$$C \subseteq \mathbf{P}^{g-1} \quad \text{of degree } 2g-2.$$

Since this embedding is naturally defined, the intrinsic properties of $C$ must be reflected in the projective geometry of its canonical model. Working this out concretely is an important and enjoyable chapter in the theory of algebraic curves.

A first connection is given by a geometric formulation of the Riemann–Roch theorem:

**Theorem 7.1.1 (Geometric Riemann–Roch).** *Let $D = P_1 + \ldots + P_d$ be an effective divisor of degree $d$ on $C$, and denote by*

$$\overline{D} \;=\; \mathrm{Span}(P_1, \ldots, P_d) \;\subseteq\; \mathbf{P}^{g-1}$$

*the projective subspace it spans.*[1] *Then*

$$\dim \overline{D} \;=\; d - 1 - r(D),$$

*where as usual $r(D) = h^0\big(C, \mathcal{O}_C(D)\big) - 1$.*

So for instance if $C$ admits a degree 3 covering $\pi : C \longrightarrow \mathbf{P}^1$ – in other words, if $C$ is *trigonal* – then each of the divisors $D_\lambda = \pi^{-1}(\lambda)$ spans a line in canonical space. Quite generally, if a canonical curve $C \subseteq \mathbf{P}^{g-1}$ possesses a divisor of degree $d$ spanning a linear space of dimension $d - 1 - r$, then $C$ carries an $r$-dimensional family of such divisors.

*Proof of Theorem 7.1.1.* The (classical) Riemann–Roch theorem states that

$$h^0\big(K_C - D\big) \;=\; g \;-\; (d+1) \;+\; h^0\big(D\big). \tag{*}$$

Since the group $H^0\big(C, \mathcal{O}_C(K_C - D)\big)$ is identified with the space of hyperplanes in $\mathbf{P}^{g-1}$ passing through $D$, the right-hand side of (*) computes the codimension of $\overline{D}$ in $\mathbf{P}^{g-1}$. The assertion follows. □

**Example 7.1.2 (The scroll determined by a pencil).** Suppose that $D$ is a divisor of degree $d$ that moves in a basepoint-free pencil, and hence defines a covering $\pi : C \longrightarrow \mathbf{P}^1$ of degree $d$. Then each of the divisors $D_\lambda = \pi^{-1}(\lambda)$ spans a subspace $\overline{D_\lambda} \subseteq \mathbf{P}^{g-1}$ of dimension $d - 2$. As $\lambda$ varies over $\mathbf{P}^1$ these sweep out a scroll

$$S \;=\; \bigcup_{\lambda \in \mathbf{P}^1} \overline{D_\lambda} \;\subseteq\; \mathbf{P}^{g-1}$$

of dimension $d-1$ containing $C$. Canonically, $S$ is the image of the projective bundle $\mathbf{P}\big(\pi_*\omega_C\big)$ on $\mathbf{P}^1$, its mapping to $\mathbf{P}^{g-1}$ arising from the isomorphism $H^0\big(C, \omega_C\big) = H^0\big(\mathbf{P}^1, \pi_*\omega_C\big)$. □

Turning to the defining equations of canonical curves, there are two classical results:

**Theorem 7.1.3 (Theorems of Noether and Petri).** *Let $C \subseteq \mathbf{P}^{g-1}$ be a non-hyperelliptic canonical curve.*

*(i).* (Noether). *$C$ is projectively normal, i.e. the canonical bundle $\omega_C$ is normally generated.*

*(ii).* (Petri). *The homogeneous ideal $I_{C/\mathbf{P}^{g-1}}$ fails to be generated by quadrics if and only if either $C$ is trigonal or a smooth plane quintic.*

[1] In case of repetitions among the $P_i$, the span is understood as usual to involve the tangent or osculating spaces to $C$ at the points in question.

Concerning the exceptional cases in Petri's statement, observe that there is one family (trigonal curves) that exists in all genera, as well as one sporadic example that occurs only in genus 6. It is immediate that if $C$ is trigonal (or a plane quintic), then $I_{C/\mathbf{P}^{g-1}}$ cannot be generated by quadrics: in the former case, for example, the trisecant lines spanned by trigonal divisors must be contained in any quadric passing through $C$.[2] Thus the essential content of Petri's theorem is that $I_{C/\mathbf{P}^{g-1}}$ is generated by quadrics in all but the stated examples. A proof of Theorem 7.1.3 is sketched in §7.1.D

It is instructive to see explicitly what happens in low genera. In the following, $S$ denotes the coordinate ring $S = \operatorname{Sym} H^0\big(C, \omega_C\big)$ of canonical space, and $R = \oplus\, H^0\big(C, \omega_C^{\otimes m}\big)$; thanks to Noether's theorem, $R = S/I_C$ unless $C$ is hyperelliptic.

**Example 7.1.4** (**Genus 3 and 4**)**.** Assume that $C$ is non-hyperelliptic.

(i). When $g = 3$, the canonical embedding $C \subseteq \mathbf{P}^2$ realizes $C$ as a smooth plane quartic.

(ii). When $g = 4$, $C \subseteq \mathbf{P}^3$ is the complete intersection of a quadric $Q$ and a cubic $F$. The quadric $Q$ is unique, and is smooth or singular depending on whether $C$ carries two or only one pencils of degree three. The Betti table of $R = S/I_C$ has the form:

| | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 1 | – | – |
| 1 | – | 1 | – |
| 2 | – | 1 | – |
| 3 | – | – | 1 |

**Example 7.1.5** (**Genus 5**)**.** Suppose that $C \subseteq \mathbf{P}^4$ is a non-hyperelliptic canonical curve of genus 5. One finds that

$$\dim H^0\big(\mathbf{P}^4, \mathcal{I}_{C/\mathbf{P}^4}(2)\big) \;=\; 3,$$

but now there are two cases.

(i). If $C$ is non-trigonal, then it is the complete intersection of three quadrics. In this case $R = S/I_C$ has the Betti table:

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1 | – | – | – |
| 1 | – | 3 | – | – |
| 2 | – | – | 3 | – |
| 3 | – | – | – | 1 |

(ii). When $C$ is trigonal, the three quadrics through $C \subseteq \mathbf{P}^4$ cut out a surface scroll $S \subseteq \mathbf{P}^4$ of degree three, and there are two linear syzyies among these quadrics. Thefore the homogeneous ideal $I_C$ requires two minimal generators in degree three, and the Betti table of $R = S/I_C$ takes the form:

[2] In fact the intersection of the quadrics through a trigonal canonical curve is precisely the two-dimensional scroll $S \subseteq \mathbf{P}^{g-1}$ that these trisecants sweep out.

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1 | – | – | – |
| 1 | – | 3 | 2 | – |
| 2 | – | 2 | 3 | – |
| 3 | – | – | – | 1 |

We refer the reader to [?] or [?] for a detailed discussion of the geometry of low genus canonical curves. □

Note that these Betti tables have exactly four rows, and they display a symmetry. This is a general fact:

**Proposition 7.1.6 (Betti symmetries of canonical curves).** *Let $C \subseteq \mathbf{P}^{g-1}$ be a non-hyperelliptic canonical curve of genus $g$, and let $R = S/I_C$ be the canonical ring of $C$. Then:*

*(i).* $K_{p,q}(C;\omega_C) \;=\; 0$ *for* $q > 3$ *or* $p > g-2$*;*

*(ii).* $\dim K_{0,0}(C;\omega_C) = \dim K_{g-2,3}(C;\omega_C) = 1$*, while all other* $K_{p,0}$ *and* $K_{p,3}$ *vanish;*

*(iii). One has*

$$\dim K_{p,1}(C;\omega_C) \;=\; \dim K_{g-2-p,2}(C;\omega_C) \quad , \quad \dim K_{p,2}(C;\omega_C) \;=\; \dim K_{g-2-p,1}(C;\omega_C).$$

Graphically, this means that the row corresponding to $q = 1$ in the Betti table read from left to right has the same entries as those in the $q = 2$ row read from right to left. This is illustrated schmatically in Figure 7.1. Observe that the Proposition implies that the grading of the minimal free resolution of $I_{C/\mathbf{P}^{g-1}}$ is completely determined by knowing the least integer $k$ for which Property $(N_k)$ fails for the canonical bundle of $C$.

*Proof of Proposition 7.1.6 .* Statement (i) is a consequence of the fact that $C \subseteq \mathbf{P}^{g-1}$ is 4-regular, while (ii) follows from (iii). As for (iii), duality (Theorem 5.1.18) and index shifting yield

$$\dim K_{p,1}(C;\omega_C) \;=\; \dim K_{g-2-p,1}(C,\omega_C;\omega_C) \;=\; \dim K_{g-2-p,2}(C;\omega_C),$$

as required. □

### 7.1.B Green's conjecture

The theorems of Noether and Petri can be rephrased as saying that if $C$ is a smooth projective curve of genus $g$, then:

(a). Property $(N_0)$ fails for the canonical bundle $\omega_C$ if and only if $C$ is hyperelliptic;

| | 0 | 1 | 2 | ... | $g-4$ | $g-3$ | $g-2$ |
|---|---|---|---|---|---|---|---|
| 0 | 1 | – | – | ... | – | – | – |
| 1 | – | □ | □ | ... | ○ | ○ | – |
| 2 | – | ○ | ○ | ... | □ | □ | – |
| 3 | – | – | – | ... | – | – | 1 |

Figure 7.1: Schematic illustataion of the Betti table of a canonical curve

(b). If $C$ is non-hyperelliptic, then Property $(N_1)$ fails for the canonical bundle $\omega_C$ if and only if $C$ trigonal or a smooth plane quintic.

If one hopes to extend these classical statements to higher syzygies, the first step is to understand the pattern behind the exceptional cases. Green realized that the right invariant to look at is the *Clifford index* of $C$.

Recall the classical theorem of Clifford: If $A$ is a line bundle on $C$ with $h^0\big(C, A\big) \geq 2$ and $h^1\big(C, A\big) \geq 2$, then

$$\deg(A) \ - \ 2 \cdot r(A) \ \geq \ 0.$$

Moreover equality holds if and only if $C$ is hyperelliptic and $A$ is a multiple of the hyperelliptic pencil. This motivates:

**Definition 7.1.7 (Clifford index).** Let $C$ be a smooth projective curve, and $A$ a line bundle on $C$. The *Clifford index* of $A$ is defined to be

$$\text{Cliff}(A) \ = \ \deg(A) \ - \ 2 \cdot r(A).$$

The Clifford index of $C$ itself is

$$\text{Cliff}(C) \ = \ \min \big\{ \text{Cliff}(A) \mid h^0(A) \geq 2, h^1(A) \geq 2 \big\}. \qquad \square$$

Thus $\text{Cliff}(C) = 0$ if and only if $C$ is hyperelliptic, and similarly it follows from a theorem of Mumford that $\text{Cliff}(C) = 1$ if and only if $C$ is trigonal or a smooth plane quintic. At the other extreme, if $C$ is a *general* curve of genus $g$, then

$$\text{Cliff}(C) \ = \ \left[ \frac{g-1}{2} \right].$$

See [66] for more information about this invariant.

These considerations led Green [88] to make the celebrated

**Conjecture 7.1.8 (Green's conjecture on canonical curves).** *The Clifford index of $C$ is equal to the least integer $k$ for which Property $(N_k)$ fails for its canonical bundle $\omega_C$.* $\square$

By duality, this is equivalent to the assertion that

$$K_{p,1}(C;\omega_C) \;=\; 0 \iff p \;\geq\; g-1-\mathrm{Cliff}(C). \tag{7.1.1}$$

As of this writing Green's conjecture remains open, but it has been the focus of a huge amount of work. We will summarize some of this in Section 7.4, but for now we mention just a few highlights. To begin with, note that Riemann–Roch implies that

$$r(A) \,+\, r(\omega_C \otimes A^*) \;=\; g \,-\, 1 \,-\, \mathrm{Cliff}(A)$$

for any line bundle $A$ on $C$. Therefore it follows from Theorem 5.3.4 that

$$K_{p,1}(C;\omega_C) \;\neq\; 0 \;\text{ when } p \;<\; g-1-\mathrm{Cliff}(C).$$

In other words:

**Proposition 7.1.9.** *Property $(N_k)$ fails for the canonical bundle $\omega_C$ when $k \leq \mathrm{Cliff}(C)$.* □

(There are elementary direct proofs, e.g. [7, Corollary 3.39], avoiding the general case of 5.3.4.) Thus the essential content of Conjecture 7.1.8 is the assertion that $(N_k)$ holds for $k < \mathrm{Cliff}(C)$.

Schreyer [172] and Voisin [187] proved the first non-classical case of Green's conjecture, namely that if $\mathrm{Cliff}(C) \geq 2$, then the canonical model of $C$ satisfies $(N_2)$. But by far the most important progress towards Conjecture 7.1.8 is Voisin's theorem that it holds for a general curve of genus $g$:

**Theorem 7.1.10 (Voisin, [190,192]).** *Let $C$ be a general curve of genus $g$, so that $\mathrm{Cliff}(C) = \left[\frac{g-1}{2}\right]$, and put $k = \mathrm{Cliff}(C) - 1$. Then property $(N_k)$ holds for the canonical bundle of $C$.*

One can view this as asserting that the resolution of $I_{C/\mathbf{P}^{g-1}}$ is "as pure as possible" subject to the constraints of Proposition 7.1.6. Specifically, if $g$ is odd, then for each $1 \leq p \leq g-3$ only one of $K_{p,1}$ and $K_{p,2}$ is non-zero. The same holds for even $g$ except that

$$\dim K_{p,1} \;=\; \dim K_{p,2} \;\neq\; 0 \quad \text{for} \quad p = \left(\frac{g-2}{2}\right).$$

Voisin's proof of Theorem 7.1.10 started with a new interpretation of syzygies involving the geometry of symmetric products or Hilbert schemes. In a tour de force, she was then able to carry out computations on the Hilbert scheme of a $K3$ surface required to establish the result. Subsequently Kemeny [117] gave a considerably simpler argument that we reproduce (for even genus) in §7.3.B. Other approaches are summarized in Section 7.4.

| | 0 | 1 | $\cdots$ | $(d-2g-1)$ | $(d-2g)$ | $\cdots$ | $(d-g-1)$ |
|---|---|---|---|---|---|---|---|
| 0 | 1 | – | $\cdots$ | – | – | $\cdots$ | – |
| 1 | – | $*$ | $\cdots$ | $*$ | $\diamond$ | $\cdots$ | $\diamond$ |
| 2 | – | – | $\cdots$ | – | $*$ | $\cdots$ | $*$ |

Figure 7.2: Betti table of curve of large degree

## 7.1.C Curves of large degree

Now we turn to curves of large degree. Continuing to assume that $g = g(C) \geq 2$, let $L = L_d$ be a line bundle on $C$ of degree $d \gg 0$, defining an embedding $C \subseteq \mathbf{P}^{d-g}$. Recall from Section 5.4.A that Green's Theorem 5.4.3 and Proposition 5.3.3 imply that

$$K_{p,2}(C; L) \neq 0 \iff d - 2g \leq p \leq d - g - 1,$$

and hence $K_{p,1}(C; L) \neq 0$ for $1 \leq p < d - 2g$. This fixes the vanishing or non-vanishing of all but $g$ entries of the Betti table of $C$. Specifically, the only undecided issue is to analyze when

$$K_{p,1}(C; L) = 0 \text{ for } p \text{ in the range } d - 2g \leq p \leq d - g - 1. \tag{7.1.2}$$

Figure 7.2 reproduces from Section 5.4.A a schematic summary of what this says: the entries with an asterisk are non-zero, whereas a diamond indicates a Koszul group whose vanishing or non-vanishing isn't so far determined.

Now recall:

$$K_{p,q}(C; L) \text{ and } K_{d-g-1-p,2-q}(C, \omega_C; L) \text{ are dual.} \tag{7.1.3}$$

So the issue in (7.1.2) is to determine for which values of $p$ in the range $0 \leq p \leq g - 1$ it happens that

$$K_{p,1}(C, \omega_C; L) = 0.$$

These groups in turn govern the beginning of the resolution of the *Arbarello–Sernesi* module of $C$, which leads to a more geometric perspective.

To wit, define

$$\mathrm{AS} = \mathrm{AS}(C; L) = \oplus_{m \geq 0} H^0\big(C, \omega_C \otimes L^{\otimes m}\big).$$

This is in the natural way a graded module over $S = \mathrm{Sym}\, H^0\big(C, L\big)$. The Arbarello–Sernesi module has $g$ generators in degree zero, given by a basis of holomorphic one-forms on $C$. Then

$$K_{0,1}(C, \omega_C; L) = \mathrm{coker}\Big(H^0(\omega_C) \otimes H^0(L) \longrightarrow H^0(\omega_C \otimes L)\Big),$$

so $K_{0,1}(C, \omega_C; L)$ measures the failure of AS to be generated in degree 0.[3] If $K_{0,1}(C, \omega_C; L) = 0$, then $K_{1,1}(C, \omega_C; L) = 0$ if and only of AS has a linear presentation

$$S(-1)^N \longrightarrow S^g \longrightarrow \mathrm{AS} \longrightarrow 0,$$

[3] As we are assuming that $d \gg 0$, it is elementary that generators for $M$ could occur only in degrees 0 and 1.

where $N = g \cdot h^0(L) - h^0(\omega_C \otimes L)$. In general, the vanishing

$$K_{p,1}(C, \omega_C; L) \;=\; 0 \quad \text{for} \quad 0 \leq p < p_0$$

is equivalent to the linearity of the first $p_0$ steps of the resolution of AS.

Always supposing that $d \gg 0$, Green showed in [?] that $K_{0,1}(C, \omega_C; L) = 0$, and he proved moreover that

$$K_{1,1}(C, \omega_C; L) \neq 0 \iff C \text{ is hyperelliptic.}$$

Inspired by Conjecture 7.1.8, it is natural to ask what is the least value of $p$ for which $K_{p,1}(C, \omega_C; L) \neq 0$. An elementary argument (Example 7.1.16) shows that if $C$ admits a branched covering $C \longrightarrow \mathbf{P}^1$ of degree $\leq p+1$, then $K_{p,1}(C, \omega_C; L) \neq 0$. This led Green and the second author to propose

**Conjecture 7.1.11** (**Gonality conjecture,** [95]). *If $d = \deg(L)$ is sufficiently large, then*

$$K_{p,1}(C, \omega_C; L) \;=\; 0 \iff p+1 \;<\; \operatorname{gon}(C).$$ [4]

The conjecture was verified for general curves (and more) by Aprodu and Voisin. Somewhat surprisingly, it later turned out that there is a very quick proof of a statement that implies 7.1.11.

To explain this, we start with

**Definition 7.1.12.** Given a line bundle $B$ on the curve $C$, one says that $B$ is *p-very ample* if the natural map

$$H^0\big(C, B\big) \longrightarrow H^0\big(C, B \otimes \mathcal{O}_\xi\big)$$

is surjective for every effective divisor $\xi$ on $X$ of degree $p+1$.

In other words, one asks that sections of $B$ separate any $p+1$ (not necessarily distinct) points on $C$. So for instance, $B$ is 1-very ample if and only it is very ample in the usual sense.

It follows from the Geometric Riemann–Roch Theorem 7.1.1 that

$$\omega_C \text{ is } p\text{-very ample} \iff \operatorname{gon}(C) \;>\; p+1$$

Therefore Conjecture 7.1.11 is a consequence a result of the present authors:

**Theorem 7.1.13** ([53]). *Fix a line bundle $B$ on $C$. Then for any line bundle $L$ of sufficiently large degree:*

$$K_{p,1}(C, B; L) \;=\; 0 \iff B \text{ is } p\text{-very ample.}$$

[4] Recall that by definition, the *gonality* of $C$ is the least degree of a covering $C \longrightarrow \mathbf{P}^1$.

The proof appears in the next section: it revolves around a small modification of the Hilbert-schematic interpretation of syzygies that Voisin introduced in connection with Theorem 7.1.10.

One would of course like to have an effective bound on the degree of $L$ in order that the conclusion of the Theorem hold. Rathmann [168] proved a very clean statement:

**Theorem 7.1.14** (**Rathmann**, [168]). *In the setting of Theorem 7.1.13, it suffices that*

$$\deg(L) \;\geq\; \max\big\{2g-1\,,\,\deg(B)+2g-1\,\big\}.$$

*In particular, the conclusion of Conjecture 7.1.11 holds as soon as* $d = \deg(L) \geq 4g-3$.

Returning to the Koszul groups $K_{p,1}(C;L)$ discussed at the beginning of the subsection, 7.1.11 and (7.1.3) imply that if $d = \deg(L) \gg 0$, then

$$K_{p,1}(C;L) \,=\, 0 \iff d-g-\mathrm{gon}(C) \,<\, p \leq d-g-1. \tag{7.1.4}$$

In fact, thanks to Rathmann's theorem, it suffices that $d \geq 4g-3$. In particular, one finds the amusing

**Corollary 7.1.15.** *The gonality of a curve is determined by the grading of the minimal resolution of the ideal* $I_C \subseteq S$ *of* $C$ *for the embedding*

$$C \,\subseteq\, \mathbf{P}^{d-g}$$

*defined by any one line bundle of sufficiently large degree.* □

**Example 7.1.16** (**Non-vanishing for weight one syzygies**). For curves of small gonality, the non-vanishing predicted by Conjecture 7.1.11 is easy to see. In fact, suppose that $C$ admits a branched covering $\pi : C \longrightarrow \mathbf{P}^1$ of degree $c$. Then in the embedding $C \subseteq \mathbf{P}^{d-g}$ defined by a line bundle $L$ of degree $d \gg 0$, the linear spans of the fibres of $\pi$ sweep out a scroll $S \subseteq \mathbf{P}^{d-g}$ of dimension $c$ containing $C$. The Eagon–Northcott resolution of $S$ (Example 1.3.23) shows that

$$K_{p,1}(S;\mathcal{O}_S(1)) \,\neq\, 0 \;\text{ for }\; 1 \,\leq\, p \,\leq\, d-g-c.$$

These are syzygies of weight one among the quadrics defining $S$, which in particular persist as syzygies among the quadrics through $C$. Thus $K_{p,1}(C;L) \neq 0$ for $p$ in the same range. By duality, this shows that

$$K_{p,1}(C,\omega_C;L) \,\neq\, 0 \;\text{ for }\; c-1 \,\leq\, p \,\leq\, d-g-2. \qquad \square$$

## 7.1.D The theorems of Noether and Petri

We briefly indicate here a proof of Theorem 7.1.3 following the paper [96] of Green and the second author, to which we refer for details. The argument is rather special, but we would be remiss not to include at least a sketch of the classical results motivating Green's conjecture.

