MATH 589 – PROBLEM SET 1

Due Tuesday February 1

(1). Let k be an algebraically closed field, and let $M_{n\times n} = \mathbf{A}^{n^2}(k)$ be the affine space of all $n \times n$ matrices with entries in k. Determine which of the following subsets of $M_{n\times n}$ are algebraic:

(a). $\mathrm{SL}(n) \ =_{\text{def}} \ \big\{ A \in M_{n\times n} \mid \det A = 1 \big\}$.

(b). $\mathrm{Diag}(n) \ =_{\text{def}} \ \big\{ A \in M_{n\times n} \mid A \text{ can be diagonalized} \big\}$.

(c). $\mathrm{Nilp}(n) \ =_{\text{def}} \ \big\{ A \in M_{n\times n} \mid A \text{ is nilpotent, i.e. } A^N = 0 \text{ for some } N \big\}$.

(2). Let P_d be the set of monic polynomials in one variable T over $\mathbf{C}$, which we identify with $\mathbf{A}^d = \mathbf{A}^d(\mathbf{C})$ in the evident way: the polynomial $f = T^d + c_1T^{d-1} + \ldots + c_d$ corresponds to $(c_1, \ldots, c_d) \in \mathbf{A}^d$. Let

$$D \ = \ \{ f \in P_d \mid f \text{ has a repeated root } \}.$$

Show that $D \subseteq \mathbf{A}^d$ is a hypersurface, the so-called discriminant locus.

HINT: Write (formally) $f = \Pi_{i=1}^d (T - \rho_i)$, so that $c_i = \pm i^{\text{th}}$ elementary symmetric polynomial in the ρ_i. Then use the fact that any symmetric polynomial in the ρ_i can be expressed as a polynomial in the c_i.

(3). Consider the subset $X \subseteq \mathbf{C}^2$ defined by

$$X \ = \ \big\{ (z, e^z) \mid z \in \mathbf{C} \big\}.$$

Prove that X is not an algebraic subset of $\mathbf{C}^2$. (Your argument shouldn't be more than a paragraph or so!)

(4). Let $n \geq 2$, and let $f \in k[x_1, \ldots, x_n]$ be a non-constant polynomial over an algebraically closed field k. Show that $X = \{f = 0\} \subseteq \mathbf{A}^n$ is infinite. When $k = \mathbf{C}$, show that X is non-compact in the classical topology. (Hint: View f as a polynomial in one of the variables whose coefficients are polynomials in the remaining ones.)

(5). For which fields k is it true that any algebraic subset in $X \subseteq \mathbf{A}^n(k)$ $(n \geq 2)$ is the zero-locus of a single polynomial $f \in k[x]$? (HINT: To get going, note that this is happens when $k = \mathbf{R}$, although there are other fields as well. Compare the previous problem for some k that you have to rule out....)

Math 589 – Problem Set 2

Due Tuesday February 8

Unless otherwise stated, we work as usual over an algebraically closed field k and if you like you can assume that k has characteristic 0.

(1). Let $A \subseteq k[t]$ be the set of polynomials whose linear term vanishes. Then A is a finitely generated reduced k-algebra, and hence $A = k[X]$ for some affine variety X. Carry out this identification explicitly, ie find X.

(2). Let $X \subseteq \mathbf{A}^2$ be the curve $xy = 1$. Prove that X is not isomorphic to $\mathbf{A}^1$.

(3). As in class, let

$$M_{n\times m}^{\leq r} \ \subseteq \ \mathbf{A}^{nm}$$

be the set of all $n \times m$ matrices of rank $\leq r$. Prove that $M_{n\times m}^{\leq r}$ is irreducible.

Hint: $GL_n(k) \times GL_m(k)$ acts with dense orbit.

(4). Let $X \subseteq \mathbf{A}^{n^2}$ be the locus

$$X \ = \ \{A \in M_{n\times n} \mid \det A = 0\}$$

(so $X = M_{n\times n}^{\leq n-1}$). Prove that X is birationally isomorphic to $\mathbf{A}^{n^2-1}$.