The argument revolves around the kernel bundle $M = M_{\omega_C}$ associated to $\omega_C$, and its dual $Q = M^*_{\omega_C}$, which sits in the exact sequence

$$0 \longrightarrow T_C \longrightarrow H^0(\omega_C)^* \otimes \mathcal{O}_C \longrightarrow Q \longrightarrow 0.$$

It follows from Serre duality that

$$K_{0,2}(C;\omega_C) = 0 \iff H^0(\omega_C)^* \longrightarrow H^0(Q) \text{ is surjective} \tag{7.1.5}$$
$$K_{1,2}(C;\omega_C) = 0 \iff \Lambda^2 H^0(\omega_C)^* \longrightarrow H^0(\Lambda^2 Q) \text{ is surjective.} \tag{7.1.6}$$

Since both maps on the right are in any event injective, the issue is to estimate the dimensions of $H^0(Q)$ and $H^0(\Lambda^2 Q)$ under the hypotheses of the Theorem. We will do this using the secant constructions introduced for the second proof of Green's Theorem 5.4.3, summarized in Figure 5.2.

For Noether's theorem, assume that $C$ is non-hyperelliptic, so that the canonical mapping is an embedding. Therefore if we take $D = x_1 + \ldots + x_{g-2}$ to be the sum of $(g-2)$ general points, then $\omega_C(-D)$ is base-point free. So as in (5.4.1) we get an exact sequence

$$0 \longrightarrow T_C(D) \longrightarrow M \longrightarrow \bigoplus_{i=1}^{g-2} \mathcal{O}_C(-x_i) \longrightarrow 0.$$

Dualizing and noting that $h^0(\omega_C(-D)) = 2$, we see that

$$h^0(Q) \;\leq\; (g-2) + 2 \;=\; g,$$

as required.

The proof of Petri's statement is a little more involved. When $g = 4$ the statement is elementary, so we take $g \geq 5$. The starting point is

**Lemma 7.1.17.** *Assume that $g \geq 5$ and that $C$ is not one of the exceptional curves in Theorem 7.1.3. Then $C$ carries an effective divisor $D$ of degree $g-1$ having the property that both $D$ and $K_C - D$ move in basepoint-free pencils.*

For the proof see [11, p. 373] or [96, §4]. We may suppose that $D = x_1 + \ldots + x_{g-1}$ consists of $(g-1)$ distinct points spanning a linear space of dimension $g-3$ in canonical space $\mathbf{P}^{g-1}$, and that any $(g-3)$ of the $x_i$ are in linear general position.

Writing $\Sigma_D$ for the secant bundle determined by $D$, we consider the exact sequence $0 \longrightarrow T_C(D) \longrightarrow M \longrightarrow \Sigma_D \longrightarrow 0$ and its dual

$$0 \longrightarrow \Sigma_D^* \longrightarrow Q \longrightarrow \omega(-D) \longrightarrow 0.$$

The plan is to take $\Lambda^2$ of this and bound the dimensions of the resulting spaces of global sections. To this end, set $D' = x_1 + \ldots + x_{g-3}$ and $E = x_{g-2} + x_{g-1}$. Then

$$\Sigma_{D'} \;=\; \bigoplus_{i=1}^{g-3} \mathcal{O}_C(-x_i),$$

and one has an exact sequence $0 \longrightarrow \mathcal{O}_C(-E) \longrightarrow \Sigma_D \longrightarrow \Sigma_{D'} \longrightarrow 0$. It follows first of all that

$$\begin{aligned} h^0\big(\Lambda^2\Sigma_D^*\big) \ &\le \ h^0\big(\Lambda^2\Sigma_{D'}^*\big) \ + \ h^0\big(\Sigma_{D'}(E)\big) \\ &= \ \binom{g-3}{2} \ + \ g-3. \end{aligned}$$

Similarly, we find that

$$\begin{aligned} h^0\big(\Sigma_D^* \otimes \omega(-D)\big) \ &\le \ h^0\big(\Sigma_{D'}^* \otimes \omega(-D)\big) \ + \ h^0\big(\omega(-D+E)\big) \\ &= \ 2\cdot(g-3) \ + \ 3. \end{aligned}$$

So all told, one sees that

$$\begin{aligned} h^0\big(\Lambda^2 Q\big) \ &\le \ \binom{g-3}{2} + 3\cdot(g-3) + 3 \\ &= \ \binom{g}{2}, \end{aligned}$$

and we are done.

**Remark 7.1.18.** Green [88] gives a different proof of Petri's statement, deducing it from his syzygetic generalization of Castelnuovo's lemma (Theorem 5.3.6).

## 7.2 The gonality theorem

In the course of her work [190] on Green's conjecture, Voisin realized that one could study syzygies via symmetric products or Hilbert schemes. This leads to geometric questions that can be more approachable than computing the cohomology of large wedge powers of a kernel bundle as in §5.2. The proof of Theorem 7.1.13, which unfolds on the symmetric product of $C$, provides a particularly simple illustration of Voisin's idea.[5]

### 7.2.A Symmetric products of curves

We review in this subsection some basic facts about the geometry of symmetric products of curves.

Let $C$ be a smooth projective curve of genus $g$. The symmetric group $S_k$ acts on the $k$-fold product $C^{\times k}$ by permuting the factors. The quotient

$$C_k \ = \ \mathrm{Sym}^k C \ =_{\mathrm{def}} \ C^{\times k}/S_k$$

[5] Voisin's arguments take place on the Hilbert scheme of a suitable $K3$ surface $X$. Interestingly, Kemeny's simplified proof (§7.3.B) returns to powers of a kernel bundle on $X$. The relevance of $K3$ surfaces to these questions is explained in §7.3.A.

is the $k^{\text{th}}$ *symmetric product* of $C$. We denote by $\pi_k : C^{\times k} \longrightarrow C$ the quotient map. This symmetric product is a smooth projective variety of dimension $k$.

We view $C_k$ as parameterizing effective divisors of degree $k$ on $C$. The mapping

$$\sigma_k : C \times C_{k-1} \longrightarrow C_k \quad , \quad \sigma_k(x, \xi') \;=\; x + \xi'$$

then realizes $C \times C_{k-1}$ as the universal cycle of degree $k$ over $C_k$. More precisely, one has the commutative diagram

$$\begin{array}{ccccc} & & C & & \\ & \overset{\mathrm{pr}_1}{\nearrow} & & \overset{\mathrm{pr}_1}{\nwarrow} & \\ C \times C_{k-1} & & \xhookrightarrow{\;\; j_k \;\;} & & C \times C_k \\ & \underset{\sigma_k}{\searrow} & & \underset{\mathrm{pr}_2}{\swarrow} & \\ & & C_k & & \end{array} \tag{7.2.1}$$

where $j_k(x, \xi') = (x, x + \xi')$. Via $j_k$, $\sigma_k^{-1}(\xi)$ is identified with the subscheme $\xi \subseteq C$. Observe that $\sigma_k$ is a finite flat branched covering of degree $k$.

A line bundle $B$ on $C$ gives rise to a *tautological vector bundle* $E_B$ on $C_k$. Specifically, put

$$E_B \;=\; E_{k,B} \;=_{\text{def}}\; \sigma_{k,*}\big(\mathrm{pr}_1^* B\big). \tag{7.2.2}$$

Thus $E_B$ is a vector bundle of rank $k$ whose fibre at a point $\xi \in C_k$ is canonically identified with the vector space $H^0\big(C, B \otimes \mathcal{O}_\xi\big)$. Note that

$$H^0\big(C_k, E_B\big) \;=\; H^0\big(C \times C_{k-1}, \mathrm{pr}_1^* B\big) \;=\; H^0\big(C, B\big).$$

Thus there is a canonical homomorphism

$$\mathrm{ev} \;:\; H^0\big(C, B\big) \otimes_{\mathbf{C}} \mathcal{O}_{C_k} \longrightarrow E_B \tag{7.2.3}$$

of vector bundles on $C_k$ that fiber by fibre is identified with the mapping

$$H^0\big(C, B\big) \longrightarrow H^0\big(C, B \otimes \mathcal{O}_\xi\big).$$

In particular, one finds:

**Lemma 7.2.1.** *The homomorphism* (7.2.3) *is surjective if and only if $B$ is* $(k-1)$*-very ample in the sense of Definition 7.1.12.* □

Next, we observe that there are two interesting line bundles $\mathcal{S}_L$ and $\mathcal{N}_L$ on $C_k$ determined by a line bundle $L$ on $C$. To begin with, consider on $C^{\times k}$ the $k$-fold exterior product

$$L^{\boxtimes k} \;=_{\text{def}}\; \mathrm{pr}_1^* L \otimes \ldots \otimes \mathrm{pr}_k^* L$$

of $k$ copies of $L$, one from each factor. Then there is a unique line bundle $\mathcal{S}_L = \mathcal{S}_{k,L}$ on $C_k$ such that

$$L^{\boxtimes k} \;=\; \pi_k^* \, \mathcal{S}_L.$$

For the other, we define

$$\mathcal{N}_L \;=\; \mathcal{N}_{k,L} \;=_{\mathrm{def}}\; \det E_{k,L}.$$

The following basic result computes the sections of these bundles.

**Proposition 7.2.2.** *Let $L$ be any line bundle on the curve $C$. Then:*

$$H^0\big(C_k, \mathcal{S}_{k,L}\big) \;=\; \mathrm{Sym}^k\, H^0\big(C, L\big). \tag{7.2.4}$$

$$H^0\big(C_k, \mathcal{N}_{k,L}\big) \;=\; \Lambda^k\, H^0\big(C, L\big). \tag{7.2.5}$$

*Proof.* It follows from the construction that

$$H^0\big(C_k, \mathcal{S}_{k,L}\big) \;=\; H^0\big(C^k, L^{\boxtimes k}\big)^{S_k} \;=\; \mathrm{Sym}^k\, H^0\big(C, L\big),$$

which gives (7.2.4). For (7.2.5), note first that taking wedge products in (7.2.3) determines a natural mapping

$$\Lambda^k\, H^0\big(C, L\big) \longrightarrow H^0\big(C_k, \mathcal{N}_{k,L}\big);$$

we will show that this is an isomorphism. To this end we assert:

**Claim 7.2.3.** There is a generically injective mapping of vector bundles

$$\pi_k^*\, E_{k,L} \longrightarrow \oplus_{i=1}^k\, \mathrm{pr}_i^* L \tag{7.2.6}$$

on $C^{\times k}$ having the property that the composition

$$H^0\big(C, L\big) \;=\; H^0\big(C_k, E_L\big) \hookrightarrow H^0\big(C^{\times k}, \pi_k^* E_L\big) \longrightarrow \oplus_{i=1}^k\, H^0\big(C, L\big)$$

is given by the diagonal mapping $s \mapsto (s, \ldots, s)$.

Granting this, we get maps

$$\Lambda^k\, H^0\big(C, L\big) \longrightarrow H^0\big(C_k, \mathcal{N}_L\big) \hookrightarrow \otimes\, H^0\big(C, L\big)$$

that identify $H^0\big(C_k, \mathcal{N}_L\big)$ with the space of alternating tensors, and (7.2.5) follows.

Turning to the Claim, denote by $\pi^i : C^{\times k} \longrightarrow C \times C_{k-1}$ the quotient by the subgroup $S_{k-1} < S_k$ that fixes the $i^{\text{th}}$ factor. Writing $\sigma^i : C \times C_{k-1} \longrightarrow C_k$ for summation, and $\mathrm{pr}_1^i : C \times C_{k-1} \longrightarrow C$, we find a surjection $(\sigma^i)^* E_L \longrightarrow (\mathrm{pr}_1^i)^* L$. This pulls back to

$$\pi_k^*\, E_{k,L} \longrightarrow \mathrm{pr}_i^* L,$$

which gives the $i^{\text{th}}$ component of (7.2.6). The second assertion of the Claim follows from the construction. □

Although this doesn't play a direct role in the gonality theorem, it is instructive to work out these construction concretely for $C = \mathbf{P}^1$. (Some of these computations will however come up in the work of Park discussed in Section 8.2.) In this case we have $C_k = \mathbf{P}^k$, viewed as the projective space $\mathbf{P}_{\text{sub}}\, H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(k)\big)$ parametrizing homogeneous forms of degree $k$ on $\mathbf{P}^1$ up to scalars. The tautological divisor

$$\mathbf{P}^1 \times \mathbf{P}^{k-1} \ \subseteq \ \mathbf{P}^1 \times \mathbf{P}^k$$

arises as the zeroes of a section of $\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^k}(k,1)$, and $\sigma : \mathbf{P}^1 \times \mathbf{P}^{k-1} \longrightarrow \mathbf{P}^k$ is given by the restriction of the second projection $\mathrm{pr}_2 : \mathbf{P}^1 \times \mathbf{P}^k \longrightarrow \mathbf{P}^k$. Now fix $b$ and set $B = \mathcal{O}_{\mathbf{P}^1}(b)$. Then

$$E_{k,B} \ = \ \mathrm{pr}_{2,*}\big(\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^{k-1}}(b,0)\big).$$

This in turn may be computed by pushing forward to $\mathbf{P}^k$ the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^k}(b-k,-1) \longrightarrow \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^k}(b,0) \longrightarrow \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^{k-1}}(b,0) \longrightarrow 0,$$

and we find

**Proposition 7.2.4.** *Assume that $b \geq k-1$. Then $E_{k,B}$ sits in an exact sequence*

$$0 \longrightarrow H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(b-k)\big) \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}^k}(-1) \longrightarrow H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(b)\big) \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}^k} \longrightarrow E_{k,B} \longrightarrow 0$$

*of vector bundles on $\mathbf{P}^k$.*

Thus $E_{k,B}$ is a rank $k$ vector bundle on $\mathbf{P}^k$, often called a *Schwartzenberger bundle*. These bundles have attracted a considerable amount of attention over the years: see for instance [?], [?], [?].

Next, consider $L = \mathcal{O}_{\mathbf{P}^1}(d)$. Then

$$\mathcal{S}_{k,L} \ = \ \mathcal{O}_{\mathbf{P}^k}(d), \tag{7.2.7}$$

and provided that $d \geq k-1$ one finds from 7.2.4 that

$$\mathcal{N}_{k,L} \ = \ \mathcal{O}_{\mathbf{P}^k}(d-k). \tag{7.2.8}$$

These identifications in turn have an interesting consequence for representations of $\mathrm{SL}(2,\mathbf{C})$. Specifically, write $\mathbf{P}^1 = \mathbf{P}(U)$ for a two-dimensional vector space $U$ and fix a trivialization of $\Lambda^2 U$. Then everything in this discussion becomes $\mathrm{SL}(U)$-equivariant, and we can identify $\mathrm{Sym}^k U$ with $\mathrm{Sym}^k U^*$, so that $\mathbf{P}^k = \mathbf{P}\big(\mathrm{Sym}^k U\big)$. Proposition 7.2.2, (7.2.7), and (7.2.8) therefore yield:

**Corollary 7.2.5** (**Hermite reciprocity**)**.** *There are canonical isomorphisms*

$$\begin{aligned} \mathrm{Sym}^d\,\mathrm{Sym}^k(U) \ &= \ \mathrm{Sym}^k\,\mathrm{Sym}^d(U) \\ \mathrm{Sym}^{d-k}\,\mathrm{Sym}^k(U) \ &= \ \Lambda^k\,\mathrm{Sym}^d(U) \end{aligned}$$

*of* $\mathrm{SL}(U)$*-modules.* □

## 7.2.B Proof of Theorem 7.1.13

We now turn to the proof of Theorem 7.1.13 and some related results. The argument revolves around a small modification of Voisin's techniques in [190], the idea being to use Proposition 7.2.2 to interpret syzygies via line bundles on a symmetric product in place of large wedge powers of vector bundles on the curve itself.

Fix then line bundles $B$ and $L$ on $C$, and an integer $p \geq 0$. Recall that $K_{p,1}(C, B; L)$ is the cohomology of the complex

$$\Lambda^{p+1} H^0(L) \otimes H^0(B) \longrightarrow \Lambda^p H^0(L) \otimes H^0(B \otimes L) \longrightarrow \Lambda^{p-1} H^0(L) \otimes H^0(B \otimes L^{\otimes 2}) \ ;$$

we wish to realize this group geometrically on a suitable symmetric product $C_k$. For this, take $k = p+1$, and twist (7.2.3) by $\mathcal{N}_L = \mathcal{N}_{p+1,L}$. We arrive at a mapping

$$\mathrm{ev}_{B,L} \ : \ H^0\big(C, B\big) \otimes_{\mathbf{C}} \mathcal{N}_L \longrightarrow E_B \otimes \mathcal{N}_L, \tag{7.2.9}$$

and thanks to (7.2.5) the $H^0$ of the term on the left is isomorphic to $\Lambda^{p+1} H^0(L) \otimes H^0(B)$. The crucial point, essentially due to Voisin [?], is:

**Proposition 7.2.6.** *The global sections of $E_B \otimes \mathcal{N}_L$ are identified with the group of Koszul cycles*

$$H^0\big(C_{p+1}, E_B \otimes \mathcal{N}_L\big) \ = \ Z_{p,1}(C, B; L).$$

*Moreover, $K_{p,1}(B, L)$ is realized as the cokernel of map $H^0(\mathrm{ev}_{B,L})$ arising from* (7.2.9)*:*

$$K_{p,1}(C, B; L) \ = \ \mathrm{coker}\Big(H^0\big(C, B\big) \otimes H^0\big(C_{p+1}, \mathcal{N}_L\big) \longrightarrow H^0\big(C_{p+1}, E_B \otimes \mathcal{N}_L\big)\Big).$$

*Proof.* With notation as in (7.2.1), it follows from the projection formula and the constructions that

$$\begin{aligned} H^0\big(C_{p+1}, E_B \otimes \mathcal{N}_L\big) \ &= \ H^0\big(C \times C_p \, , \, \mathrm{pr}_1^* B \otimes \sigma_{p+1}^* \mathcal{N}_L\big) \\ &= \ H^0\big(C \times C_p \, , \, j_{p+1}^* (\mathrm{pr}_1^* B \otimes \mathrm{pr}_2^* \mathcal{N}_L) \, \big). \end{aligned}$$

Moreover the map induced by (7.2.9) on global sections is identified with the restriction

$$H^0\big(C \times C_{p+1}, B \boxtimes \mathcal{N}_L\big) \longrightarrow H^0\big(C \times C_{p+1}, \, (B \boxtimes \mathcal{N}_L) | (C \times C_p) \, \big).$$

Now we assert

**Claim 7.2.7.** On $C \times C_p$ one has an isomorphism

$$\sigma_{p+1}^* \big(\mathcal{N}_{p+1,L}\big) \ = \ \big(L \boxtimes \mathcal{N}_{p,L}\big)(-D), \tag{7.2.10}$$

where $D \subseteq C \times C_p$ is the image of $j_p : C \times C_{p-1} \hookrightarrow C \times C_p$.

Granting this it follows that $H^0\big(C\times C_p\,,\, j_{p+1}^*(\mathrm{pr}_1^* B\otimes \mathrm{pr}_2^*\mathcal{N}_L)\,\big)$ is identified with

$$\ker\Big(H^0\big(C\times C_p, (B\otimes L)\boxtimes \mathcal{N}_{p,L}\big)\longrightarrow H^0\big(C\times C_{p-1}, (B\otimes L^{\otimes 2})\boxtimes \mathcal{N}_{p-1,L}\big),$$

and the Proposition follows. As for the Claim, fix any line bundle $B$ on $C$. Arguing as in the derivation of (7.2.6) – by comparing the two maps $C\times C\times C_p \longrightarrow C\times C_{p+1}$ – one finds on $C\times C_p$ a natural homomorphism of vector bundles

$$\sigma_{p+1}^*\, E_{p+1,B} \;\longrightarrow\; \mathrm{pr}_1^* B\,\oplus\,\mathrm{pr}_2^*\, E_{p,B}$$

that drops rank by one at a general point of $D$. The Claim is then established by setting $B=L$ and taking determinants. □

We can now give the proof of the main assertion of Theorem 7.1.13. Specifically, suppose that $B$ is a $p$-very ample line bundle on $C$. Then (Lemma 7.2.1) the evaluation mapping $e_{p+1,B}$ in (7.2.3) is surjective, and therefore its kernel $M_B = M_{p+1,B}$ sits in an exact sequence

$$0\longrightarrow M_B\longrightarrow H^0\big(C,B\big)\otimes_{\mathbf{C}}\mathcal{O}_{C_{p+1}}\longrightarrow E_B\longrightarrow 0.$$

To establish the vanishing of $K_{p,1}(C,B;L)$ it suffices thanks to Proposition 7.2.6 to prove that

$$H^1\big(C_{p+1}, M_B\otimes\mathcal{N}_L\big) \;=\; 0. \tag{*}$$

So the issue is to show that (*) holds when $d=\deg(L)\gg 0$, and this follows from the next Lemma. The non-vanishiing of $K_{p,1}$ when $B$ is not $p$-very ample is analyzed in Proposition 7.2.9.

**Lemma 7.2.8.** *Let $\mathcal{F}$ be an arbitrary coherent sheaf on $C_{p+1}$. There exists an integer $d_0 = d_0(\mathcal{F})$ having the property that if $L$ is any line bundle of degree $d\geq d_0$ on $C$, then*

$$H^i\big(C_{p+1},\mathcal{F}\otimes\mathcal{N}_L\big) \;=\; 0 \;\; \textit{for} \;\; i\,>\,0. \tag{*}$$

*Proof.* As above, let $\pi=\pi_{p+1}: C^{\times(p+1)}\longrightarrow C_{p+1}$ be the quotient map. It suffices to prove the analogous vanishing for the group

$$H^i\big(C^{\times(p+1)},\pi^*(\mathcal{F}\otimes\mathcal{N}_L)\big),$$

since this contains the group appearing in (*) as a summand. Arguing as in the proof of Proposition 7.2.2, one sees that

$$\pi^*\mathcal{N}_L \;=\; \big(L\boxtimes\ldots\boxtimes L\big)(-\Delta),$$

where $\Delta=\sum\Delta_{i,j}\subseteq C^{\times(p+1)}$ is the sum of the pairwise diagonals. So the assertion follows from Serre vanishing on $C^{\times(p+1)}$. Alternatively, one can argue directly on $C_{p+1}$ using the observation that $\mathcal{N}_{L\otimes A}=\mathcal{N}_L\otimes\mathcal{S}_A$: see [53, Lemma 1.2]. □

Finally, we consider what happens when $B$ fails to be $p$-very ample. Define

$$\gamma_p(B) \;=\; \dim \big\{ \xi \in C_{p+1} \mid H^0(B) \longrightarrow H^0(B \otimes \mathcal{O}_\xi) \text{ is not surjective } \big\}.$$

**Proposition 7.2.9.** *There is a polynomial $P(d)$ of degree $\gamma_p(B)$ in $d = \deg(L)$ such that*

$$\dim K_{p,1}(C, B; L) \;=\; P(d) \quad \textit{when } \; d \gg 0.$$

*Sketch of Proof.* Consider the exact sequence

$$0 \longrightarrow \ker(e_{p+1,B}) \longrightarrow H^0\big(C, B\big) \otimes_{\mathbf{C}} \mathcal{O}_{C_{p+1}} \longrightarrow E_B \longrightarrow \operatorname{coker}(e_{p+1,B}) \longrightarrow 0 \qquad (*)$$

of sheaves on $C_{p+1}$. Twisting through by $\mathcal{N}_L$, the previous Lemma implies that all the terms in (*) have vanishing higher cohomology when $d = \deg(L) \gg 0$. It then follows from Propostion 7.2.6 that

$$K_{p,1}(C, B; L) \;=\; \dim H^0\big(C_{p+1}, \operatorname{coker}(e_{p+1,B}) \otimes \mathcal{N}_L\big)$$

for $d \gg 0$. But $\gamma_p(B)$ is the dimension of the support of $\operatorname{coker}(e_{p+1,B})$, and one checks that this $h^0$ is given by a polynomial of degree $\gamma$ in $d = \deg(L)$. (See [53] for details.) □

## 7.2.C Rathmann's theorem

We conclude with a few words about Rathmann's strengthening Theorem 7.1.14 of the gonality theorem. Assuming that $B$ is $p$-very ample, he proves that $K_{p,1}(C, B; L) = 0$ as soon as

$$H^1\big(C, L\big) \;=\; H^1\big(C, L \otimes B^*\big) \;=\; 0.$$

Referring to [168] for details, we content ourselves here with a couple of brief remarks about his strategy.

Rathmann's first step is to rephrase the question on the Cartesian products $C^k$ of $C$ rather than its symmetric products. (Compare Proposition 6.1.5.) Denote by

$$\mathrm{pr}_i \,:\, C^k \longrightarrow C \quad , \quad \pi_i : C^k \longrightarrow C^{k-1}$$

the $i^{\text{th}}$ projection and the map forgetting the $i^{\text{th}}$ factor. Writing $\Delta_{i,j}$ for the codimension one $i = j$ diagonal, consider on $C^{p+1} \times C$ the divisor

$$\Sigma \;=_{\text{def}}\; \sum_{i=1}^{p+1} \Delta_{i,p+2};$$

this plays the role of the universal family over $C^{p+1}$. Then given a line bundle $B$ on $C$, one defines the rank $(p+1)$ vector bundle

$$E'_{p+1,B} \;=\; \pi_{p+2,*}\big(\mathrm{pr}^*_{p+2} \otimes \mathcal{O}_\Sigma\big)$$

on $C^{p+1}$. As before there is a natural map $\text{ev}_{p+1,B} : H^0(B) \otimes_{\mathbf{C}} \mathcal{O}_{C_{p+1}} \longrightarrow E'_{p+1,B}$ that is surjecive provided that $B$ is $p$-very ample, which we henceforth assume. Set

$$M'_{p+1,B} \;=\; \ker(\text{ev}_{p+1,B}) \;=\; \pi_{p+2,*}\,\big(\text{pr}^*_{p+2}(B) \otimes \mathcal{O}_{C^{p+1}\times C}(-\Sigma)\big).$$

Finally, given a line bundle $L$ on $C$, put

$$\mathcal{N}' \;=\; \mathcal{N}'_{p+1,L} \;=\; \big(\otimes_{i=1}^{p+1} \text{pr}^*_i(L)\big) \otimes \mathcal{O}_{C^{p+1}}(-\Delta),$$

where $\Delta$ is the sum of all the pairwise diagonals on $C^{p+1}$. As above, to prove the vanishing of $K_{p,1}(C, B; L)$, it suffices to establish

$$H^1\Big(C^{p+1}, M'_{p+1,B} \otimes \mathcal{N}'_{p+1,L}\Big) \;=\; 0.$$

Rathmann deduces this as a special case of more general statement:

**Theorem 7.2.10.** *Assume that $B$ is $p$-very ample, and that $H^1\big(C, L\big) = H^1\big(C, L \otimes B^*\big) = 0$. Then*

$$H^k\Big(C^{p+1}, \Lambda^j M'_{p+1,B} \otimes \mathcal{N}'_{p+1,L}\Big) \;=\; 0$$

*for every $k, j > 0$.*

His beautiful idea is to argue by increasing induction on $p$ and decreasing induction on $j$. What makes this possible is that working on the Cartesian product gives rise to various relations among the sheaves in play. For example, the kernel bundles of weights $p$ and $p+1$ sit in an exact sequence:

$$0 \longrightarrow M'_{p+1,B} \longrightarrow \pi^*_{p+1} M'_{p,B} \longrightarrow \text{pr}^*_{p+1}(B) \otimes \mathcal{O}_{C^{p+1}}(-\Sigma_{i=1}^{p}\, \Delta_{i,p+1}) \longrightarrow 0$$

of bundles on $C^{p+1}$. Along the way, one encounters some terms where diagonals appear with positive coefficients, which in higher dimensions would cause problems. However Rathmann observes that as one is dealing with curves, these can be made to contribute increased positivity when one takes direct images under suitable projections. These techniques have found other applications as well, for instance in the work [56] of the first author and collaborators on the singularities and syzygies of secant varieties of curves.

## 7.3 General canonical curves

This section is devoted to the proof of Voisin's Theorem 7.1.10 (in the case of even genus) following Kemeny's considerably simplified approach [117]. Both of these authors work on a $K3$ surface in order to be able to get their hands on curves of generic Clifford index, so we start in the first subsection with a review of Brill-Noether theory for curves on a $K3$.

## 7.3.A Brill–Noether theory for curves on a $K3$ surface

Given a smooth projective curve $C$ of genus $g \geq 2$, and an integer $d \geq 1$, denote by $\text{Pic}^d(C)$ the $g$-dimensional abelian variety parameterizing (isomorphism classes of) line bundles of degree $d$ on $C$. A great deal of interesting geometry is captured by the varieties

$$W^r_d(C) \ =_{\text{def}} \left\{ A \in \text{Pic}^d(C) \mid r(A) \geq r \right\}$$

of special linear series on $C$. These are closed subvarieties of $\text{Pic}^d(C)$ having expected dimension

$$\rho = \rho(g, r, d) \ =_{\text{def}} \ g \ - \ (r+1)(g - d + r).$$

They were intensively studied in the late 1970s and 1980s, and the following statement summarizes the work of numerous authors:

**Theorem 7.3.1** (**Brill-Noether Theorem**). *Let $C$ be an arbitrary curve of genus $g$. Then*

$$\rho \geq 0 \implies W^r_d(C) \neq \varnothing;$$
$$\rho > 0 \implies W^r_d(C) \text{ is connected.}$$

*If $C$ is a general curve of genus $g$, then*

$$\dim W^r_d(C) \ = \ \rho(g, r, d)$$

*for all $r$ and $d$. Moreover, $\text{Sing}(W^r_d(C)) = W^{r+1}_d(C)$ and $W^r_d(C)$ is irreducible when $\rho > 0$ on a general curve.*

In particular, on a general curve $C$, $W^r_d(C) = \varnothing$ if $\rho < 0$ (which implies as stated above that $\text{Cliff}(C) = [\frac{g-1}{2}]$).

We refer to [11] for references and a much fuller and more precise discussion. While the results for arbitrary curves ultimately flow from considerations of positivity, the assertions for general curves are of a different nature. The conditions in question being open in families, it suffices to exhibit one Brill–Noether general curve of each genus. But until very recently [9] no concrete examples were known, and the original arguments went via degenerations. However the second author observed in [124] that curves on a $K3$ surface having Picard number $= 1$ behave generally from the point of view of Brill-Noether theory. So it was natural to study Green's conjecture via such surfaces.

Consider then a polarized $K3$ surface $(X, L)$ of genus $g \geq 2$. In other words, $X$ is a smooth projecive surface with $H^1\big(X, \mathcal{O}_X\big) = 0$ and $\omega_X = \mathcal{O}_X$, and $L$ is an ample line bundle on $X$ with

$$c_1(L)^2 \ = \ 2g - 2.$$

Then $L$ is globally generated, defining a morphism $\phi_L : X \longrightarrow \mathbf{P}^g$ whose hyperplane sections are the canonical mappings of the curves $C \in |L|$. The bundle $L$ is very ample unless $\phi_L$

realizes $X$ as a double cover of a surface of minimal degree; this happens exactly when the curves in question are hyperelliptic.

Fix now a smooth curve $C \in |L|$. The plan is to study the Brill–Noether problem on $C$ by building a vector bundle on $X$ starting from a linear series on $C$; this bundle will play the central role in Kemeny's proof. Specifically, consider a line bundle $A$ on $C$ with

$$\deg(A) = d \ , \ r(A) = r.$$

We will assume that both $A$ and $\omega_C \otimes A^*$ are globally generated; this is harmless for Theorem 7.3.1 since removing a base-point from $A$ or $\omega_C \otimes A^*$ lowers the Brill-Noether number. Via extension by zero, we may view $A$ as a globally-generated $\mathcal{O}_X$-module, giving rise to a surjective homomorphism

$$\mathrm{ev} : H^0(A) \otimes_{\mathbf{C}} \mathcal{O}_X \longrightarrow A.$$

We set $F = F_{C,A} = \ker(\mathrm{ev})$. This kernel is locally free since $A$ is locally Cohen-Macaulay of codimension one, and it appears in the basic exact sequence:

$$0 \longrightarrow F_{C,A} \xrightarrow{u} H^0(A) \otimes_{\mathbf{C}} \mathcal{O}_X \longrightarrow A \longrightarrow 0 \tag{7.3.1}$$

of sheaves on $X$. Put $E = E_{C,A} = F^*_{C,A}$. Thus $E_{C,A}$ is a vector bundle of rank $r+1 = h^0(C, A)$ on $X$, whose basic properties are summarized in the following Lemma.[6]

**Lemma 7.3.2.** *One has* $c_1(E) = [C]$ *and* $\deg c_2(E) = d$. *The bundle* $E_{C,A}$ *sits in an exact sequence*

$$0 \longrightarrow H^0(A)^* \otimes_{\mathbf{C}} \mathcal{O}_X \longrightarrow E_{C,A} \longrightarrow \omega_C \otimes A^* \longrightarrow 0, \tag{7.3.2}$$

*and it is globally generated. Moreover* $\mathrm{Hom}(E, \mathcal{O}_X) = 0$.

*Proof.* The homomorphism $u$ in (7.3.1) drops rank exactly on $C$, which shows that $\det F = \mathcal{O}_X(-C)$. One can compute $c_2(E)$ by counting the number of points at which a general map $F \longrightarrow \mathcal{O}_X^r$ drops rank, and one sees from the defining sequence that this happens exactly along the divisor on $C$ of a section of $A$. Recalling that

$$\mathcal{E}xt^1_{\mathcal{O}_X}\big(A, \omega_X\big) \ = \ \omega_C \otimes A^*,$$

(7.3.2) follows by taking the transpose of (7.3.1). Finally, $E$ is certainly globally generated away from $C$ thanks to (7.3.2), but since $\omega_C \otimes A^*$ is basepoint-free by assumption and $H^1\big(X, \mathcal{O}_X\big) = 0$, the same sequence gives global generation also at the points of $C$. Finally, (7.3.1) shows that $H^0(F) = \mathrm{Hom}(E, \mathcal{O}_X) = 0$. □

The essential connection with Brill–Noether theory arises from:

[6] Because of [124] and [139] the $E_{C,A}$ have come to be known in the literature as "Lazarsfeld–Mukai bundles."

**Lemma 7.3.3.** *With $C$ and $A$ as above, one has*

$$\begin{aligned}\chi(X, E^* \otimes E) &= 2 \cdot h^0\big(X, E^* \otimes E\big) - h^1\big(X, E^* \otimes E\big) \\ &= 2 - 2 \cdot \rho(A),\end{aligned}$$

*where $\rho(A)$ denotes the Brill–Noether number $\rho(g, r, d)$ with $r = r(A)$ and $d = \deg(A)$.*

*Proof.* The first equality is a consequence of Serre duality. For the second, Hirzebruch–Riemann–Roch, and the multiplicitivity of Chern characters, gives that

$$\chi(X, E^* \otimes E) \;=\; \int \big(\mathrm{ch}(E^*) \cdot \mathrm{ch}(E) \cdot \mathrm{Td}(X)\big)_2.$$

The stated formula then follows with a computation. □

In his fundamental paper [139], Mukai observed that the well-behaved vector bundles on a $K3$ surface are those that are *simple*, i.e. that have only trivial endomorphisms. In our setting, this is guaranteed by a condition on curves linearly equivalent to $C$.

**Lemma 7.3.4.** *Assume that every curve in the linear series $|L|$ is reduced and irreducible. Then*

$$H^0\big(X, E \otimes E^*\big) \;=\; \mathbf{C} \cdot \mathrm{id}_E.$$

*Sketch of Proof.* This is established by essentially the same argument that proves the simplicity of a stable vector bundle. Suppose to the contrary that $E$ carries an endomorphism $u : E \longrightarrow E$ that is not a scalar multiple of the identity. Take $\lambda \in \mathbf{C}$ to be an eivenvalue of $u(x)$ for some $x \in X$, and set $u' = u - \lambda \cdot \mathrm{id}_E$. Then $u' \neq 0$ and $u'$ drops rank at $x$; therefore $\det u' = 0$, i.e. $u'$ drops rank identically. Now let $N = \mathrm{im}(u')$ and $P = \mathrm{coker}(u')$. Then

$$[C] \;=\; c_1(E) \;=\; c_1(N) + c_1(P).$$

But $N$ and $P$, being quotients of $E$, are represented by effective curves, and one checks using the last assertion of the previous Lemma that neither of their first Chern classes can be trivial. Thus we have exhibited $C$ as being linearly equivalent to a non-trivial sum of effective curves, a contradiction. (For more details, we refer to [124].) □

The hypothesis of Lemma 7.3.4 is certainly satisfied if $C$ generates the Picard group of $X$. Hence combining the previous two lemmas, we find:

**Corollary 7.3.5.** *Assume that*

$$\mathrm{Pic}(X) \;=\; \mathbf{Z} \cdot [C].$$

*Then $\rho(A) \geq 0$ for every line bundle $A$ on $C$ with $A$ and $\omega_C \otimes A^*$ globally generated.* □

As noted above, by removing basepoints, one deduces the same statement for every line bundle $A$ on $C$.

What gives the present discussion its punch is that examples exist in all genera:

**Theorem.** *For every genus $g \geq 2$, there exist polarized $K3$ surfaces $(X, L)$ of genus $g$ with*

$$\mathrm{Pic}(X) \;=\; \mathbf{Z} \cdot [L].$$

This is a well-known consequence of the Hodge theory of $K3$ surfaces: see for instance [?]. So it follows from this discussion that $\rho(A) \geq 0$ for every line bundle $A$ on a general curve $C$ of genus $g \geq 2$. Of course it's not easy when $g$ is large to write down a $K3$ surface of Picard number one. Hence the present approach to Theorem 7.3.1, while non-degenerational, cannot claim to be explicit.

Still assuming that $\mathrm{Pic}(X) = \mathbf{Z} \cdot [L]$, we remark for later reference that when $\rho = 0$ the corresponding bundle $E$ is unique. Specifically, suppose that $r$ and $d$ are integers such that $\rho(g, r, d) = 0$. Consider smooth curves $C, C' \in |L|$, and line bundles $A \in W^r_d(C)$ and $A' \in W^r_d(C')$. We assert that then

$$E_{C,A} \;\cong\; E_{C',A'}.$$

In fact, the computation of Lemma 7.3.3 implies that there exists a non-zero homomorphism $u : E_{C,A} \longrightarrow E_{C',A'}$, which must be an isomorphism as in the proof of Lemma 7.3.4.

For the most part, this is all that's needed for Kemeny's proof. However it may be worthwhile to indicate informally how one deduces the other assertions of the Brill–Noether theorem for curves generating the Picard group of $X$ (and hence for general curves of that genus). Mukai [139] shows that the moduli space $\mathcal{M}$ of simple bundles on $X$ with the invariants of $E_{C,A}$ is smooth of dimension $= 2 \cdot \rho(A)$. Now consider the set of pairs

$$\mathcal{G}^r_d \;=_{\mathrm{def}} \big\{ (E, V) \mid E \in \mathcal{M} \ ,\ V \subseteq H^0(E) \text{ a subspace of dim} = r + 1 \big\}.$$

The projection to $\mathcal{M}$ realizes $\mathcal{G}^r_d$ as an open subset in a Grassmanian bundle, and one finds that $\mathcal{G}^r_d$ is non-singular, with

$$\dim \mathcal{G}^r_d \;=\; g + \rho(g, r, d).$$

On the other hand, the vector bundle homomorphsm $V \otimes_{\mathbf{C}} \mathcal{O}_X \longrightarrow E$ drops rank on a curve $C' \in |C|$. This gives rise to a support morphism

$$\mathcal{G}^r_d \longrightarrow |C|,$$

of relative dimension $\rho(A)$, whose fibre over a general curve $C' \in |C|$ is (a resolution of) $W^r_d(C')$. This (essentially) shows that $\dim W^r_d(C') = \rho$, and that this set is smooth away from $W^{r+1}_d(C')$. For a more formal argument, see for instance [124] or [149].

**Remark 7.3.6 (Wahl mapping).** While curves generating the Picard group of a $K3$ surface are Brill–Noether general, there are other cohomological properties with respect to which they are quite special. The most interesting of these is the so-called Wahl mapping

$$\gamma_C : \Lambda^2 H^0(\omega_C) \longrightarrow H^0(\omega_C^3) \quad , \quad \eta_1 \wedge \eta_2 \mapsto \eta_1 \cdot d\eta_2 - \eta_2 \cdot d\eta_1.$$

Wahl [193] observed that this homomorphism cannot be surjective if $C$ lies on a $K3$ surface (see also [18]). On the other hand, it was established by Ciliberto, Harris and Miranda [40] that $\gamma_C$ is surjective on a general curve of genus $g$ when $g = 10$ or $g \geq 12$. In fact, in their recent paper [10], Arbarello–Bruno–Sernesi show that the failure of $\gamma_C$ to be surjective in appropriate genera comes close to characterizing curves that lie on a $K3$. □

## 7.3.B Kemeny's proof of Voisin's theorem

This subsection is devoted to an outline of Kemeny's proof of Voisin's Theorem 7.1.10 for canonical curves of even genus. Our presentation follows [117, §1]. For the case of odd genus, which is more involved, we refer to that paper.