Hint: Let V be an n-dimensional vector space over k. It may be helpful to note that a linear transformation $V \longrightarrow V$ of rank $\leq n-1$ is determined by specifying a one-dimensional subspace $K \subset V$ together with a linear transformation $V/K \longrightarrow V$.

MATH 589 – PROBLEM SET 3

Due Thursday Feb. 17

Unless otherwise stated, we work as usual over an algebraically closed field k.

(1). Consider the quasi-projective variety

$$X = \mathbf{A}^n - \{0\}.$$

Assuming that $n \geq 2$ show that the ring of regular functions on X is $k[x_1, \ldots, x_n]$. Deduce that X is not (isomorphic to) an affine algebraic set.

(2). Consider the curve

$$\{Y^2Z - X^3 - X^2Z = 0\} \subseteq \mathbf{P}^2.$$

Draw the (restriction of) this curve in each of the affine planes $U_X = \{X \neq 0\}$, $U_Y = \{Y \neq 0\}$ and $U_Z = \{Z \neq 0\}$. Indicate how the pictures fit together, i.e. how asymptotes in one view are reflected in another.

(3). Assume that $\mathrm{char}(k) \neq 2$, and consider a quadric hypersurface $Q \subseteq \mathbf{P}^n$.[1] Prove that after a linear change of coordinates Q is defined by the equation

$$X_0^2 + \ldots + X_r^2 = 0$$

for some integer $1 \leq r \leq n$ (called the *rank* of Q). How can one recognize the rank of Q in terms of geometric data? (You are free – and encouraged – for this problem to use facts from algebra about the classification of quadratic forms over k.)

(4). Given an algebraic set $X \subseteq \mathbf{P}^n$, show that X can be cut out by homogeneous polynomials all having the same degree, say d. (Note that we do not assert that these polynomials actually generate the full homogeneous ideal of X.)

[1] In other words, Q is defined by an equation of degree 2.

Math 589 – Problem Set 4
Due Thursday Feb 24

Throughout we work unless otherwise stated over an algebraically closed field k (which when convenient you can assume does not have too small characteristic – but be specific about this).

(1). Let $X \subseteq \mathbf{P}^n$ be a hypersurface of degree d (ie $X = \{F_d = 0\}$ for some homogeneous polynomial of degree d). Show that $\mathbf{P}^n - X$ is (isomorphic to) an affine variety.

(2). (a). Show that a subset $X \subseteq \mathbf{P}^n \times \mathbf{P}^m$ is Zariski closed if and only if it is defined by a collection of bihomogeneous polynomials.

(b). Recall that via the Segre mapping one can identify $\mathbf{P}^1 \times \mathbf{P}^1$ with the quadric surface $XZ = YW$ in $\mathbf{P}^3$: so we will call this surface $\mathbf{P}^1 \times \mathbf{P}^1$. Now fix $d \geq 2$ and consider the curve $C_d \subseteq \mathbf{P}^3$ arising as the image of the mapping:

$$\mathbf{P}^1 \longrightarrow \mathbf{P}^3 \quad , \quad [S, T] \mapsto [S^d, S^{d-1}T, ST^{d-1}, T^d].$$

Thus $C_d \subseteq \mathbf{P}^1 \times \mathbf{P}^1$. Write down a single bihomogeneous polynomial (of suitable bi-degrees) that defines C_d as a subset of $\mathbf{P}^1 \times \mathbf{P}^1$.

(3). Let

$$f(x) \;=\; a_0x^d + \ldots + a_d \quad , \quad g(x) = b_0x^e + \ldots + b_e$$

be polynomials of degrees d and e over an algebraically closed field (with $a_0 \neq 0, b_0 \neq 0$). Write P_i for the vector space of polynomials of degree $\leq i$ in x.

(a). Prove that f and g have a common root if and only if the mapping

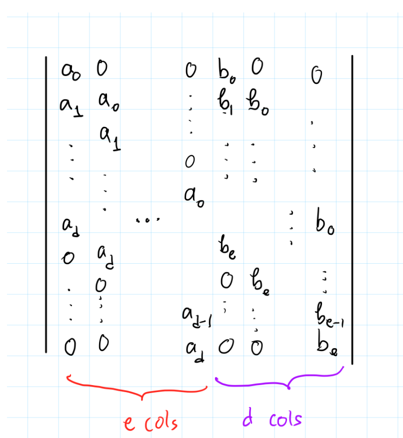
$$\mu : P_{e-1} \oplus P_{d-1} \longrightarrow P_{d+e-1}$$

defined by

$$\mu(p, q) \;=\; p(x)f(x) \;+\; q(x)g(x)$$

has a non-trivial kernel.

(b). The *resultant* $\mathrm{Res}(f, g)$ of f and g is defined to be the determinant:



Prove that f and g have a common root if and only if $\mathrm{Res}(f,g)=0$.

(c). Consider the algebraic set

$$Z \subseteq \mathbf{A}^1 \times \mathbf{P}^1$$

defined by the two equations

$$a_0(t)X^d + \ldots + a_d(t)Y^d \quad , \quad b_0(t)X^e + \ldots + b_e(t)Y^e.$$

Find an equation for $\mathrm{pr}_2(Z) \subseteq \mathbf{A}^1$,

(4). Let S_d denote the vector space of all homogeneous polynomials of degree d in $(n+1)$ variables, and consider as in class the mapping

$$\mu_{a,b} : \mathbf{P}(S_a) \times \mathbf{P}(S_b) \longrightarrow \mathbf{P}(S_{a+b}) \quad , \quad (A,B) \mapsto A \cdot B.$$

Prove that this is a morphism of varieties. Let $R_{a,b} \subseteq \mathbf{P}(S_{a+b})$ denote the image. Prove that if $a \neq b$, then there is a non-empty Zariski-open subset of $R_{a,b}$ over which $\mu_{a,b}$ is one-to-one. However show that $\mu_{a,b}$ is not globally one-to-one over $R_{a,b}$. What happens if $a = b$?

Math 589 – Problem Set 5

Due Tuesday March 22

(1). (a). Let

$$\phi : \mathbf{P}^a \times \mathbf{P}^b \longrightarrow \mathbf{P}^n$$

be the morphism determined by $n+1$ bihomogeneous forms of type $(1,1)$ which do not simultaneously vanish on $\mathbf{P}^a \times \mathbf{P}^b$. Then ϕ is finite over its image. (Compare Theorem 8 on p. 65 of Shafarevich.) A similar statement holds for products of ≥ 3 projective spaces.

(b). Let $f : \mathbf{A}^d \longrightarrow \mathbf{A}^d$ be the morphism given by

$$t \;=\; (t_1, \ldots, t_d) \;\mapsto\; \big(\sigma_1(t), \ldots, \sigma_d(t)\big),$$

where $\sigma_1, \ldots, \sigma_d$ are the elementary symmetric functions in $t_1, \ldots, t_d$.[1] Prove that f is finite.

(2). Let $f : X \longrightarrow Y$ be a dominant morphism of irreducible (quasi-projective) varieties. Prove that

$$\dim Y \;\leq\; \dim X.$$

(3). Given $d \geq 2$, denote by $\mathbf{P}^{N(d)}$ the projective space parametrizing all plane curves of degree d. (So $N(d) = \binom{d+2}{2} - 1$.)

(a). Show that if $d = 2$, then there is a dense open subset $U \subset \mathbf{P}^5$ such that all conics corresponding to points in U are projectively equivalent, i.e. differ by a linear change of coordinates.

(b). On the other hand, prove that the analogous statement fails when $d \geq 3$.

Note: You may grant that $\mathrm{SL}(n+1)$ is irreducible. You will want to observe that it has dimension $= (n+1)^2 - 1$.

(4). Find the dimension of the space $M^{\leq r}_{n \times m}$ of all $n \times m$ matrices of rank $\leq r$.

[1] Equivalently, if you view the target $\mathbf{A}^d$ as parameterizing monic polynomials of degree d in one variable X, then f is the map that takes a d-tuple $(t_1, \ldots, t_d)$ to the polynomial $(X + t_1) \cdot \ldots \cdot (X + t_d)$.