We start with some preliminary remarks and reductions. To prove Green's conjecture for a general curve of genus $g = 2k$, one needs to exhibit any curve $C$ of that genus with $K_{k,1}(C;\omega_C) = 0$. To this end, fix as in the previous subsection polarized $K3$ surface $(X, L)$ of genus $g = 2k$ with $\mathrm{Pic}(X) = \mathbf{Z}\cdot[L]$. Since the syzygies of the embedding $X \subseteq \mathbf{P}H^0(L) = \mathbf{P}^{2k}$ defined by $L$ restrict to the syzygies of a hyperplane section, it suffices to prove that

$$K_{k,1}(X;L) \ = \ 0.$$

In other words, one needs to prove the exactness of

$$\Lambda^{k+1}H^0(L) \longrightarrow \Lambda^k H^0(L)\otimes H^0(L) \longrightarrow \Lambda^{k-1}H^0(L)\otimes H^0(L^{\otimes 2}). \tag{*}$$

Consider now the kernel bundle $M_L$ on $X$ associated to $L$ (Definition 5.2.1); it appears in the sequence

$$0 \longrightarrow \Lambda^{k+1}M_L \longrightarrow \Lambda^{k+1}H^0(L)\otimes_{\mathbf{C}}\mathcal{O}_X \longrightarrow \Lambda^k M_L \otimes L \longrightarrow 0.$$

Recalling that $H^0\big(X, \Lambda^k M_L \otimes L\big) = Z_{k,1}(X;L)$ and that $H^1(X, \mathcal{O}_X) = 0$, one sees that

$$\text{Exactness of } (*) \quad \Longleftrightarrow \quad H^1\big(X, \Lambda^{k+1}M_L\big) \ = \ 0.$$

Voisin's result therefore follows from:

**Theorem 7.3.7 (Kemeny,** [117]**).** *Let $(X, L)$ be any polarized $K3$ surface of genus $g = 2k$ with $\mathrm{Pic}(X) = \mathbf{Z}\cdot[L]$. Then*

$$H^1\big(X, \Lambda^{k+1}M_L\big) \ = \ 0.$$

The next few pages sketch Kemeny's proof of this result.

**Secant constructions.** We have seen on several occasions – for example in the second proof of Green's Theorem 5.4.3, or in Section 7.1.D – that sheaves arising from secant planes can carry useful information about syzygies. One of the key new ideas of Kemeny's proof is to exploit these constructions in families. However as in [117] it will be helpful by way of motivation to start with the pointwise picture.

Observe to begin with that when $g = 2k$, curves $C \in |L|$ have gonality $= k+1$ and carry finitely many line bundles $A \in W^1_{k+1}(C)$. These give rise as explained in the previous section to a globally generated rank two vector bundle $E = E_{C,A}$ on $X$ (independent of the choice of $C$ and $A$). One has

$$\deg c_2(E) \ = \ k+1 \ \text{ and } \ h^0\big(X, E\big) \ = \ k+2.$$

The hypothesis that $\text{Pic}(X) = \mathbf{Z} \cdot [L]$ implies (by the argument leading to Lemma 7.3.4) that every section of $E$ vanishes at only finitely many points. If $Z_t = \text{Zeroes}(t)$ is the zero-scheme of $t \in H^0(E)$, then $Z_t$ is a cycle of length $k+1$ spanning a linear space of dimension $k-1$ in $\mathbf{P}^{2k} = \mathbf{P}H^0(L)$.

Fix one such section $t \in H^0\big(X, E\big)$, and put

$$U_t \;=\; H^0\big(X, L \otimes \mathcal{I}_{Z_t}\big) \;\;,\;\; W_t \;=\; H^0\big(X, L\big)/U_t.$$

Thus $U_t$ and $W_t$ are vector spaces of dimensions $k{+}1$ and $k$ respectively, and $\mathbf{P}(W_t) \subseteq \mathbf{P}H^0(L)$ is the secant plane spanned by $Z_t$. Writing $V_X$ for the trivial vector bundle on $X$ modeled on a vector space $V$, the natural evaluation maps sit in an exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (U_t)_X & \longrightarrow & H^0(L)_X & \longrightarrow & (W_t)_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L \otimes \mathcal{I}_{Z_t} & \longrightarrow & L & \longrightarrow & L \otimes \mathcal{O}_{Z_t} & \longrightarrow & 0 \end{array} \tag{7.3.3}$$

of sheaves on $X$. Denote by $S_t$ and $\Sigma_t$ the kernels of the first and third vertical maps: these are torsion-free sheaves of rank $k$, and while $\Sigma_t$ isn't locally free we could make it so by first blowing up $Z_t$. In any event, we get an exact sequence:

$$0 \longrightarrow S_t \longrightarrow M_L \longrightarrow \Sigma_t \longrightarrow 0. \tag{7.3.4}$$

Now recall that a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of bundles on a variety determines for any $r \geq 1$ long exact sequences

$$\ldots \longrightarrow \text{Sym}^2 A \otimes \Lambda^{r-2} B \longrightarrow \; A \otimes \Lambda^{r-1} B \longrightarrow \;\; \Lambda^r B \longrightarrow \;\; \Lambda^r C \longrightarrow 0 \tag{7.3.5}$$

We apply this to (7.3.4) with $r = k+1$. If we pretend that that $\Sigma_t$ is locally free, then $\Lambda^{k+1}\Sigma_t = 0$ and this gives:

$$\ldots \longrightarrow \text{Sym}^2 S_t \otimes \Lambda^{k-1} M_L \longrightarrow S_t \otimes \Lambda^k M_L \longrightarrow \Lambda^{k+1} M_L \longrightarrow 0.$$

So we would be reduced to proving vanishings of the form

$$H^1(S_t \otimes \Lambda^k M_L) \;=\; H^2(\text{Sym}^2 S_t \otimes \Lambda^{k-1} M_L) \;=\; \ldots \;=\; 0.$$

On the other hand, one could hope that the Koszul sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\;t\;} E \longrightarrow L \otimes \mathcal{I}_{Z_t} \longrightarrow 0$$

would let us get our hands on $S_t$.

Unfortunately, we don't know whether it is possible implement this plan directly. Instead, Kemeny lets $t$ vary over $H^0\big(X, E\big)$, so that one ends up with a similar picture on $X \times \mathbf{P}^{k+1}$. His beautiful discovery is that here the Künneth formula gives many vanishings for free.

**Globalization.** The next step is to globalize the discussion just completed. Denote by

$$\mathbf{P} \;=\; \mathbf{P}^{k+1} \;=\; \mathbf{P}_{\text{sub}} H^0(E)$$

the projective space of one-dimensional *subspaces* of $H^0\big(X, E\big)$, and write

$$p : X \times \mathbf{P} \longrightarrow X \quad , \quad q : X \times \mathbf{P} \longrightarrow \mathbf{P}$$

for the two projections. We consider the universal zero-scheme

$$Z =_{\text{def}} \big\{(x, s) \mid s(x) = 0\big\} \;\subseteq\; X \times \mathbf{P}.$$

The first projection $Z \longrightarrow X$ is a projective bundle thanks to the global generation of $E$, and the second $Z \longrightarrow \mathbf{P}$ is finite and flat. Put

$$U =_{\text{def}} q^* q_* \big(p^* L \otimes \mathcal{I}_Z\big) \quad , \quad W \;=\; q^* q_*(p^* L) \,\big/\, U.$$

These are vector bundles on $X \times \mathbf{P}$ of ranks $k+1$ and $k$ respectively. Noting that $q^* q_*(p^* L) = H^0(L)_{X \times \mathbf{P}}$ is the trivial bundle modeled on $H^0(L)$, we arrive at the exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & H^0(L)_{X \times \mathbf{P}} & \longrightarrow & W & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & p^* L \otimes \mathcal{I}_Z & \longrightarrow & p^* L & \longrightarrow & p^* L \otimes \mathcal{O}_Z & \longrightarrow & 0 \end{array} \tag{7.3.6}$$

globalizing (7.3.3). Later we will blow up $Z$, but for the moment we make some calculations on $X \times \mathbf{P}$ involving the bundle $U$. (The reader wishing to see how these lead to the theorem could peek ahead at the next paragraph, but we want to emphasize that the actual computations take place on $X \times \mathbf{P}$ prior to any blowing up.)

Specifically, observe that $Z \subseteq X \times \mathbf{P}$ is the zero-locus of a "universal section"

$$q^* \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow p^* E,$$

giving rise to a Koszul complex

$$0 \longrightarrow q^* \mathcal{O}_{\mathbf{P}}(-2) \longrightarrow p^* E \otimes q^* \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow p^* L \otimes \mathcal{I}_Z \longrightarrow 0. \tag{7.3.7}$$

Defining $V = q_*(p^* L \otimes \mathcal{I}_Z)$, we find that $U = q^* V$ where $V$ sits in the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-2) \longrightarrow H^0(E) \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow V \longrightarrow 0 \tag{7.3.8}$$

of bundles on $\mathbf{P}$. (In particular this means that $V = T_{\mathbf{P}}(-2)$, but we don't explicitly need this.) We see also that $R^1 q_*(p^* L \otimes \mathcal{I}_Z) = \mathcal{O}_{\mathbf{P}}(-2)$.

Kemeny's first crucial observation is the following:

**Lemma 7.3.8.** *Consider the composition*

$$\mathrm{Sym}^{k+1}U \hookrightarrow \mathrm{Sym}^k U \otimes U \longrightarrow \mathrm{Sym}^k U \otimes (p^*L \otimes \mathcal{I}_Z),$$

*where the second arrow comes from the left-hand column in* (7.3.6). *Then the induced map*

$$H^k\Big(X \times \mathbf{P}, \mathrm{Sym}^{k+1}U\Big) \xrightarrow{\cong} H^k\Big(X \times \mathbf{P}, \mathrm{Sym}^k U \otimes (p^*L \otimes \mathcal{I}_Z)\Big). \tag{7.3.9}$$

*is an isomorphism.*

*Proof.* Taking symmetric products in (7.3.8) gives for every $\ell \geq 1$ the short exact sequence

$$0 \longrightarrow \mathrm{Sym}^{\ell-1}H^0(E) \otimes \mathcal{O}_{\mathbf{P}}(-\ell-1) \longrightarrow \mathrm{Sym}^{\ell}H^0(E) \otimes \mathcal{O}_{\mathbf{P}}(-\ell) \longrightarrow \mathrm{Sym}^{\ell}(V) \longrightarrow 0 \tag{*}$$

of bundles on $\mathbf{P}$. For $\ell = k$ this shows that $H^i\big(\mathbf{P}, \mathrm{Sym}^k V\big) = 0$ for all $i$, and that

$$H^i\big(\mathbf{P}, \mathrm{Sym}^k V \otimes \mathcal{O}_{\mathbf{P}}(-2)\big) \ = \ 0,$$

for $i \leq k-1$. When $\ell = k+1$, one finds that

$$H^k\big(\mathbf{P}, \mathrm{Sym}^{k+1}V\big) \ = \ \mathrm{Sym}^k\, H^0(E),$$

while the other cohomology groups of $\mathrm{Sym}^{k+1}V$ vanish. Similarly, putting $\ell = k$ and tensoring through by $V$, one sees that the natural map

$$H^k\big(\mathbf{P}, \mathrm{Sym}^{k+1}V\big) \xrightarrow{\cong} H^k\big(\mathbf{P}, \mathrm{Sym}^k V \otimes V\big) \tag{7.3.10}$$

is an isomorphism.

We now use the Leray spectral sequence to compute the homomorphism appearing in (7.3.9). By the projection formula,

$$\begin{aligned} q_*(\mathrm{Sym}^{k+1}U) \ = \ R^2q_*(\mathrm{Sym}^{k+1}U) \ &= \ \mathrm{Sym}^{k+1}V \\ q_*\big(\mathrm{Sym}^k U \otimes (p^*L \otimes \mathcal{I}_Z)\big) \ &= \ \mathrm{Sym}^k V \otimes V \\ R^1q_*\big(\mathrm{Sym}^k U \otimes (p^*L \otimes \mathcal{I}_Z)\big) \ &= \ \mathrm{Sym}^k V \otimes \mathcal{O}_{\mathbf{P}}(-2) \end{aligned}$$

and the remaining direct images vanish. It follows that (7.3.9) is identified with

$$H^k\big(\mathbf{P}, q_*(\mathrm{Sym}^{k+1}U)\big) \longrightarrow H^k\big(\mathbf{P}, q_*\big(\mathrm{Sym}^k U \otimes (p^*L \otimes \mathcal{I}_Z)\big)\big),$$

and we have just seen that this is an isomorphism. □

**Remark 7.3.9.** Note for later reference that a similar (but simpler) computation shows that

$$H^{k+1}\big(X \times \mathbf{P}, \mathrm{Sym}^{k+1}U\big) \ = \ H^{k+1}\big(\mathbf{P}, \mathrm{Sym}^{k+1}V\big) \ = 0.$$

The Künneth theorem particularly comes into the argument via:

**Lemma 7.3.10.** *Let $F$ be any vector bundle on $X$. Then*

$$H^i\big(X \times \mathbf{P}, p^*F \otimes \mathrm{Sym}^i U\big) \;=\; 0 \quad \text{for } 1 \le i \le k.$$

*If in addition $H^0\big(X, F\big) = 0$, then*

$$H^{i-1}\Big(X \times \mathbf{P}, p^*F \otimes \mathrm{Sym}^{i-1}U \otimes (p^*L \otimes \mathcal{I}_Z)\Big) \;=\; 0 \quad \text{for } 1 \le i \le k.$$

*Proof.* We pull back to $X \times \mathbf{P}$ the sequence (*) in the proof of Lemma 7.3.8 to get:

$$0 \longrightarrow \mathrm{Sym}^{\ell-1} H^0(E) \otimes q^*\mathcal{O}_{\mathbf{P}}(-\ell-1) \longrightarrow \mathrm{Sym}^{\ell} H^0(E) \otimes q^*\mathcal{O}_{\mathbf{P}}(-\ell) \longrightarrow \mathrm{Sym}^{\ell}(U) \longrightarrow 0. \quad (**)$$

The first assertion follows from tensoring through by $p^*F$ and applying Künneth. For the second, tensor the case $\ell = i-1$ of (**) by the short exact sequence

$$0 \longrightarrow p^*F \otimes q^*\mathcal{O}_{\mathbf{P}}(-2) \longrightarrow p^*(E \otimes F) \otimes q^*\mathcal{O}_{\mathbf{P}}(-1) \longrightarrow p^*F \otimes p^*L \otimes \mathcal{I}_Z \longrightarrow 0.$$

One finds that that the sheaf $P =_{\mathrm{def}} p^*F \otimes \mathrm{Sym}^{i-1}U \otimes (p^*L \otimes \mathcal{I}_Z)$ in question admits a resolution

$$0 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P \longrightarrow 0,$$

where $P_a$ $(0 \le a \le 2)$ is a bundle on $X \times \mathbf{P}$ having the shape

$$P_a \;=\; V_a \otimes_{\mathbf{C}} p^*(F_a) \otimes q^*\mathcal{O}_{\mathbf{P}}(-i-a)$$

for some vector spaces $V_a$, and bundles $F_a$ on $X$ with $F_2 = F$. The assertion again follows from Künneth: the hypothesis on $F$ comes in when $i = k$ to guarantee that $H^{k+1}\big(X \times \mathbf{P}, P_2\big) = 0$. □

**Conclusion of the proof.** We are now ready to finish the proof of Theorem 7.3.7. The idea is to blow up along $Z$ so that all of the sheaves in the picture become locally free, but to reduce to the calculations just completed on $X \times \mathbf{P}$.

Turning to details, consider the blowing-up

$$b \,:\, B \;=\; \mathrm{Bl}_Z\big(X \times \mathbf{P}\big) \longrightarrow X \times \mathbf{P}$$

of $X \times \mathbf{P}$ along $Z$, and denote by $D \subset B$ the exceptional divisor. Thanks to the smoothness of $Z$, $B$ is non-singular and

$$b_*\mathcal{O}_B(-D) \;=\; \mathcal{I}_Z \quad , \quad R^i b_*\mathcal{O}_B(-D) = 0 \;\text{ for } i > 0.$$

Write $\widetilde{p} = p \circ b : B \longrightarrow X$ and $\widetilde{q} = q \circ b : B \longrightarrow \mathbf{P}$ for the natural projections.

We wish to construct on $B$ the analogue of (7.3.6). To this end, put

$$\widetilde{U} \;=\; \widetilde{q}_* \widetilde{q}^* \big(\widetilde{p}^* L \otimes \mathcal{O}_B(-D)\big) \quad , \quad \widetilde{W} \;=\; H^0(L)_B \,\big/\, \widetilde{U}.$$

It follows from the projection formula that $\widetilde{U} = b^*U$, and one finds the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{U} & \longrightarrow & H^0(L)_B & \longrightarrow & \widetilde{W} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widetilde{p}^* L \otimes \mathcal{O}_B(-D) & \longrightarrow & \widetilde{p}^* L & \longrightarrow & \widetilde{p}^* L \otimes \mathcal{O}_D & \longrightarrow & 0. \end{array} \tag{7.3.11}$$

Denote by $\widetilde{S}$ and $\widetilde{\Sigma}$ the kernels of the first and third vertical maps. These are vector bundles of rank $k$, the local freeness of $\widetilde{\Sigma}$ being a consequence of the fact that $D$ is a divisor on $B$. One has a short exact sequence

$$0 \longrightarrow \widetilde{S} \longrightarrow \widetilde{p}^* M_L \longrightarrow \widetilde{\Sigma} \longrightarrow 0 \tag{7.3.12}$$

of bundles on $B$, and since $R\widetilde{p}_* \mathcal{O}_B = \mathcal{O}_X$, Kemeny's Theorem 7.3.7 is equivalent to the vanishing

$$H^1\big(B, \Lambda^{k+1} \widetilde{p}^* M_L\big) \;=\; 0. \tag{7.3.13}$$

As suggested above, we attack this by applying (7.3.5) to take the $(k+1)^{\text{st}}$ wedge power of (7.3.12). Since $\Lambda^{k+1}\widetilde{\Sigma} = 0$, this leads to a long exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Sym}^{k+1}\widetilde{S} \longrightarrow \operatorname{Sym}^{k}\widetilde{S} \otimes \widetilde{p}^* M_L \longrightarrow \operatorname{Sym}^{k-1}\widetilde{S} \otimes \Lambda^2 \widetilde{p}^* M_L \longrightarrow \ldots \\ \ldots \longrightarrow \operatorname{Sym}^{2}\widetilde{S} \otimes \Lambda^{k-1} \widetilde{p}^* M_L \longrightarrow \widetilde{S} \otimes \Lambda^{k} \widetilde{p}^* M_L \longrightarrow \Lambda^{k+1} \widetilde{p}^* M_L \longrightarrow 0. \end{aligned}$$

So to prove (7.3.13), we are reduced to establishing on $B$ the vanishings :

$$H^1\big(\widetilde{S} \otimes \Lambda^k \widetilde{p}^* M_L\big) = H^2\big(\operatorname{Sym}^2 \widetilde{S} \otimes \Lambda^{k-1} \widetilde{p}^* M_L\big) = \ldots = H^k\big(\operatorname{Sym}^k \widetilde{S} \otimes \widetilde{p}^* M_L\big) = 0 \tag{7.3.14}$$

$$H^{k+1}\big(\operatorname{Sym}^{k+1}\widetilde{S}\big) \;=\; 0. \tag{7.3.15}$$

For this we start with the short exact sequence of locally free sheaves

$$0 \longrightarrow \widetilde{S} \longrightarrow \widetilde{U} \longrightarrow \widetilde{p}^* L \otimes \mathcal{O}_B(-D) \longrightarrow 0$$

arising from the left-hand column of (7.3.11). Taking symmetric powers, we get for every $\ell \geq 1$ an exact sequence

$$0 \longrightarrow \operatorname{Sym}^\ell \widetilde{S} \longrightarrow \operatorname{Sym}^\ell \widetilde{U} \longrightarrow \operatorname{Sym}^{\ell-1} \widetilde{U} \otimes \widetilde{p}^* L \otimes \mathcal{O}_B(-D) \longrightarrow 0$$

and hence also

$$0 \to \widetilde{p}^* F \otimes \operatorname{Sym}^\ell \widetilde{S} \longrightarrow \widetilde{p}^* F \otimes \operatorname{Sym}^\ell \widetilde{U} \longrightarrow \widetilde{p}^* F \otimes \operatorname{Sym}^{\ell-1} \widetilde{U} \otimes \big(\widetilde{p}^* L \otimes \mathcal{O}_B(-D)\big) \to 0 \tag{*}$$

for any bundle $F$ on $X$. Now recall that

$$\widetilde{U} \,=\, b^*U \quad , \quad Rb_*\big(\widetilde{p}^*L \otimes \mathcal{O}_B(-D)\big) \,=\, p^*L \otimes \mathcal{I}_Z.$$

Thus for any vector bundle $F$ on $X$, the projection formula yields identifications

$$H^i\Big(B, \widetilde{p}^*F \otimes \mathrm{Sym}^\ell \widetilde{U}\Big) \,=\, H^i\Big(X \times \mathbf{P}, p^*F \otimes \mathrm{Sym}^\ell U\Big)$$
$$H^i\Big(B, \widetilde{p}^*F \otimes \mathrm{Sym}^{\ell-1}\widetilde{U} \otimes \widetilde{p}^*L \otimes \mathcal{O}_B(-D)\Big) \,=\, H^i\Big(X \times \mathbf{P}, p^*F \otimes \mathrm{Sym}^{\ell-1}U \otimes p^*L \otimes \mathcal{I}_Z\Big)$$

compatible with the natural maps on cohomology. Looking at the long exact sequence from (*) with this in mind, the vanishings (7.3.14) follow from Lemma 7.3.10, while (7.3.15) is a consequence of Lemma 7.3.8 and Remark 7.3.9. This completes the proof of Kemeny's theorem.

**Remark 7.3.11** (**The geometric syzygy conjecture**)**.** Fix $(X, L)$ of genus $g = 2k$ as above, with $\mathrm{Pic}(X) = \mathbf{Z} \cdot [L]$, and let $C \in |L|$ be a general curve. Kemeny [117, §2] also proves that the last non-vanishing group $K_{k-1,1}(C; \omega_C)$ is spanned by classes of minimal rank $k$ in the sense of Section 5.3.B. It follows that the same statement holds for a general curve of genus $2k$. In other words, extremal syzygies are spanned by those of geometric origin. That this should be so had become known as the geometric syzygy conjecture.