MATH 589 – PROBLEM SET 6
Due Tuesday April 5

(1). (a). Let $X \subseteq \mathbf{P}^{n+1}$ be a hypersurface, so $\dim X = n$. Let $V, W \subseteq X$ be closed subvarieties. Show that if $\dim V + \dim W \geq n+1$, then V and W must meet. Prove that in general the inequality cannot be improved.

(b). We saw in class that if $C \subseteq \mathbf{P}^2$ is a conic curve of maximal rank 3 (i.e. if C is non-singular), then $C \cong \mathbf{P}^1$, and similarly a quadric surface $Q \subseteq \mathbf{P}^3$ of maximal rank 4 is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. This suggests the conjecture that if $Q \subseteq \mathbf{P}^{n+1}$ is a quadric hypersurface of maximal rank $n+2$, then

$$Q \ \cong \ \mathbf{P}^1 \times \ldots \times \mathbf{P}^1 \quad (n \text{ times}).$$

Settle this conjecture one way or the other, i.e. either prove it or give a counter-example.

(2). (a). Show that for every $d \geq 2$ and $n \geq 2$, there exist singular hypersurfaces of degree d in $\mathbf{P}^n$ that have only finitely many singular points.

(b). Prove that if $n \geq 2$, then a non-singular hypersurface in $\mathbf{P}^n$ is irreducible.

(c). Is the analogue of (b) true for codimension 1 subvarieties of $\mathbf{P}^n \times \mathbf{P}^n$?

(3). Let

$$\Delta \ = \ M_{n\times n}^{\leq n-1} \ \subseteq \ M_{n \times n} \ = \ \mathbf{A}^{n^2}$$

denote the set of all singuler $n \times n$ matrices. (So Δ is the hypersurface defined by the vanishing of the determinant of an $n \times n$ matrix of variables.) Find the singular locus of Δ, and the multiplicity of Δ at each of its singular points.

(4). Let $\mathbf{G} = \mathbf{G}(1,3)$ be the Grassmannian parameterizing lines in $\mathbf{P}^3$, and fix a line

$$\ell_0 \subseteq \mathbf{P}^3.$$

Let $\Sigma \subset \mathbf{G}$ be the subset of $\mathbf{G}$ corresponding to all lines meeting ℓ_0, so that (as we've seen before) Σ is a hyperplane section of $\mathbf{G}$.

(a). Show that Σ has a unique singular point (at the point of $\mathbf{G}$ corresponding to ℓ_0). In fact, Σ' is a quadric $\mathbf{Q}$ of rank 4 in $\mathbf{P}^4$

(b). Let $\mathbf{Q} \subset \mathbf{P}^4$ be a quadric of rank 4 in $\mathbf{P}^4$, with singular point o$\in \mathbf{Q}$. Let $\mathbf{Q}'$ be the proper transform of $\mathbf{Q}$ under the blowing up of o$\in \mathbf{P}^4$. Show that $\mathbf{Q}'$ is non-singular, and that the fibre of

$$\nu : \mathbf{Q}' \longrightarrow \mathbf{Q}$$

over o is (isomorphic to) $\mathbf{P}^1 \times \mathbf{P}^1$.

(c). Returning to the realization Σ of $\mathbf{Q}$ in (a), define

$$\Sigma' \subseteq \mathbf{G} \times \ell_0$$

to be the incidence correspondence

$$\Sigma' \ = \ \big\{ ([\ell], x) \mid x \in \ell \cap \ell_0 \big\}$$

Show that Σ' is smooth, and that the projection

$$\mu : \Sigma' \longrightarrow \Sigma$$

is an isomorphism away from $[\ell_0]$, while $\mu^{-1}[\ell_0]$ is a copy of $\mathbf{P}^1$. This map is called the "small resolution" of $\mathbf{Q} = \Sigma$.

Math 589 – Problem Set 7
Due Thursday April 14

(1). Let $X \subseteq \mathbf{C}^3$ be the hypersurface $x^2 + y^4 + z^4 = 0$, and denote by $X' \subseteq \mathrm{Bl}_0(\mathbf{C}^3)$ the proper transform of X under the blowing up of the origin. Show that X has an isolated singularity, but that X' is singular along a curve.