**Remark 7.3.12** (**Voisin's proof**)**.** Besides the original papers [190, 192], the reader interested in exploring Voisin's original proof of Theorem 7.1.10 might consult [19] for an introduction, and Chapters 5 and 6 of the lectures [7] of Aprodu–Nagel for a detailed account.

**Remark 7.3.13.** Rathmann [?] has given a somewhat different account of Kemeny's argument.

## 7.4 Complements

In this final section, we survey without proof some further developments and applications.

**Syzygies and singularities of secant varieties to curves.** An interesting avenue of investigation is to extend some of these results to secant varieties of curves. Suppose then that $C$ is a smooth projective curve of genus $g \geq 2$ and that $L$ is a very ample line bundle of degree $d \geq 2g + 1$ defining a linearly normal embedding

$$C \,\subseteq\, \mathbf{P}^r \,=\, \mathbf{P}^{d-g}.$$

Besides $C$ itself, one can consider the secant variety $\mathrm{Sec}(C) \subseteq \mathbf{P}^r$, defined to be the Zariski closure of all secant lines joining points of $C$. More generally, one has the $k^{\mathrm{th}}$ secant variety $\mathrm{Sec}_m(C) \subseteq \mathbf{P}^r$, which is the Zariski closure of the union of all $(m+1)$-secant $m$-planes to $C$.

This is an irreducible variety with $\dim \mathrm{Sec}_m(C) = 2m+1$ provided that $d - g \geq 2m+1$. It is the image of the natural map

$$\mu_m : \mathbf{P}(E_{m+1,L}) \longrightarrow \mathbf{P}^r \ = \ \mathbf{P}H^0(L)$$

where $E_{m+1,L}$ is the tautological bundle (7.2.2) associated to $L$ on the symmetric produce $C_{m+1}$.

Starting with work of Sidman–Vermiere [178] and Ullery [185] in the case $m = 1$, the singularities and syzygies of these varieties have attracted attention. It is elementary that $\mathrm{Sec}_m(C)$ cannot be cut out by hypersurfaces of degrees $\leq m+1$. Therefore one is led to extend Definitions 5.4.1 and 6.1.1 by saying that a projectively normal variety $V \subseteq \mathbf{P}^r$ satisfies Property $(N_{\ell,k})$ if its homogeneous ideal $I_V$ is generated in degree $\ell$, and the first $k$ steps of the resolution of $I_V$ are linear.

In the paper [56], the first author, Niu and Park generalize Green's Theorem 5.4.3 by establishing:

**Theorem 7.4.1.** *Assume that*

$$d \ \geq \ 2g \, + \, 2m \, + \, 1 \, + \, p.$$

*Then* $\mathrm{Sec}_m(C) \subseteq \mathbf{P}^r$ *is arithmetically Cohen–Macaulay, and it satisfies Property* $(N_{m+2,p})$.

The proof of the Theorem then involves a delicate inductive study of the geometry of the map $\nu_m$ as well as the cohomological properties of $E_{m+1,L}$. Building on [56], Choe, Kwak and Park [39] have recently established for $\mathrm{Sec}_m(C)$ an analogue of the statement (7.1.4) of the gonality theorem. Interestingly, besides the classical gonality of $C$ one also looks at minimals degrees of maps $C \longrightarrow \mathbf{P}^s$ for $s > 1$.

**The tangent developable surface of a rational normal curve.** The original proofs of Theorem 7.3.1 for general curves, by Griffiths–Harris [100] and Gieseker [85], proceeded via degenerations: these authors deformed the canonical curve to a rational curve with nodes, allowing them to study the question on that more concrete model. It turned out, however, that the nodes themselves had to be chosen generally, which added an extra layer of complexity. But shortly thereafter, Eisenbud and Harris [64] discovered that *cuspidal* rational curves of degree $2g-2$ in $\mathbf{P}^{g-1}$ behave Brill-Noether generally independent of the location of the cusps. It happened that the work of Eisenbud–Harris appeared at around the same time that Green was formulating his conjecture on canonical curves, and Kieran O'Grady noticed a very nice way to put the two together.

Namely, consider a rational normal curve $\Gamma \subseteq \mathbf{P}^g$ of degree $g$. The tangent developable surface of $\Gamma$ is the union of all its embedded tangent lines:

$$T \ = \ \mathrm{Tan}(\Gamma) \ \subseteq \ \mathbf{P}^g.$$

Very concretely, $T$ is the image of the map $\nu : \mathbf{P}^1 \times \mathbf{P}^1 \longrightarrow \mathbf{P}^g$ given matricially by

$$\nu\big([s,t] \times [u,v]\big) \ = \ [\, u \ \ v\,] \cdot \mathrm{Jac}(\mu),$$

where $\mathrm{Jac}(\mu)$ is the $2 \times (g+1)$ matrix of partials of $\mu = [\, s^g, s^{g-1}t, \ldots, st^{g-1}, t^g]$. In particular, $T$ is a rational surface of degree $2g-2$ having cusps along $\Gamma$ but no other singularities. Hence a hyperplane section of $T$ is a rational curve of degree $2g-2$ in $\mathbf{P}^{g-1}$ with $g$ cusps, i.e. it is precisely the type of curve that Eisenbud–Harris considered.

O'Grady suggested therefore that one should study the syzygies of $T$. These were easily calculated for many values of $g$ using early versions of the program `Macaulay`, and it became clear experimentally that the resolution of the homogeneous ideal $I_{T/\mathbf{P}^g}$ had exactly the numerical shape predicted for general canonical curves. So modulo some technical details about carrying out the reduction, it was understood by the mid 1980's that the generic case of Conjecture 7.1.8 would follow from a solution to the down-to-earth

> Problem: Show that the syzygies of the homogeneous ideal $I_{T/\mathbf{P}^g}$ of the tangent developable surface have the expected shape.

(For example, Eisenbud's notes [58] from around 1990 mention this approach.)

However for many years the problem remained open. It was only in 2018 that Aprodu, Farkas, Papadima, Raicu and Weyman were able to give an affirmative solution [4]. One of the key new ideas was to relate the question to their vanishing theorem for Koszul modules (Theorem 6.2.13). See [55] for a geometrically–oriented account of this work. While the basic idea is quite clean, some of the verifications in [4] remained a bit painful. Recently Jinhyung Park [156] used the results [56] discussed in the previous paragraph to give a quicker approach to filling in the details.

**The theorem of Hirschowitz–Ramanan.** In their influential paper [111], dealing with curves of odd genus $g = 2k+1 \geq 5$, Hirschowitz and Ramanan computed the class of the virtual divisor on the moduli space $\mathfrak{M}_g$ parametrizing curves $C$ for which $K_{k,1}(C;\omega_C) \neq 0$, i.e which carry non-generic syzygies. They showed that it coincided with (the expected multiple of) the divisor of curves of gonality $k+1$. Once Voisin's Theorem 7.1.10 guaranteed that the locus in question was actually a divisor, this led to the following

**Theorem 7.4.2.** *Let $C$ be a smooth curve of genus $2g+1 \geq 5$ with $K_{k,1}(C;\omega_C) \neq 0$. Then $C$ carries a pencil of degree $k+1$.*

Theorem 7.4.2 has been generalized and applied in several ways. For example, Aprodu [1] uses a variant (along with ideas of Voisin from [192]) to prove that Green's conjecture holds for any curve whose Brill–Noether loci have linear growth:

**Theorem 7.4.3.** *Let $C$ be a curve of genus $g$ having gonality $c \leq [g/2]+2$. Assume that*

$$\dim, W^1_{c+n}(C) \ \leq \ n \quad \textit{for all } 0 \ \leq \ n \ \leq \ g - 2c + 2.$$

*Then* $\mathrm{Cliff}(C) = c-2$*, and $C$ satisfies Green's conjecture.*

In another direction, Farkas [72] shows that most of the important effective divisors studied on $\mathfrak{M}_g$ and its cousins are of Koszul-theoretic type. We refer to [116], [3] and [73] for nice surveys of recent work on these questions.

**Curves on a $K3$ surface.** Let $(X, L)$ be an arbitrary polarized $K3$ surface. Green observed in [88] that his Conjecture 7.1.8 would imply that all curves $C \in |L|$ have the same Clifford index (since the canonical syzygies of any such $C$ are the restriction of those of $X$). The constancy of Clifford index within a linear series was subsequently established by Green and the second author in [97] via a more careful analysis of the vector bundles introduced in Section 7.3.A.

More recently, Aprodu and Farkas [2] proved the striking

**Theorem 7.4.4.** *Green's conjecture holds for every smooth curve lying on a $K3$ surface.*

The proof combines arguments involving the bundles $E_{C,A}$ with Theorem 7.4.3 and related ideas.

**The secant conjecture.** Recalling from Section 5.4.A the classical result of Castelnuovo, Mattuck and Mumford that a line bundle $L$ of degree $\geq 2g+1$ on a curve $C$ of genus $g$ is normally generated, it is natural to ask what one can say for bundles of smaller degree. Green and the second author used vector bundles of rank 2 to prove:

**Theorem 7.4.5.** *Let $L$ be a very ample line bundle on $C$ with*

$$\deg(L) \ \geq \ 2g \,+\, 1 \,-\, 2 \cdot h^1(L) \,-\, \mathrm{Cliff}(C)$$

*(and hence $h^1(L) \leq 1$). Then $L$ is normally generated, i.e. satisfies $(N_0)$.*

Green's conjecture on canonical curves then suggested:

**Conjecture 7.4.6.** *In the situation of the Theorem, assume that*

$$\deg(L) \ \geq \ 2g \,+\, 1 \,+\, p \,-\, 2 \cdot h^1(L) \,-\, \mathrm{Cliff}(C).$$

*Then Property $(N_p)$ fails for $L$ if and only if $L$ fails to be $(p+1)$-very ample, i.e. $\phi_L$ embeds $C$ with a $(p+2)$-secant $p$-plane.*

When $H^1(L) \neq 0$ this reduces to 7.1.8; in the remaining case it has become known as the Secant Conjecture. The most striking progress to date is due to Farkas and Kemeny [74] who show that it holds generically:

**Theorem 7.4.7.** *The secant conjecture holds for a general non-special line bundle of degree $d$ on a general curve $C$ of genus $g$.*

One of the interesting inputs to the proof is an application of Theorem 7.4.4 on a carefully chosen lattice-polarized $K3$ surface. We again refer to the survey [116] for more details.

## 7.5 Notes ⋄

# Lecture 8

# Asymptotic Syzygies in Higher Dimensions

In this lecture, we study the asymptotic behavior of the syzygies of a fixed variety under increasingly positive Veronese-type embeddings. The theme is that the resulting Betti tables exhibit a certain uniformity, and in dimensions $n \geq 2$ they are far from being pure.

Concerning notation to describe the growth of a function $f(d)$ of a natural number $d \in \mathbf{N}$, we refer to the summary of Notation and Conventions at the end of the Introduction. In particular, we say that

$$f \in \Theta(d^q)$$

if there exist positive real numbers $C_1, C_2 > 0$ such that

$$C_1 \cdot d^q \leq f(d) \leq C_2 \cdot d^q$$

for all sufficiently large $d$.

## 8.1 Overview

We start with an overview of the questions to be discussed in this Lecture.

Let $X$ be a smooth irreducible complex projective variety of dimension $n$. We wish to study the syzygies of $X$ as a function of the positivity of an embedding line bundle. To this end, fix divisors $A$ and $P$ on $X$, with $A$ ample and $P$ arbitrary and put:

$$L_d = \mathcal{O}_X(dA + P). \tag{8.1.1}$$

We will always assume that $d$ is sufficiently large so that $L_d$ is very ample, defining an embedding:

$$X \subseteq \mathbf{P}H^0(L_d) = \mathbf{P}^{r_d}, \quad \text{where } r_d = h^0(L_d) - 1.$$

It follows from asympotic Riemann–Roch that $r_d \sim C \cdot d^n$ for a positive constant $C = C(X, A)$ depending on $X$ and $A$. It is useful to fix in addition a line bundle $B$, and to consider more generally the Koszul cohomology groups $K_{p,q}\big(X, B; L_d\big)$ as $d \to \infty$.

When $d \gg 0$, syzygies of weight $q = 0$ or $q \geq n+1$ are easily analyzed:

**Proposition 8.1.1.** *If $d$ is sufficiently large, then:*

(*i*) $K_{p,q}(X, B; L_d) \;=\; 0$ *for all* $p \geq 0$ *and all* $q \geq n+2$*;*

(*ii*) $K_{p,0}(X, B; L_d) \;\neq\; 0 \;\iff\; p \;\leq\; r(B)$*;*

(*iii*) $K_{p,n+1}(X, B; L_d) \;\neq\; 0$ *if and only if*

$$r_d \;-\; n \;-\; r(\omega_X \otimes B^*) \;\leq\; p \;\leq r_d - n.$$

If $H^0\big(X, \omega_X \otimes B^*\big) = H^n\big(X, B\big)^* = 0$, then (iii) means that $K_{p,n+1}\big(X, B; L_d\big) = 0$ for all $p$.

*Proof.* The first assertion follows from considerations of Castelnuovo–Mumford regularity: for $d \gg 0$, $B$ is $(n+1)$-regular with respect to $L_d$. Statement (ii) is a consequence of Theorem 5.3.1 and Proposition 5.3.3. After replacing $B$ by $\omega_X \otimes B^*$, (iii) follows from (ii) by duality (Theorem 5.2.11).

$$H^{q-1+i}\big(X, \Lambda^{p_0+q-2} M_d \otimes L_d \otimes B(-i)\big) \;=\; K_{p_0}$$

□

Thus the interesting question is to analyze the $K_{p,q}(X, B; L_d)$ in the range $1 \leq q \leq n$. The result (Proposition 6.2.8) of Ottaviani–Paoletti on the Veronese surface, asserting that

$$K_{p,2}\big(\mathbf{P}^2; \mathcal{O}_{\mathbf{P}^2}(d)\big) \;\neq\; 0 \;\; \text{for} \;\; 3d \;-\; 2 \leq p \;\leq\; r_d - 2,$$

suggests that the Koszul groups in question should satisfy non-vanishings. The following theorem, due to the authors, shows that this is indeed the case.

**Theorem 8.1.2** (**Asymptotic non-vanishing theorem**, [52]). *Fix $q \in [1, n]$. There exist real numbers $C_1, C_2 > 0$, depending on $X, B, A$ and $P$ with the property that for $d \gg 0$*

$$K_{p,q}(X, B; L_d) \;\neq\; 0$$

*for every value of $p$ in the range*

$$C_1 \cdot d^{q-1} \;\leq p \;\leq\; r_d \;-\; C_2 \cdot d^{n-1}. \tag{8.1.2}$$

To get a feeling for what this says, fix $q \in [1, n]$ and define

$$w_q(d) \;=\; \frac{\big\{p \in [1, r_d] \mid K_{p,q}(X, B; L_d) \neq 0\big\}}{r_d}.$$

Thus $w_d(q)$ measures the proportion of potentially non-zero weight $q$ Koszul groups that are actually non-zero. Recalling that $r_d \in \Theta(d^n)$, the Theorem implies that

$$\lim_{d\to\infty} w_q(d) \ = \ 1. \tag{8.1.3}$$

Pictorially this means that as $d \to \infty$, the $q^{\text{th}}$ row of the Betti diagram becomes almost entirely filled with non-zero entries.

The authors conjectured in [52] that the lower bound in (8.1.2) is best possible in the sense that Theorem 8.1.2 should be accompanied by an asymptotic vanishing theorem. This was established by Jinhyung Park:

**Theorem 8.1.3** (**Asymptotic vanishing theorem**, [155])**.** *Keeping notation as in the previous Theorem, there exists a number $C_3 > 0$ such that if $d \gg 0$ then*

$$K_{p,q}(X, B; L_d) \ = \ 0 \quad \textit{for} \ \ p \ \leq \ C_3 \cdot d^{q-1}.$$

Taking $q \geq 2$ this recovers the linearity theorem of Section 6.1, but Park's result is much stronger. In addition, Park shows that the Theorem remains true assuming that $B$ is an arbitrary coherent sheaf on $X$.

The proof of Theorem 8.1.2 in [52] combined secant constructions with a rather lengthly inductive argument involving hyperplane sections. Very recently, Park realized that his asymptotic vanishing theorem leads to a much quicker approach: we will outline this in the next subsection. He also obtained a very nice precision of 8.1.2 to the effect that the non-vanishing occurs for values of $p$ lying in a connected interval:

**Theorem 8.1.4** ([154])**.** *Fix $q \in [1, n]$ and if $q \geq 2$ assume for simplicity that $H^{q-1}\big(X, B\big) = 0$. Then there exist functions*

$$c_q(d) \ \in \ \Theta(d^{q-1}) \quad , \quad c'_q(d) \ \in \ \Theta(d^{n-q})$$

*with the property that for $d \gg 0$:*

$$K_{p,q}(X, B : L_d) \ \neq \ 0 \ \Longleftrightarrow \ c_q(d) \ \leq \ p \ \leq \ r_d - c'_q(d).$$

When $H^{q-1}\big(X, B\big) \neq 0$ the same assertion holds, except that in this case $c'_q(d) = q - 1$. Taken together, these results provide a fairly complete picture of the asymptotic vanishing and non-vanishing of Koszul groups. They are the subject of Section 8.2.

The case when $X = \mathbf{P}^n$ and $L_d = \mathcal{O}_{\mathbf{P}^n}(d)$ – in other words, Veronese syzygies – is particularly interesting. Here there is an effective statement:

**Theorem 8.1.5.** *Fix $b \geq 0$ and $q \in [1, n]$. Then*

$$K_{p,q}\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(b); \mathcal{O}_{\mathbf{P}^n}(d)\big) \ \neq \ 0$$

*for any*

$$d \ \geq \ b + q + 1$$

*and all $p$ in the range*

$$\binom{d+q}{q} - \binom{d-b-1}{q} - q \ \leq \ p \leq \ \binom{d+n}{n} - \binom{d+n-q}{n-q} + \binom{n+b}{q+b} - q - 1. \quad (8.1.4)$$

For example, taking $n = 2$ and $b = 0$ one recovers the result (Proposition 6.2.8) of Ottaviani–Paoletti that $K_{p,2}(\mathbf{P}^2; \mathcal{O}_{\mathbf{P}^2}(d)) \neq 0$ for $3d - 2 \leq p \leq \binom{d+2}{2} - 3$.

Theorem 8.1.5 was originally established in [52] by keeping explicit track of the constructions used in that paper; Weyman (unpublished) independently obtained the case $b = 0$. Subsequently the authors and Erman [49] found a simpler approach that reduces the question to computations with monomials: we outline this in Section 8.3. Interestingly enough, these two very different methods lead to exactly the same bounds. This is one of the rationales behind

**Conjecture 8.1.6.** *The statement of Theorem 8.3 is sharp. In other words,*

$$K_{p,q}\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(b); \mathcal{O}_{\mathbf{P}^n}(d)\big) \ = \ 0$$

*when $p$ lies outside the range specified in* (8.1.4).

When $q \geq 2$ and $b = 0$, this reduces to the conjecture of Ottaviani–Paoletti discussed in Section 6.2.A.

Finally, what about the Betti numbers themselves? Write

$$k_{p,q}(X, B; L_d) \ =_{\text{def}} \ \dim K_{p,q}\big(X, B; L_d\big). \quad (8.1.5)$$

When $n = 1$ and $L_d$ is a line bundle of degree $d$ on the curve $X$, one can combine Green's Theorem 5.4.3 with a calculation of Euler characteristics to compute (almost all of) the $k_{p,1}(X; L_d)$: see Figure 1 in the Introduction for a plot of two examples with $d = 80$. The dominant term is a binomial coefficient, and Stirling's formula then shows that the Betti numbers approach a normal distribution:

**Proposition 8.1.7.** *Choose a sequence $\{p_d\}$ of integers such that*

$$p_d \ \longrightarrow \ \frac{r_d}{2} + a \cdot \frac{\sqrt{r_d}}{2}$$

*for some fixed number $a$ (i.e. $\lim_{d\to\infty} \frac{2p_2 - r_d}{\sqrt{r_d}} = a$). Then as $d \to \infty$:*

$$\left(\frac{1}{2^{r_d}} \cdot \sqrt{\frac{2\pi}{r_d}}\right) \cdot k_{p_d,1}\big(X; L_d\big) \ \longrightarrow \ e^{-a^2/2}.$$

We conjecture that the same pattern holds universally:

**Conjecture 8.1.8.** *Returning to a smooth projective variety $X$ of dimension $n$ and $L_d$ as in* (8.1.1)*, fix $q \in [1, n]$. Then there is a normalizing function $F_q(d)$ (depending on $X$ and geometric data) such that*

$$F_q(d) \cdot k_{p_d,q}(X; L_d) \ \longrightarrow \ e^{-a^2/2}$$

*as $d \to \infty$ and $p_d \to \frac{r_d}{2} + a \cdot \frac{\sqrt{r_d}}{2}$.*

One expects slightly more generally that an analogous statement is true for the dimensions $k_{p,q}(X, B; L_d)$ for fixed $B$. As of this writing, the Conjecture is not known for any variety $X$ of dimension $\geq 2$. However we outline in Section 8.4 some indirect probabilistic evidence.

## 8.2 Asymptotic non-vanishing and vanishing theorems

This section is devoted to the asymptotic non-vanishing and vanishing theorems. We start with Park's Theorem 8.1.3. In the second subsection, we use this to sketch a proof of Theorem 8.1.2.

### 8.2.A Park's asymptotic vanishing theorem

The first input to Park's proof of Theorem 8.1.3 is an observation of Raicu [163] to the effect it suffices to consider the special case when $X = \mathbf{P}^{n_1} \times \mathbf{P}^{n_2} \times \mathbf{P}^{n_3}$ is a product of three projective spaces. We will discuss Raicu's result at the end of the subsection. Granting this reduction for now, Theorem 8.1.3 then follows from the case $k = 3$ of:

**Theorem 8.2.1** ([155], Theorem 1.2)**.** *Fix $k \geq 1$, positive integers $n_1, \ldots, n_k$, $d_1, \ldots, d_k$ and arbitrary integers $b_1, \ldots, b_k$. Set*

$$\begin{aligned} X \ &= \ \mathbf{P}^{n_1} \ \times \ \ldots \ \times \ \mathbf{P}^{n_k} \\ B \ &= \ \mathcal{O}_{\mathbf{P}^{n_1}}(b_1) \boxtimes \ldots \boxtimes \mathcal{O}_{\mathbf{P}^{n_k}}(b_k) \\ L \ &= \ \mathcal{O}_{\mathbf{P}^{n_1}}(d_1) \boxtimes \ldots \boxtimes \mathcal{O}_{\mathbf{P}^{n_k}}(d_k). \end{aligned}$$

*Fix $2 \leq q \leq 1 + \sum n_i$, and write $b = \min\{b_i\}$, $d = \min\{d_i\}$. If $d + b \geq 0$ then*

$$K_{p,q}\big(X, B; L\big) \ = \ 0 \quad \textit{for every} \ \ p \ \leq \ \frac{1}{n_1! \cdots n_k!} \cdot \big(d^{q-1} \ + \ b \cdot d^{q-2}\big). \tag{8.2.1}$$

For simplicity we will focus on the case $k = 1$ and $B = \mathcal{O}_{\mathbf{P}^n}$, and we will content ourselves with proving that $K_{p,q} = 0$ for $p \leq \text{const} \cdot d^{q-1}$. The general case is similar, but notationally more involved.

We start with some notation and preliminary observations. Fix $d$ and write $M_d = M_{n,d}$ for the kernel bundle on $\mathbf{P}^n$ associated to $\mathcal{O}_{\mathbf{P}^n}(d)$. Then (Proposition 5.2.9)

$$K_{p,q}\big(\mathbf{P}^n; \mathcal{O}_{\mathbf{P}^n}(d)\big) \ = \ H^{q-1}\big(\mathbf{P}^n, \Lambda^{p+q-1} M_d \otimes \mathcal{O}_{\mathbf{P}^n}(d)\big), \tag{8.2.2}$$

so the issue is to prove the vanishing of this cohomology group for an appropriate range of $p$. Park's beautiful idea is to use the identification of $\mathbf{P}^n$ as the $n^{\text{th}}$ symmetric product of $\mathbf{P}^1$, and to consider the degree $n$ finite covering

$$\sigma : \mathbf{P}^{n-1} \times \mathbf{P}^1 \longrightarrow \mathbf{P}^n \quad , \quad (\xi, x) \mapsto \xi + x \tag{8.2.3}$$

realizing $\mathbf{P}^{n-1} \times \mathbf{P}^1$ as the universal family of degree $n$ divisors on $\mathbf{P}^1$ (Section 7.2.A). Pulling back the data in play under $\sigma$ then enables him to set up an induction on $n$.

Turning to details, note to begin with:

$$\sigma^* \mathcal{O}_{\mathbf{P}^n}(1) \ = \ \mathcal{O}_{\mathbf{P}^{n-1}}(1) \boxtimes \mathcal{O}_{\mathbf{P}^1}(1), \tag{8.2.4}$$

$$\sigma_* \left(\mathrm{pr}_2^* \, \mathcal{O}_{\mathbf{P}^1}(n-1)\right) \ = \ \mathcal{O}_{\mathbf{P}^n}^n. \tag{8.2.5}$$

In fact, the first assertion is clear while the second follows from the description (Proposition 7.2.4) of the tautological bundle on $\mathbf{P}^n$ associated to a line bundle on $\mathbf{P}^1$. The projection formula then shows that killing the cohomology group appearing in (8.2.2) is equivalent to proving the vanishing of

$$H^{q-1}\Big(\mathbf{P}^{n-1} \times \mathbf{P}^1, \big(\Lambda^{p+q-1}(\sigma^* M_{n,d}) \otimes \mathcal{O}_{\mathbf{P}^{n-1}}(d)\big) \boxtimes \mathcal{O}_{\mathbf{P}^1}(d+n-1)\Big) \tag{8.2.6}$$

(since the sheaf here pushes down under $\sigma$ to a direct sum of copies of $\Lambda^{p+q-1} M_{n,d} \otimes \mathcal{O}_{\mathbf{P}^n}(d)$).

The next point is:

**Lemma 8.2.2.** *There is a short exact sequence of vector bundles on* $\mathbf{P}^{n-1} \times \mathbf{P}^1$ :

$$0 \longrightarrow \oplus \ \mathcal{O}_{\mathbf{P}^{n-1}} \boxtimes \mathcal{O}_{\mathbf{P}^1}(-n) \longrightarrow \sigma^* M_{n,d} \longrightarrow M_{n-1,d} \boxtimes \mathcal{O}_{\mathbf{P}^1}(d) \longrightarrow 0, \tag{8.2.7}$$

*the term on the left being a direct sum of copies of* $\mathcal{O}_{\mathbf{P}^{n-1}} \boxtimes \mathcal{O}_{\mathbf{P}^1}(-n)$.

*Proof.* Start with the exact sequence

$$0 \longrightarrow M_{1,d} \longrightarrow H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d)\big) \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}^1} \longrightarrow \mathcal{O}_{\mathbf{P}^1}(d) \longrightarrow 0$$

of bundles on $\mathbf{P}^1$ and take symmetric powers to construct the sequence

$$0 \longrightarrow \mathrm{Sym}^n(M_{1,d}) \longrightarrow \mathrm{Sym}^n\big(H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d))\big) \otimes \mathcal{O}_{\mathbf{P}^1} \longrightarrow \mathrm{Sym}^{n-1}\big(H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d))\big) \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}^1}(d) \longrightarrow 0. \tag{*}$$

Recalling that $M_{1,d} = \mathcal{O}_{\mathbf{P}^1}^d(-1)$, we see that the term on the left is a direct sum of copies of $\mathcal{O}_{\mathbf{P}^1}(-n)$. On the other hand, pulling back under $\sigma$ the evaluation morphism for $\mathcal{O}_{\mathbf{P}^n}(d)$, one gets on $\mathbf{P}^{n-1} \times \mathbf{P}^1$ an exact sequence

$$0 \longrightarrow \sigma^*(M_{n,d}) \longrightarrow H^0\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)\big) \otimes \mathcal{O}_{\mathbf{P}^{n-1} \times \mathbf{P}^1} \longrightarrow \mathcal{O}_{\mathbf{P}^{n-1}}(d) \boxtimes \mathcal{O}_{\mathbf{P}^1}(d) \longrightarrow 0. \tag{**}$$

But now recall (Hermite Reciprocity, Corollary 7.2.5) that there are canonical isomorphisms

$$\begin{aligned} \mathrm{Sym}^n H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d)\big) &= \mathrm{Sym}^d H^0\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)\big) &= H^0\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)\big) \\ \mathrm{Sym}^{n-1} H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d)\big) &= \mathrm{Sym}^d H^0\big(\mathbf{P}^{n-1}, \mathcal{O}_{\mathbf{P}^n}(1)\big) &= H^0\big(\mathbf{P}^{n-1}, \mathcal{O}_{\mathbf{P}^{n-1}}(d)\big). \end{aligned}$$

Using these, one can splice together the pullback to $\mathbf{P}^{n-1} \times \mathbf{P}^1$ of (*) with (**) to get the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{pr}_2^*(\mathrm{Sym}^n M_{1,d}) & \longrightarrow & \mathrm{Sym}^n H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d)) & \longrightarrow & \mathrm{Sym}^{n-1}\big(H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d))\big) \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}^1}(d) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow{\scriptstyle \varepsilon} & & \\ 0 & \longrightarrow & \sigma^*(M_{n,d}) & \longrightarrow & \mathrm{Sym}^d H^0\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)\big) & \longrightarrow & \mathcal{O}_{\mathbf{P}^{n-1}}(d) \boxtimes \mathcal{O}_{\mathbf{P}^1}(d) & \longrightarrow & 0. \end{array}$$

Observing that

$$\ker(\varepsilon) \;=\; M_{n-1,d} \boxtimes \mathcal{O}_{\mathbf{P}^1}(d),$$

the result then follows from the snake lemma. □

We now give the proof of Park's theorem.

*Proof of Theorem 8.2.1.* Recall that we treat the case $k = 1$ and $B = \mathcal{O}_{\mathbf{P}^n}$, and the issue is to prove the existence of $C > 0$ with the property that the groups in (8.2.6) vanish for $0 \le p \le C \cdot d^{q-1}$. The exact sequence (8.2.7) gives rise to a filtration of $\Lambda^{p+q-1}(\sigma^* M_{n,d})$ having graded pieces

$$\Lambda^i M_{n-1,d} \boxtimes \mathcal{O}_{\mathbf{P}^1}\big(id - (p+q-1-i)n\big) \quad (0 \le i \le p+q-1).$$

So we get a filtration of $\big(\Lambda^{p+q-1}(\sigma^* M_{n,d}) \otimes \mathcal{O}_{\mathbf{P}^{n-1}}(d)\big) \boxtimes \mathcal{O}_{\mathbf{P}^1}(d+n-1)$ with graded pieces

$$\mathrm{Gr}_i \;=\; \big(\Lambda^i M_{n-1,d} \otimes \mathcal{O}_{\mathbf{P}^{n-1}}(d)\big) \boxtimes \mathcal{O}_{\mathbf{P}^1}(a_i)$$

$(0 \le i \le p+q-1))$, where

$$a_i \;=\; i(d+n) + (2-p-q)n + (d-1).$$

By induction on $n$, we can assume the existence of $C_1 > 0$ such that if $j = q-2$ or $j = q-1$, then

$$H^j\big(\mathbf{P}^{n-1}, \Lambda^i M_{n-1,d} \otimes \mathcal{O}_{\mathbf{P}^{n-1}}(d)\big) \;=\; 0 \quad \text{for} \quad 0 \le i \le C_1 \cdot d^j$$

when $d \gg 0$. Therefore $H^{q-1}\big(\mathbf{P}^{n-1} \times \mathbf{P}^1, \mathrm{Gr}_i\big) = 0$ for $i \le C_1 \cdot d^{q-2}$. On the other hand, after possibly adjusting $C_1$ we can find $0 < C < C_1$ so that $H^1\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(a_i)\big) = 0$ when

$$C_1 \cdot d^{q-2} \;\le\; i \;\le\; p+q-1 \;\le\; C \cdot d^{q-1}.$$

All told, it follows (using Künneth) that

$$H^{q-1}\big(\mathbf{P}^{n-1} \times \mathbf{P}^1, \mathrm{Gr}_i\big) \;=\; 0 \;\; \text{for} \;\; 0 \le i \le p+q-1 \le C \cdot d^{q-1},$$

completing the proof. □

**Raicu's reduction.** In his paper [163] on representation stability for syzygies of Segre–Veronese varieties, Raicu made the very nice observation that Theorem 8.1.3 follows from the case $k = 3$ of Theorem 8.2.1. In order to illustrate the idea, we will sketch the reduction under some simplifying hypotheses. We follow the presentation of [155, §3.3].

Consider then a smooth projective variety $X$ of dimension $n$, and let $L = \mathcal{O}_X(1)$ be a normally generated very ample line bundle on $X$. We will outline the proof of the following

**Assertion 8.2.3.** Granting the case $k = 1$ of Theorem 8.2.1, there exists $C > 0$ such that if $d \gg 0$ and $q \geq 2$, then

$$K_{p,q}\big(X; \mathcal{O}_X(d)\big) \ = \ 0 \quad \text{for} \ \ 0 \ \leq \ p \ \leq \ C \cdot d^{q-1}.$$

The three-factor instance of Theorem 8.2.1 comes into the picture in order to deal with line bundles having the more general shape envisioned in (8.1.1).

For the proof, we begin by considering the embedding

$$X \ \subseteq \ \mathbf{P}^m \ = \ \mathbf{P}H^0\big(\mathcal{O}_X(1)\big)$$

defined by the given line bundle. Composing with the $d$-fold Veronese embedding $\mathbf{P}^m \subseteq \mathbf{P}^{N_d}$, we arrive at a fibre square:

$$\begin{array}{ccc} X & \hookrightarrow & \mathbf{P}^m =_{\text{def}} Y \\ {\scriptstyle |\mathcal{O}_X(d)|}\big\downarrow & & \big\downarrow{\scriptstyle |\mathcal{O}_{\mathbf{P}^m}(d)|} \\ \mathbf{P}^{r_d} & \xhookrightarrow[\text{section}]{\text{linear}} & \mathbf{P}^{N_d}, \end{array}$$

where we write $Y \subseteq \mathbf{P}^{N_d}$ for the Veronese image of $\mathbf{P}^m$. The embedding $X \subseteq \mathbf{P}^{r_d}$ defined by $\mathcal{O}_X(d)$ is a linear space section of the embedding $Y \subseteq \mathbf{P}^{N_d}$.

We can view $\mathcal{O}_X$ as a coherent sheaf on $Y = \mathbf{P}^m$, on $\mathbf{P}^{N_d}$ and on $\mathbf{P}^{r_d}$. It is an elementary general fact (Proposition 8.2.4 or Corollary 8.2.6 in the next subsection) that the resolution of $X$ in $\mathbf{P}^{N_d}$ – which is governed by the Koszul groups $K_{p,q}(\mathbf{P}^m, \mathcal{O}_X; \mathcal{O}_{\mathbf{P}^m}(d))$ – is the tensor product of the resolution of $X$ in $\mathbf{P}^{r_d}$ with the Koszul complex of the linear forms defining $\mathbf{P}^{r_d}$ in $\mathbf{P}^{N_d}$. Therefore, for fixed $q$:

$$\min\big\{p \ | \ K_{p,q}\big(X; \mathcal{O}_X(d)\big) \neq 0\big\} \ = \ \min\big\{p \ | \ K_{p,q}\big(\mathbf{P}^m, \mathcal{O}_X; \mathcal{O}_{\mathbf{P}^m}(d)\big) \neq 0\big\}. \tag{8.2.8}$$

Raicu's idea is that one can study the groups on the right via the syzygies of $X$ in $\mathbf{P}^m$.

Specifically, consider (the sheafification $\mathcal{P}_\bullet$ of) a minimal free resolution of $\mathcal{O}_X$ in $\mathbf{P}^m$:

$$\begin{array}{ccccccccc} \ldots \longrightarrow & \oplus\, \mathcal{O}_{\mathbf{P}^m}(-b_{2,j}) & \longrightarrow & \oplus\, \mathcal{O}_{\mathbf{P}^m}(-b_{1,j}) & \longrightarrow & \mathcal{O}_{\mathbf{P}^m} & \longrightarrow \mathcal{O}_X \longrightarrow 0. \\ & \| & & \| & & \| & \\ & \mathcal{P}_2 & & \mathcal{P}_1 & & \mathcal{P}_0 & \end{array} \tag{8.2.9}$$

Note that the $b_{i,j}$ depend only on the embedding $X \subseteq \mathbf{P}^m$ defined by $\mathcal{O}_X(1)$, and not on $d$. One can view this as an exact sequence of sheaves on $\mathbf{P}^{N_d}$, and this gives rise to a change of rings (homological) spectral sequence with

$$E_1^{i,j} \ = \ K_{j,p+q-j}\big(\mathbf{P}^m, \mathcal{P}_i; \mathcal{O}_{\mathbf{P}^m}(d)\big)$$

converging to $K_{p,q}\big(\mathbf{P}^m, \mathcal{O}_X; \mathcal{O}_{\mathbf{P}^m}(d)\big)$ (cf.[88, §1.d], [163, Appendix A], or [179, 061Y]). Fixing $p$ and $q$, it follows from this that if

$$K_{p-i,q+i}\big(\mathbf{P}^m, \mathcal{P}_i; \mathcal{O}_{\mathbf{P}^m}(d)\big) \ = \ 0 \text{ when } \ i+j=p, \tag{*}$$

then $K_{p,q}\big(\mathbf{P}^m, \mathcal{O}_X; \mathcal{O}_{\mathbf{P}^m}(d)\big) = 0$. But $\mathcal{P}_i = \oplus\, \mathcal{O}_{\mathbf{P}^m}(-b_{i,j})$ is a direct sum of line bundles, and the Koszul cohomology $K_{p+i,q-i}\big(\mathbf{P}^m, \mathcal{O}_{\mathbf{P}^m}(-b_{i,j}); \mathcal{O}_{\mathbf{P}^m}(d)\big)$ of each summand is governed by Theorem 8.2.1. So for fixed $q$ we can find $C > 0$ such that (*) holds for all $0 \leq p \leq C \cdot d^{q-1}$, as required.

## 8.2.B The asymptotic non-vanishing theorem

This subsection is devoted Park's proof [154] of Theorem 8.1.2 and some remarks on his extension Theorem 8.1.4. As in [52] the argument proceeds by induction on dimension. So we will start with some remarks on Koszul groups determined by a subvariety and the possibility of lifting cohomology classes to the ambient space.

**Algebraic preliminaries.** Consider a variety $X$ and a subvariety $\overline{X} \subseteq X$. A very positive embedding $X \subseteq \mathbf{P}^r$ typically maps $\overline{X}$ to a linear subspace $\mathbf{P}^{\overline{r}} \subseteq \mathbf{P}^r$, and we wish to compare the syzygies of $\overline{X}$ in $\mathbf{P}^r$ with those of its embedding in $\mathbf{P}^{\overline{r}}$. It is cleanest in the first instance to formulate the discussion algebraically.

Suppose then that $V$ is a vector space of dimension $r+1$, $V' \subseteq V$ is a subspace of dimension $s$, and put $\overline{V} = V/V'$ with $\dim \overline{V} = \overline{r}+1$, so that $\overline{r} = r - s$. Write $S = \mathrm{Sym}(V)$ and $\overline{S} = \mathrm{Sym}(\overline{V})$ for the corresponding polynomial rings. Consider now a finitely generated $\overline{S}$-module $\overline{E}$. We may also view $\overline{E}$ as an $S$-module having the property that $V' \cdot \overline{E} = 0$. One therefore arrives at two Koszul cohomology groups

$$K_{p,q}(\overline{E}; V) \ \text{ and } \ K_{p,q}(\overline{E}, \overline{V}),$$

corresponding respectively to the $S$- and $\overline{S}$-module structures on $\overline{E}$. We aim to relate these.

To this end, fix a subspace $V'' \subseteq V$ complementary to $V'$ splitting the exact sequence

$$0 \longrightarrow V' \longrightarrow V \longrightarrow \overline{V} \longrightarrow 0. \tag{8.2.10}$$

**Proposition 8.2.4.** *Having fixed a splitting $V'' \subseteq V$ of* (8.2.10)*, one has*

$$K_{p,q}(\overline{E}; V) \ \cong \ \bigoplus_{j=0}^{p} \Big(\Lambda^j V' \otimes K_{p-j,q}(\overline{E}; \overline{V})\Big). \tag{8.2.11}$$

*Proof.* The Koszul group $K_{p,q}(\overline{E};V)$ is computed as the cohomology of the complex

$$\ldots \longrightarrow \Lambda^{p+1}V \otimes \overline{E}_{q-1} \longrightarrow \Lambda^{p}V \otimes \overline{E}_{q} \longrightarrow \Lambda^{p-1}V \otimes \overline{E}_{q+1} \longrightarrow \ldots \tag{8.2.12}$$

Write

$$\Lambda^p V \;=\; \Lambda^p\big(V' \oplus V''\big) \;=\; \bigoplus_{j=0}^{p} \Big(\Lambda^j V' \otimes \Lambda^{p-j} V''\Big).$$

The hypothesis that $V' \cdot \overline{E} = 0$ implies that (8.2.12) splits as the direct sum (over $j$) of the complexes

$$\ldots \longrightarrow \Lambda^j V' \otimes \Lambda^{p+1-j} V'' \otimes \overline{E}_{q-1} \longrightarrow \Lambda^j V' \otimes \Lambda^{p-j} V'' \otimes \overline{E}_{q} \longrightarrow \Lambda^j V' \otimes \Lambda^{p-1-j} V'' \otimes \overline{E}_{q+1} \longrightarrow \ldots$$

The assertion follows. □

Keeping the same notation, consider a short exact sequence of finitely generated graded $S$-modules

$$0 \longrightarrow E' \longrightarrow E \longrightarrow \overline{E} \longrightarrow 0 \tag{8.2.13}$$

with $V' \cdot \overline{E} = 0$. The long exact sequence of Tor (Example 5.1.5) gives rise to homomophisms:

$$\begin{aligned} \theta_{p,q} \;&:\; K_{p+1,q-1}(\overline{E};V) \longrightarrow K_{p,q}(E';V) \\ \theta'_{p,q} \;&:\; K_{p,q}(E;V) \longrightarrow K_{p,q}(\overline{E};V). \end{aligned} \tag{8.2.14}$$

One of Park's crucial observations is that the non-vanishing of these for a specific value of $p$ implies its non-vanishing for many values of the index.

**Proposition 8.2.5.** *Assume that*

$$\theta_{p_0,q} \neq 0 \;\;\textit{for some}\;\; 0 \leq p_0 \leq r - \overline{r} - 1.$$

*Then $\theta_{p,q} \neq 0$ for every integer $p$ with $p_0 \leq p \leq r - \overline{r}$.*

Park also proves an analogous statement for $\theta'$ (working from the top down), but we don't require this.

*Proof of Proposition 8.2.5.* By induction it suffices to treat the case $p = p_0 + 1$. Let

$$\gamma_0 \;\in\; K_{p_0+1,q-1}(\overline{E};V)$$

be a class with $\theta_{p_0,q}(\gamma_0) \neq 0$. Using the decomposition of Proposition 8.2.4 we can assume that

$$\gamma_0 \;=\; v'_1 \wedge \ldots \wedge v'_{j_0} \otimes \alpha_0,$$

where $\alpha_0 \in K_{p_0+1-j_0,q-1}(\overline{E};\overline{V}) = K_{p_0+1-j_0,q-1}(\overline{E},V'')$. Using the numercal hypotheses on $p$ we choose a vector $v'_{j_0+1} \in V'$ such that $v'_1 \wedge \ldots \wedge v'_{j_0} \wedge v'_{j_0+1} \neq 0$. Then set

$$\gamma \;=\; v'_1 \wedge \ldots \wedge v'_{j_0} \wedge v'_{j_0+1} \otimes \alpha_0 \;\in\; \Lambda^{j_0+1}V' \otimes K_{p_0+1-j_0,q-1}(\overline{E},\overline{V}) \;\subseteq\; K_{p_0+2,q-1}(\overline{E},V).$$

The Proposition will follow once we verify:

$$\theta_{p_0+1,q}(\gamma) \neq 0. \tag{8.2.15}$$

For this, recall (Example 5.1.4) that for any graded $S$-module $F$ there are canonical homomorphisms

$$\mu_{p,q} \,:\, K_{p,q}(F;V) \longrightarrow V \otimes K_{p-1,q}(F;V).$$

These are deduced from the natural maps $\Lambda^p V \otimes F_q \longrightarrow V \otimes \Lambda^{p-1} V \otimes F_q$, and they commute with the connecting homomorphisms determined by exact sequences of $S$-modules. So in our setting we get a commutative diagram:

$$\begin{array}{ccc} K_{p_0+2,q-1}(\overline{E};V) & \xrightarrow{\theta_{p_0+1,q}} & K_{p_0+1,q}(E';V) \\ \Big\downarrow{\scriptstyle \mu_{p_0+2,q-1}} & & \Big\downarrow{\scriptstyle \mu_{p_0,q}} \\ V \otimes K_{p_0+1,q-1}(\overline{E};V) & \xrightarrow[1_V \otimes \theta_{p_0,q}]{} & V \otimes K_{p_0,q}(E';V). \end{array}$$

Now

$$\mu_{p_0+2,q-1}(\gamma) \;=\; \sum_{k=1}^{j_0+1} (-1)^k \, v'_k \otimes (v'_1 \wedge \ldots \wedge \widehat{v'_k} \wedge \ldots \wedge v'_{j_0+1}) \otimes \alpha_0 + (v'_1 \wedge \ldots \wedge v'_{j_0+1}) \otimes \mu_{p_0+1-j_0,q-1}(\alpha_0),$$

where we view $\mu_{p_0+1-j_0,q-1}(\alpha_0)$ as an element of

$$V'' \otimes K_{p_0-j_0,q-1}(\overline{E},\overline{V}) \;\subseteq\; V \otimes K_{p_0-j_0,q-1}(\overline{E},V).$$

Therefore there is only one term in the sum on the right that lies in the subspace

$$\langle v'_{j_0+1} \rangle \otimes K_{p_0+1,q-1}(\overline{E};V) \;\subseteq\; V \otimes K_{p_0+1,q-1}(\overline{E};V),$$

namely $\pm(v'_{j_0+1}) \otimes \gamma_0$, and by assumption this maps to a non-zero class under $1_V \otimes \theta_{p_0,q}$. The commutativity of the diagram then implies that $\theta_{p_0+1,q}(\gamma) \neq 0$, and (8.2.15) is proved. □

**Geometric application.** We return now to the geometric setting. As before, let $X$ be a smooth irreducible projective variety of dimension $n \geq 2$, and let $B$ be a line bundle on $X$. We assume for simplicity that

$$H^1\big(X,B\big) = 0, \tag{8.2.16}$$

referring to [154] for the small modifications required when this group doesn't vanish. As before we write $L_d = \mathcal{O}_X(dA+P)$ and we silently assume that $d$ is large enough so that twisting by $L_d$ kills various higher cohomology groups that might otherwise show up. Put $V = V_d = H^0\big(X,L_d\big)$ and set

$$E(X,B;L_d) \;=\; \bigoplus_{m\geq 0} H^0\big(X, B \otimes L_d^{\otimes m}\big).$$

We view this as a graded module over $S = \mathrm{Sym} V_d$.

Choose next a very ample line bundle $\mathcal{O}_X(1)$, and fix a smooth divisor

$$\overline{X} \in |\mathcal{O}_X(1)|.$$

Observe that $\overline{X}$ is independent of $d$. Write

$$V' = V'_d = H^0\big(X, L_d(-1)\big),$$

denote by $\overline{B}$ and $\overline{L_d}$ the restrictions of $B$ and $L_d$ to $\overline{X}$, and put

$$\overline{V} = \overline{V_d} = H^0\big(\overline{X}, \overline{L_d}\big).$$

Thanks to (8.2.16), restriction to $\overline{X}$ gives rise to an exact sequence

$$0 \longrightarrow E(X, B; L_d) \longrightarrow E(X, B(1); L_d) \longrightarrow E(\overline{X}, \overline{B(1)}; L_d) \longrightarrow 0 \tag{8.2.17}$$

of graded $S$-modules, where $E(\overline{X}, \overline{B(1)}; L_d)$ is the indicated twisted section ring on $\overline{X}$ viewed as a module over $S = \mathrm{Sym} V_d$. Assuming also – as we may – that $H^1\big(X, B(-1)\big) = 0$, we get a second exact sequence

$$0 \longrightarrow E(X, B(-1); L_d) \longrightarrow E(X, B; L_d) \longrightarrow E(\overline{X}, \overline{B}; L_d) \longrightarrow 0. \tag{8.2.18}$$

The right-hand modules in both of these sequences are annihilated by $V'$, so we are in the situation of Proposition 8.2.4. Therefore:

**Corollary 8.2.6.** *Under the stated hypotheses, one has isomorphisms:*

$$K_{p,q}(\overline{X}, \overline{B}; L_d) \cong \bigoplus_{j=0}^{p} \Big(\Lambda^j V' \otimes K_{p-j,q}(\overline{X}, \overline{B}; \overline{L_d})\Big),$$

*and similarly with $B$ replaced by $B(1)$.* □

The plan now is to use Park's vanishing theorem to study the maps

$$\begin{aligned} \theta_{p,q} &: K_{p+1,q-1}(\overline{X}, \overline{B(1)}; L_d) \longrightarrow K_{p,q}(X, B; L_d) \\ \theta'_{p,q} &: K_{p,q}(X, B; L_d) \longrightarrow K_{p,q}(\overline{X}, \overline{B}; L_d) \end{aligned} \tag{8.2.19}$$

arising from (8.2.17) and (8.2.18). However this requires some additional care in the choice of the auxilliary bundle $\mathcal{O}_X(1)$.

**Lemma 8.2.7.** *We can find $\mathcal{O}_X(1)$ in such a manner that for every $d \gg 0$, the kernel bundle $M_d$ admits a resolution*

$$\ldots \longrightarrow W_{2,d} \otimes \mathcal{O}_X(-3) \longrightarrow W_{1,d} \otimes \mathcal{O}_X(-2) \longrightarrow W_{0,d} \otimes \mathcal{O}_X(-1) \longrightarrow M_d \longrightarrow 0, \tag{8.2.20}$$

*where the $W_{i,d}$ are finite-dimensional vector spaces depending on $d$.*

*Proof.* If $H$ is a sufficiently positive divisor, then $\mathcal{O}_X$ and $M_d$ are $(n+1)$-regular with respect to $H$ provided that $d$ is large. By Proposition 3.2.7, $M_d$ then sits at the end of a long exact sequence having the shape

$$\ldots \longrightarrow W_{1,d} \otimes \mathcal{O}_X(-2(n+1)H) \longrightarrow W_{0,d} \otimes \mathcal{O}_X(-(n+1)H) \longrightarrow M_d \longrightarrow 0.$$

Take $\mathcal{O}_X(1) = \mathcal{O}_X((n+1)H)$. □

Finally we sketch Park's proof of asymptotic non-vanishing.

*Proof of Theorem 8.1.2.* There is no loss in supposing that $n \geq q \geq 2$. By induction on dimension, we may assume the Theorem known for $K_{p,q}(\overline{X}, \overline{B}; \overline{L_d})$. In addition, we choose $\mathcal{O}_X(1)$ satisfying the conclusion of the previous Lemma.

We first show that

$$K_{p,q}(X, B; L_d) \neq 0 \;\text{ for some }\; p \in \Theta(d^{q-1}). \tag{*}$$

In fact, consider the exact sequence (Example 5.1.5)

$$K_{p,q}(X, B; L_d) \xrightarrow{\theta'_{p,q}} K_{p,q}(\overline{X}, \overline{B}; L_d) \longrightarrow K_{p-1,q+1}(X, B(-1); L_d).$$

By Park's Vanishing Theorem 8.1.3, the group on the right vanishes for $p \leq C \cdot p^q$, while by our induction hypothesis (and Corollary 8.2.6) the group in the middle is $\neq 0$ for $p \geq C' \cdot d^{q-1}$. This proves (*).

We would now like to use Proposition 8.2.5 to show that $K_{p,q} \neq 0$ for a large range of $p$. However this is not automatic, because we do not know that the class just constructed lies in the image of $\theta_{p,q}$. To circumvent this, let $p_0 = c_q(d)$ be the *smallest* index such that

$$K_{p_0,q}(X, B; L_d) \neq 0.$$

Thanks to (*) and 8.1.3, we know that $c_q(d) \in \Theta(d^{q-1})$.

Consider next the exact sequence

$$K_{p_0+1,q-1}(\overline{X}, \overline{B(1)}; L_d) \xrightarrow{\theta_{p_0,q}} K_{p_0,q}(X, B; L_d) \longrightarrow K_{p_0,q}(X, B(1); L_d).$$

It suffices to prove:

$$K_{p_0,q}(X, B(1); L_d) = 0. \tag{8.2.21}$$

In fact, this implies that $\theta_{p_0,q} \neq 0$, and then the Theorem follows from Proposition 8.2.5.

Since $q \geq 2$, we have (Proposition 5.2.9)

$$K_{p_0,q}(X, B(1); L_d) = H^{q-1}\big(X, \Lambda^{p_0+q-1} M_d \otimes L_d \otimes B(1)\big),$$

so it suffices to prove the vanishing of the group on the right. For this, start with the long exact sequence (8.2.20) and tensor through by $\Lambda^{p_0+q-2}M_d \otimes L_d \otimes B(1)$. Since $\Lambda^{p_0+q-1}M_d$ is a summand of $\Lambda^{p_0+q-1}M_d \otimes M_d$, (8.2.21) will follow if we show

$$\begin{aligned} H^{q-1}\big(X, \Lambda^{p_0+q-2}M_d \otimes L_d \otimes B\big) &= H^{q}\big(X, \Lambda^{p_0+q-2}M_d \otimes L_d \otimes B(-1)\big) \\ &= H^{q+1}\big(X, \Lambda^{p_0+q-2}M_d \otimes L_d \otimes B(-2)\big) \\ &\vdots \\ &= H^{n}\big(X, \Lambda^{p_0+q-2}M_d \otimes L_d \otimes B(q-1-n)\big) \ = \ 0. \end{aligned}$$

Now $H^{q-1}\big(X, \Lambda^{p_0+q-2}M_d \otimes L_d \otimes B\big) = K_{p_0-1,q}(X, B; L_d)$, so this group vanishes by our choice of $p_0$. On the other hand,

$$H^{q-1+i}\big(X, \Lambda^{p_0+q-2}M_d \otimes L_d \otimes B(-i)\big) = K_{p_0-1-i,q+i}(X, B(-i); L_d),$$

and when $i \geq 1$ and $d \gg 0$, these groups vanish for $p_0 \in \Theta(d^{q-1})$ thanks to the asymptotic non-vanishing theorem. Thus (8.2.21) holds when $d$ is sufficiently large, and we are done. □

Finally, we say a brief word about Park's Theorem 8.1.4. The argument just completed produced the required lower bound $c_q(d)$. The construction of $c'_q(d)$ proceeds along similar lines, building on an analogue of Proposition 8.2.5 for $\theta'_{p,q}$. The actual calculations however become a little more involved.

## 8.2.C Complements

There are various directions in which the results of this section have been extended.

**Effective non-vanishing for hyper-adjoint syzygies.** Let $A$ be a very ample divisor on a smooth projective variety $X$ of dimension $n$, and take $L_d = \mathcal{O}_X(dA)$. Xin Zhou [195] gives an effective version of Theorem 8.1.2 when $B = K_X + bA$ and $b \geq n+1$. Fixing very ample divisors $H_1, \ldots, H_c$ on $X$, write

$$\phi(H_1, \ldots, H_c; L_d) \ = \ h^0\big(Z, L_d|Z_c\big),$$

where $Z$ is the complete intersection of general divisors $H'_i \in |H_i|$. Now set:

$$\begin{aligned} n_d &= \phi\big(-K_X - (n-q)A + B; A, \ldots, A; L_d\big) \\ N_d &= \phi\big((d-q)A - B, A, \ldots, A; L_d\big), \end{aligned}$$

where $A$ appears $n-q$ times in the first expression and $q$ times in the second. In principle one could use a Koszul complex and Riemann–Roch to compute $n_d$ and $N_d$ explicitly.

Zhou proves:

**Theorem 8.2.8.** *Fix $1 \leq q \leq n$. Then for sufficiently large $d$*

$$K_{p,q}(X, B; L_d) \neq 0$$

*for every value of $p$ satisfying*

$$n_d - q \leq p \leq r(L_d) - N_d - q.$$

The idea is to render effective the arguments appearing in [52]. The hypothesis on $B$ is used to guarantee the global generation and vanishing of higher cohomology of various auxiliary divisors that appear along the way.

**Subdivisions of simplicial complexes.** Recall (Section 3.2.C) that a simplicial complex $\Delta$ on $[n]$ determines a Stanley–Reisner monomial ideal $I_\Delta \subseteq \mathbf{C}[z_1, \ldots, z_n]$. In their paper [43], Conca, Juhnke-Kubitzke and Welker study monomial ideals corresponding to subdivisions of $\Delta$. These authors find that these subdivisions have the same sort of asymptotic behavior as those described by Theorem 8.1.2. For example, they prove the following analogue of (8.1.3):

**Theorem 8.2.9.** *Let $\Delta$ be a simplicial complex of dimension $n-1 > 0$, and denote by $\Delta(d)$ either the iterated barycentric subdivision or edgewise subdivision of $\Delta$. Write $\mathbf{C}[\Delta(d)]$ for coordinate ring of this new simplicial complex, and fix $1 \leq q \leq n-1$. Then*

$$\lim_{p \to \infty} \frac{\#\big\{p \mid k_{p,q}(\mathbf{C}[\Delta(d))\big\}}{\operatorname{pdim}(\mathbf{C}[\Delta(d)])} = 1.$$

It would be interesting to know whether a statement along the lines of Park's asymptotic non-vanishing theorem holds in this setting.

## 8.3 Veronese varieties

This section is devoted to the proof of Theorem 8.1.5 and some related matters. We follow the approach of Erman and the authors from [49], which considerably simplified the original arguments in [52]. To lighten notation, set

$$K_{p,q}(n, b; d) =_{\text{def}} K_{p,q}\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(b); \mathcal{O}_{\mathbf{P}^n}(d)\big),$$

and we write $k_{p,q}(n, b; d) = \dim K_{p,q}(n, b; d)$ for the dimension of this vector space. Since

$$K_{p,q}(n, b; d) = K_{p,q+1}(n, b-d; d),$$

we may – and always do – assume that $0 \leq b \leq d-1$.

### 8.3.A Veronese syzygies via monomials

Denote by $S = \mathbf{C}[z_0, \ldots, z_n]$ the homogeneous coordinate ring of $\mathbf{P}^n$. Then $K_{p,q}(n, b; d)$ is by definition the homology of the Koszul complex

$$\cdots \longrightarrow \Lambda^{p+1} S_d \otimes S_{(q-1)d+b} \longrightarrow \Lambda^p S_d \otimes S_{qd+b} \longrightarrow \Lambda^{p-1} S_d \otimes S_{(q+1)d+b} \longrightarrow \cdots . \tag{8.3.1}$$

The essential idea of [49] is to mod out by a convenient regular sequence, reducing thereby to elementary computations with monomials.

Specifically, for fixed $d$, set

$$\overline{S} \ = \ S/\big(z_0^d, \ldots, z_n^d\big).$$

Slightly abusively, we view $\overline{S}$ as the ring generated by monomials $z_i$ in which all the variables have exponent $\leq d-1$, with multiplication determined by the vanishing of the $d^{\text{th}}$ power of each variable. Note that $\mathbf{C}^*$ acts on $\overline{S}$ in the natural way, so it makes sense to talk about monomials in this ring.

Consider now the Koszul complex:

$$\cdots \longrightarrow \Lambda^{p+1} \overline{S}_d \otimes \overline{S}_{(q-1)d+b} \xrightarrow{\partial_{p+1}} \Lambda^p \overline{S}_d \otimes \overline{S}_{qd+b} \xrightarrow{\partial_p} \Lambda^{p-1} \overline{S}_d \otimes \overline{S}_{(q+1)d+b} \longrightarrow \cdots . \tag{8.3.2}$$

Since $z_0^d, \ldots, z_n^d$ is a regular sequence in $\mathrm{Sym}(S_d)$, it follows (Example 5.1.17) that the homology of (8.3.1) restricts to the homology of (8.3.2) under the homomorphism $\mathrm{Sym}(S_d) \longrightarrow \mathrm{Sym}(\overline{S}_d)$. In particular,

$$K_{p,q}(n, b; d) \neq 0 \iff \text{(8.3.2) has non-zero homology at the indicated term.}$$

While it seems to be difficult to exhibit explicitly cycles representing homology classes for (8.3.1), we shall see momentarily that this is easily accomplished for (8.3.2)

The proof of Theorem 8.1.5 requires a certain amount of book-keeping, so it seems worthwhile to begin by walking through the first non-trivial case. Specifically, we indicate (following [49]) how to use the approach of that paper to establish the non-vanishing

$$K_{3d-2,2}\big(\mathbf{P}^2; \mathcal{O}_{\mathbf{P}^2}(d)\big) \ \neq \ 0$$

of Ottaviani–Paoletti (Proposition 6.2.8). In other words, we need to exhibit a non-vanishing cohomology class for (8.3.2) in the case $n = 2$ and $b = 0$.

To begin with, fix distinct monomials

$$m_1, \ldots m_{3d-2} \in S_d$$

of degree $d$, each divisible by $z_0$ or $z_1$. Since $z_0^d = z_1^d = 0$ in $\overline{S}_d$, it follows that

$$c \ =_{\text{def}} \ m_1 \wedge \ldots \wedge m_{3d-2} \otimes z_0^{d-1} z_1^{d-1} z_2^2 \tag{*}$$

is a cycle for the complex

$$\cdots \longrightarrow \Lambda^{3d-1}\overline{S}_d \otimes \overline{S}_d \longrightarrow \Lambda^{3d-2}\overline{S}_d \otimes \overline{S}_{2d} \longrightarrow \Lambda^{3d-2}\overline{S}_d \otimes \overline{S}_{3d} \longrightarrow \cdots . \qquad (**)$$

We need to show that we can ensure that $c$ represents a non-zero cohomology class upon suitable choices of the $m_i$.

Observe to this end that the monomial $z_0^{d-1}z_1^{d-1}z_2^2$ has precisely $3d-2$ monomial divisors of degree $d$ with exponents $\leq d-1$, namely:

$$\begin{gathered} z_0^{d-1}z_1 \, , \; z_0^{d-2}z_1^2 \, , \; \ldots \, , \; z_0^2 z_1^{d-0} \, , \; z_0 z_1^{d-1} \\ z_0^{d-1}z_2 \, , \; z_0^{d-2}z_1 z_2 \, , \; \ldots \, , \; z_0 z_1^{d-2} z_2 \, , \; z_1^{d-1}z_2 \\ z_0^{d-2}z_2^2 \, , \; z_0^{d-3}z_1 z_2^2 \, , \; \ldots \, , \; z_0 z_1^{d-3} z_2^2 \, , \; z_1^{d-2}z_2^2 \; . \end{gathered}$$

(The arrangement in rows follows the power of $z_2$ that appears.) We claim that taking these for the $m_i$ in (*) guarantees that $c \not\sim 0$. In fact, suppose that $c$ were to appear even as a term in the Koszul boundary of an element

$$e \; = \; n_0 \wedge n_1 \wedge \, \ldots \, \wedge n_{3d-2} \, \otimes \, g,$$

where the $n_j$ and $g$ are monomials of degree $d$. After re-indexing we can suppose that

$$c \; = \; n_1 \wedge \ldots \wedge n_{3d-2} \, \otimes \, n_0 g.$$

Thus the $\{n_j\}$ with $j \geq 1$ must be a re-ordering of the monomials $\{m_i\}$ dividing $z_0^{d-1}z_1^{d-1}z_2^2$. On the other hand, $n_0 g = z_0^{d-1}z_1^{d-1}z_2^2$, so $n_0$ is also such a divisor. Therefore $e = 0$, a contradiction. Observe that if $m_{3d-1}, \ldots, m_p$ are additional monomials that annihilate $z_0^{d-1}z_1^{d-1}z_2^2 \in \overline{S}$, then a similar argument shows that

$$\big(m_1 \wedge \ldots \wedge m_{3d-2} \wedge m_{3d-1} \wedge \ldots \wedge m_p\big) \otimes z_0^{d-1}z_1^{d-1}z_2^2$$

represents a non-zero Koszul class, and different choices of $m_{3d-1}, \ldots, m_p$ yield linearly independent classes.

Returning to the general case of Theorem 8.1.5, we sketch – closely following [49] – how the argument just completed extends to arbitrary $n$ and $0 \leq b \leq d-1$. Given a collection $P \subseteq \overline{S}_d$ of monomials, we denote by $\Lambda^p P \subseteq \Lambda^p \overline{S}_d$ the set of all wedge products of $p$ elements in $P$. We write $\det P$ for the wedge (in some fixed order) of all of the monomials in question. Now fix a non-zero monomial $g \in \overline{S}_{qd+b}$, and denote by

$$D_g \, , \, Z_g \; \subseteq \overline{S}_d$$

the set of degree $d$ monomials that respectively divide or annihilate $g$ in $\overline{S}$. Then just as in the special case worked out above, one has:

**Lemma 8.3.1.** (*i*). *If $\xi \in \Lambda^p Z_g$, then*

$$\xi \otimes g \ \in \ \Lambda^p \overline{S}_d \otimes \overline{S}_{qd+b}$$

*is a cycle for the complex* (8.3.2).

(*ii*). *Let $\xi \in \Lambda^s \overline{S}_d$ be a wedge of degree $d$ monomials having the property that* $\det D_g \wedge \xi \neq 0$. *Then*

$$(\det D_g \wedge \xi) \otimes g \ \in \ \Lambda^{(\# D_g)+s}\, \overline{S}_d \otimes \overline{S}_{qd+b}$$

*is not a boundary in that complex.* □

This immediately implies:

**Corollary 8.3.2.** *Given $q, d$ and $0 \leq b < d$, let $g \in \overline{S}_{qd+b}$ be a monomial such that $D_g \subseteq Z_g$, and let $\xi \in \Lambda^s \overline{S}_d$ be a wedge of $s$ distinct monomials lying in $Z_g - D_g$.*[1] *Then*

$$(\det D_g \wedge \xi) \otimes g$$

*represents a non-zero cohomology class for* (8.3.2). *In particular,*

$$K_{p,q}(n, b; d) \neq 0$$

*for every $p$ satisfying*

$$\# D_g \leq p \leq \# Z_g. \tag{8.3.3}$$

It remains only to specufy a convenient choice of $g$ and to explicate the inequality (8.3.3).

*Sketch of Proof of Theorem 8.1.5.* Recalling that we assume $b + q \leq d - 1$, we take

$$g = z_0^{d-1} \cdot z_1^{d-1} \cdot \ldots \cdot z_{q-1}^{d-1} \cdot z_q^{b+q} \in \overline{S}_{qd+b}.$$

Then evidently $D_g \subseteq Z_g$. Write $s_d = \dim \overline{S}_d$, so that $s_d = \binom{n+d}{d} - (n+1)$. Among the $s_d$ monomials in $\overline{S}_d$, those *not* lying in $Z_g$ are those that map to non-zero elements in the quotient

$$\overline{S}/(z_0, \ldots, z_{q-1}, z_q^{d-b-q}).$$

With a little work, this leads to the statement that

$$\# Z_g = \binom{d+n}{n} - \binom{d+n-q}{n-q} + \binom{n+b}{q+b} - q - 1.$$

Similarly, one verifies that $\# D_g$ is given by the left-hand side of (8.1.4). We refer to [49, pp. 216-217] for details. □

**Remark 8.3.3.** Theorem 2.1 of [49] involves a slightly more general statement that does not require $d \geq b + q - 1$. §3 of that paper also uses similar computations to give an explicit non-vanishing result for any Veronese re-embeddings of arithmetically Cohen-Macaulay variety $X \subseteq \mathbf{P}^m$.

[1] We allow the possibility that $s = 0$, in which case we take $\xi = 1$.

## 8.3.B Complements

In this subsection, we indicate without proofs some further developments concerning the syzygies of Veronese varieties.

**Schur asymptotics.** The Koszul cohomology groups of different embeddings of the projective space $\mathbf{P} = \mathbf{P}(V)$ are representations of the semi-simple group $\mathrm{SL}(V)$, and it is natural to ask whether one can say anything about their decomposition into irreducible pieces. Weyman [194, §7.2], for example, recovers the result (Proposition 6.2.8) of Ottaviani–Paoletti from this perspective.

In [80], Fulger and Zhou analyze the Schur asymptotics of $K_{p,1}\big(\mathbf{P}(V); \mathcal{O}_{\mathbf{P}(V)}(d)\big)$. Specifically, they prove:

**Theorem 8.3.4.** *Fix* $p$*, and assume that* $\dim V > p$*. Then as* $d \to \infty$*, the number of distinct irreducible representations of* $\mathrm{SL}(V)$ *appearing in this* $K_{p,1}$ *is given by a function having order of growth precisely* $d^p$*, i.e.*

$$\#\left\{ \begin{matrix} \text{Distinct irreducible representations of } \mathrm{SL}(V) \text{ appearing in} \\ K_{p,1}\big(\mathbf{P}(V); \mathcal{O}_{\mathbf{P}(V)}(d)\big) \end{matrix} \right\} \in \Theta(d^p).$$

They also prove that, counting multiplicities, the number of irreps grows like $d^{\binom{p}{2}}$. In fact, consider the Koszul complex

$$\Lambda^{p+1} S^d V \longrightarrow \Lambda^p S^d V \otimes S^d V \longrightarrow \Lambda^{p-1} S^d V \otimes S^{2d} V$$

computing the $K_{p,1}$ in question. The observation of Fulger and Zhou is that thanks to Pieri's rule, any irreducible representation appearing in the term on the right must correspond to a Young diagram with at most $p$ rows. Therefore any representation indexed by a diagram with $p+1$ rows that occurs in the middle term must lie in the subspace of Koszul cycles. They then use arguments involving convex geometry to estimate the number of these. Note that if one fixes $p$ and lets $d \to \infty$, then Green's Theorem 6.2.1 shows that only $K_{p,1}$ appear. It would of course be interesting to say something when both $p$ and $d$ grow, but at the moment this seems out of reach.

**Characteristic dependence.** Although we always work over $\mathbf{C}$, one can of course consider Veronese embeddings of projective space defined over any field $k$. It is then natural to ask whether the resulting Betti numbers depend for example on the characteristic of $k$. Following Booms-Peot, Erman and Yang [26], one says that Veronese syzygies have $\ell$-torsion for a prime number $\ell$ if the Betti numbers over a field of characteristic $\ell$ differ from those of the Veronese over $\mathbf{Q}$. Non-zero torsion is known to occur for the two-fold Veronese ([?]), but for $d > 2$ the situation is currently uncharted.

The paper [26] considers the analogous question for the Stanley–Reisner ideals of random flag complexes, which one can view as a combinatorial analogue of Veronese embedding. The main result is that with high probability torsion always appears. This leads Booms-Peot and co-authors to make:

**Conjecture 8.3.5.** *Assume $n \geq 7$. Then as $d \to \infty$, the number of primes $\ell$ such that $\ell$-torsion appears in the Betti table of the $d$-fold Veronese embedding of $\mathbf{P}^n$ becomes unbounded.*

We remark that the monomial arguments appearing above work perfectly well over any field, and one can easily write down cycles for (8.3.2) that are visibly characteristic dependent. However it's not clear that they represent non-trivial cohomology classes that don't arise in characteristic zero.

**Products of projective space.** It is natural to ask whether one can extend these monomial arguments to study other varieties. As noted in Remark 8.3.3, this works nicely for Veronese re-embeddings of arithmetically Cohen–Macaulay varieties. In her papers [27, 28], Bruce considers the more subtle cases of Hirzebruch surfaces and products of projective spaces. Consider $X = \mathbf{P}^{n_1} \times \mathbf{P}^{n_2}$, and for $\vec{d} = (d_1, d_2)$, write

$$L_{\vec{d}} = \mathcal{O}_{\mathbf{P}^{n_1}}(d_1) \boxtimes \mathcal{O}_{\mathbf{P}^{n_2}}(d_2).$$

The main result of [27] gives an expicit range of $p$ for which the Koszul groups $K_{p,q}(X; L_{\vec{d}}) \neq 0$. This allows one to study the asymptotics for example when $d_1$ is fixed and $d_2 \to \infty$. As above the proof proceeds by an Artinian reduction, but here the argument is more subtle because there is no obvious regular sequence consisting of monomials that one can use.

**The vanishing conjecture.** Conjecture 8.1.6 predicts that $K_{p,q}(n, b; d) = 0$ outside the range covered by Theorem 8.1.5. When $q = 0$ this follows from Green's vanishing Theorem 5.3.1, and then one deduces the case $q = n$ by duality. When $p = 1$ the conjecture was recently established by Kemeny [115], who reduces the statement to the case $n = 2$. As far as we know, other instances of the conjecture remain open.

## 8.4 Conjectures on Betti numbers

In this section we consider the asymptotics of the Betti numbers associated to a very positive embedding of a given variety. We start in dimension $n = 1$, which is the one setting where actual results are known. Our discussion follows [48] and [54], and for simplicity we mainly focus on the "untwisted" setting corresponding to $B = \mathcal{O}_X$.

Suppose then that $X$ is a smooth projective curve of genus $g$, and let $L_d$ be a line bundle of degree $d \gg 0$ on $X$. Write

$$r_d \ = \ h^0(X, L_d) \ - \ 1 \ = \ d - g,$$

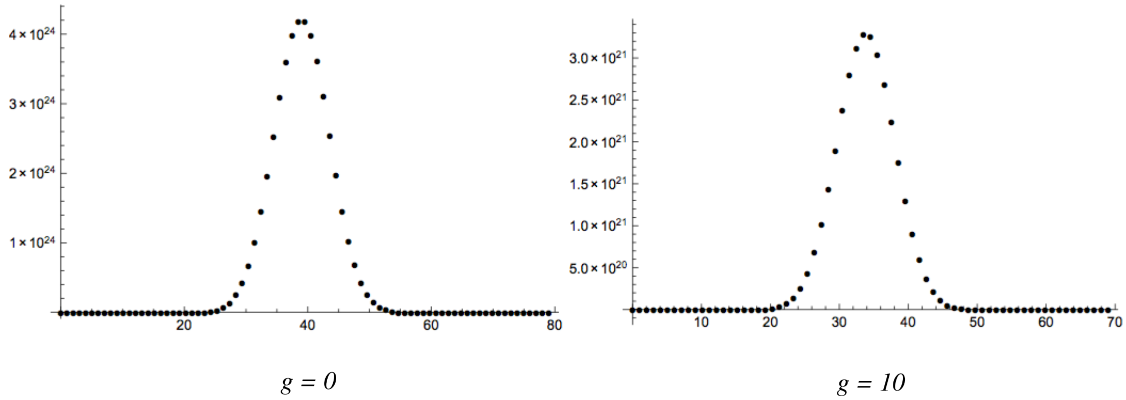



Figure 8.1: Betti numbers of curves of degree 80

and denote by $M_d$ the kernel bundle corresponding to $L_d$. Thus $M_d$ has degree $= -d$ and rank $r_d$. By the results of §5.2, the Betti numbers $k_{p,q}(X; L_d)$ are computed via the cohomology of $M_d$:

$$\begin{aligned} k_{p,1}(X; L_d) &= h^0(X, \Lambda^p M_d \otimes L_d) - \dim \Lambda^{p+1} H^0(L_d) \\ k_{p-1,2}(X; L_d) &= h^1(X, \Lambda^p M_d \otimes L_d). \end{aligned}$$

Furthermore, by Green's theorem on curves of large degree, $k_{p-1,2}(X; L_d) = 0$ for $p \leq r_d - g$. In other words, for $p \leq r_d - g$, one finds that

$$k_{p,1}\big(X; L_d\big) = \chi\big(X, \Lambda^p M_d \otimes L_d\big) - \binom{r_d + 1}{p+1}.$$

This Euler characteristic can in turn be computed by Riemann–Roch. The slope of $M_d$ is given by $\mu(M_d) = \frac{-d}{d-g}$, and hence

$$\begin{aligned} \chi\big(X, \Lambda^p M_d \otimes L_d\big) &= \operatorname{rank}(\Lambda^p M_d) \cdot \big(p \cdot \mu(M_d) + \mu(L_d) + 1 - g\big) \\ &= \binom{d-g}{p} \left( \frac{-pd}{d-g} + (d+1-g) \right). \end{aligned}$$

Writing $\binom{r_d+1}{p+1} = \binom{d-g}{p} \cdot \left(\frac{d+1-g}{p+1}\right)$, one arrives at

**Proposition 8.4.1.** *For $p \leq r_d - g$, one has*

$$k_{p,1}\big(X; L_d\big) = \binom{d-g}{p} \left( \frac{-pd}{d-g} + (d+1-g) - \frac{d+1-g}{p+1} \right). \qquad (*)$$

The results of Section 7.2 imply that the Betti numbers $k_{p,1}(X; L_d)$ are no longer uniform when $r_d - d < p \leq r_d - 1$. On the other hand, $K_{p,1}(X; L_d)$ is a sub-quotient of $\Lambda^p H^0(L_d) \otimes H^0(L_d)$, and therefore

$$k_{p,1}(X; L_d) \leq \binom{d+1-g}{p} \cdot (d+1-g).$$

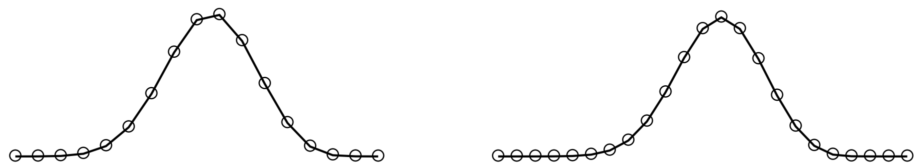


Figure 8.2: Plot of $k_{p,1}$ for 5-fold and 6-fold embeddings of $\mathbf{P}^2$ (Figure by D. Erman)

For $p \in [d+1-2g, d-g-1]$ this becomes a polynomial upper bound of degree $\leq g+1$ in $d$ that is much smaller than the value of $k_{p,1}$ for $p \approx \frac{d-g}{2}$. This being said, Proposition 8.1.7 appearing in Section 8.1 follows from (*) using Sterling's formula. In Figure 8.1, we reproduce from the Introduction plots of these Betti numbers for degree 80 embeddings of curves of genera 0 and 10.

What about varieties of dimension $\geq 2$? As of this writing, there is not a single such variety whose Betti asymptotics are actually known. However the philosophy of the present lecture is that the asymptotic behavior of the syzygies of large degree embeddings of any variety tend to behave uniformly. In particular, it seems reasonable to expect that the Betti numbers themselves should present a predictable picture. This being so, the simplest guess is that after suitable normalization they should approximate a Gaussian curve. Conjecture 8.1.8 is the outcome of this train of thought.

One can try to use `Macaulay2` to compute examples, but this turns out to become impractical quite quickly. Bruce et al [29] calculated the syzygies of the $d$-fold Veronese embedding of $\mathbf{P}^2$ for $d \leq 6$, but for $d = 6$ this already required (as of 2020) very serious computational resources and pre-processing. Their results are summarized in Figure 8.2 (kindly transmitted by Dan Erman). At least they are visually consistent with the Conjecture.

It is also not hard to establish lower and upper bounds for Betti numbers that are both Gaussian in shape. For example, using the constructions of the previous section, one finds that

$$\binom{\frac{d(d-1)}{2}}{p} \;\leq\; k_{p,2}\big(\mathbf{P}^2; \mathcal{O}_{\mathbf{P}^2}(d)\big) \;\leq\; \binom{\frac{(d+2)(d+1)}{2}}{p} \cdot (2d+1)(d+1),$$

but unfortunately the two sides do not match up.

In the absence of actual examples, one can look for probabilistic or exploratory evidence in favor of Conjecture 8.1.8. This was the approach taken in [48], which studied "random" Betti tables using Boij–Söderberg theory (Lecture 2). Closely following the exposition in [54, §3], we explain how this goes in the simplest setting.

Namely, consider the Koszul groups $K_{p,q}(\mathbf{P}^2, B; L_d)$, where $B = \mathcal{O}_{\mathbf{P}^2}(-1)$ and $L_d = \mathcal{O}_{\mathbf{P}^2}(d)$. Then $K_{p,q} = 0$ for $q \neq 1, 2$, so that the Betti table has only two rows. In this case Theorem 2.2.8 guarantees that the Betti numbers $k_{p,q}(\mathbf{P}^2, B; L_d)$ can be expressed as a rational linear combination of those of pure modules whose Betti tables consist of $i$ non-zero entries in the $q = 1$ row followed by $r_d - i$ non-zero entries in the $q = 2$ row. In other words, there exist modules $\Pi_i$ $(1 \leq i \leq r_d)$ with

$$K_{p,1}(\Pi_i) \neq 0 \Leftrightarrow 0 \leq p < i \quad , \quad K_{p,2}(\Pi_i) \neq 0 \Leftrightarrow i \leq p \leq r_d$$

together with non-negative rational numbers $x_i = x_i(\mathbf{P}^2, B; L_d)$, such that

$$k_{p,q}(\mathbf{P}^2, B; L_d) \ = \ \sum_{i=0}^{r_d} x_i \cdot k_{p,q}(\Pi_i). \tag{8.4.1}$$

We call $\{x_i\}$ the Boij–Söderberg coefficients of the Betti table of $B$ with respect to $L_d$, but of course we don't know much about their values.

Now for arbitrary $x_i \geq 0$, the right-hand side of (8.4.1) defines the Betti numbers of a module with given Boij–Söderberg coefficients. One can think of these as modeling the Betti table of an algebraic surface in $\mathbf{P}^{r_d}$. In order to test whether or not the behavior predicted by Conjecture 8.1.8 is somehow "typical," we ask what happens if we choose the $x_i$ at random.

By scaling we can suppose that $x_i \in [0, 1]$, so consider the hyper-cube $\Omega_r = [0, 1]^r$ parameterizing $r$-tuples of Boij–Söderberg coefficients. Given

$$x \ = \ \{x_i\} \ \in \ \Omega_r,$$

denote by

$$k_{p,q}(x) \ = \ \sum_{i=0}^{r} x_i \cdot k_{p,q}(\Pi_i) \tag{8.4.2}$$

the entries of the corresponding two-rowed Betti table. Stated somewhat informally, it is established in [48] that with high probability the picture predicted by the Conjecture holds for such a random Betti table.

**Theorem 8.4.2.** *Fix $q = 1$ or $q = 2$. Then as $r \to \infty$, with probability $= 1$ the Betti numbers $b_{p,q}(x)$ defined in* (8.4.2) *become normally distributed when $x \in \Omega_r$ is chosen uniformly at random.*

We refer to [48] for the precise statement. A similar result holds for the Betti tables modeling the syzygies of varieties of dimensions $\geq 3$. It is also established in [48] that the conclusion of the Theorem is quite robust, in the sense that it holds also for if $x = \{x_i\}$ is sampled with respect to many other probability measures on $\Omega_r$.

**Remark 8.4.3** (**Asymptotic Boij–Söderberg cofficients**)**.** It is natural to ask whether the actual Boij–Söderberg coefficients $x_i(\mathbf{P}^2, B; L_d)$ (along with the relevant values of $i$) can be normalized in such a way that they arise as the values of a "nice" function. Figure 5 of [29] seems promising in this respect.

Another avenue of investigation, of independent interest, is to consider families of simplicial complexes whose Stanley–Reisner ideals can be expected to model large degree embeddings. Recall that a *flag complex* is a simplicial complex $\Delta$ with the property that any set of vertices pairwise connected by edges forms a face of $\Delta$. For example, barycentric subdivisions of polytopes form flag complexes. Recalling the results [43] of Conca et al on syzygies of subdivisions quoted in Theorem 8.2.9, it is natural to ask about the Betti numbers of the ideals of flag complexes. In their nice paper [71], Erman and Yang study random choices of such complexes. We briefly summarize their results.

Note first that given a graph $G$, there is a canonical maximal flag complex whose 1-skeleton is $G$: one adjoins a $k$-simplex to every $(k+1)$-clique in $G$. On the other hand, there is a well-studied notion of a random graph. Namely, given an integer $r > 0$ and $0 < p < 1$, an Erdős–Réyni random graph $G(r, p)$ is the graph obtained by starting with $r$ vertices and joining each pair with probability $p$. Denote by $\Delta = \Delta(r, p)$ the corresponding flag complex. Erman and Yang [71, Corollary 1.5] establish:

**Theorem 8.4.4.** *Fix a real number $0 < c < 1$, and let $\Delta = \Delta(r, c/r)$ be a random flag complex. Choose a sequence $\{p_r\}$ of integers converging to $\frac{r}{2} + \frac{a\sqrt{r}}{2}$. Then*

$$\frac{\sqrt{2\pi}}{(1-c)\cdot 2^r\sqrt{r}} \cdot k_{p_r,1}(S/I_\Delta) \longrightarrow e^{-a^2/2}$$

*in probability.*

Erman and Yang also show that if $n > 0$ is an integer such that $r^{-1/q} \ll (c/r) \ll 1$, then the analogue of (8.1.3) holds for the $q^{\text{th}}$ row of the Betti table of $S/I_\Delta$.

The basic idea of the proof of the Theorem is to use Hochster's Theorem 5.4.4 to reduce the question to investigating the number of connected components of a random flag coomplex $\Delta(i, c/r)$. It seems to be unknown whether an analogue of 8.4.4 holds for the Betti numbers $k_{p.q}$ with $q > 1$.

## 8.5 Notes ⋄

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