(2). Let X be a smooth real manifold, and let $\mathcal{H}^1$ denote the presheaf on X defined by

$$\mathcal{H}^1(U) = H^1\big(U, \mathbf{R}\big)$$

(singular or de Rham cohomology, as you prefer), with the natural restriction maps. Is $\mathcal{H}^1$ a sheaf?

(3). Let $O \in \mathbf{P}^n$ be a fixed point, let $\mathbf{L} = \mathbf{P}^n - \{O\}$, and let

$$p : \mathbf{L} \longrightarrow \mathbf{P}^{n-1}$$

be the morphism given by linear projection from O. Show that p realizes $\mathbf{L}$ as the total space of a line bundle over $\mathbf{P}^{n-1}$, and find its transition functions with respect to the standard open covering of $\mathbf{P}^{n-1}$ by copies of $\mathbf{A}^{n-1}$.

(4). Denote by $\mathbf{B}$ the complement of the diagonal $\Delta \subseteq \mathbf{P}^1 \times \mathbf{P}^1$, and let

$$\pi : \mathbf{B} \longrightarrow \mathbf{P}^1$$

be projection onto the first factor. Thus

$$\pi^{-1}(p) \ = \ \mathbf{P}^1 \ - \ \{p\} \ \cong \ \mathbf{A}^1.$$

Show that π is a Zariski-locally trivial $\mathbf{A}^1$ bundle, ie that for a suitable open covering $\{U_i\}$ of $\mathbf{P}^1$,

$$\pi^{-1}(U_i) \ \cong \ U_i \times \mathbf{A}^1$$

under an identification that realizes π as projection to the first factor. On the other hand, prove that $\mathbf{B}$ is not the total space of a line bundle over $\mathbf{P}^1$. What is happening here on the level of transition functions? Can you find an analogous construction of a Zariski-locally trivial $\mathbf{A}^n$ bundle over $\mathbf{P}^n$ that is not the total space of a vector bundle?

Math 589 – Problem Set 8
Due Tuesday April 24

(0). (Not to write up). Think through the connection between divisors and line bundles stated in class.

(1). Let $\mu : X \longrightarrow \mathbf{P}^2$ be the blowing up of a point $P \in \mathbf{P}^2$, let $E \subseteq X$ be the exceptional divisor, and let $L = \mathcal{O}_X(E)$ be the line bundle corresponding to E.

(i). Write down transition functions for L with respect to a convenient affine cover of X.

(ii). Show that $\dim \Gamma\big(X, \mathcal{O}_X(mE)\big) = 1$ for every $m \geq 0$. What happens when $m < 0$?

(2). Let $(X, \mathcal{O}_X)$ be a variety (over an algebraically closed field k), and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Show that $\mathcal{F}$ is the sheaf of sections of a vector bundle if and only if $\mathcal{F}$ is locally free of some rank r, ie every point $x \in X$ has a neighborhood $U = U_x$ with the property that $\mathcal{F}|U \cong \mathcal{O}_X^r$, ie the restriction of $\mathcal{F}$ to U is free of rank r.

(3). (Fibres of a sheaf.) Let $(X, \mathcal{O}_X)$ be an algebraic variety, as always defined over an algebraically closed field k. Given a point $x \in X$, denote by $\mathfrak{m} = \mathfrak{m}_x \subseteq \mathcal{O}_X$ the ideal (sheaf) of all functions vanishing at x. Then

$$k(x) =_{\text{def}} \mathcal{O}_X/\mathfrak{m} = \mathcal{O}_{\{x\}}$$

is the structure sheaf of x, which we can think of as a copy of the ground field $k = k[\{x\}]$ supported at x. Via extension by zero, we view $k(x)$ as a coherent $\mathcal{O}_X$-module. (This $k(x)$ is often called a "sky-scraper sheaf," since one visualizes it as a one-dimensional vector space sticking out of X at the point x.)

Now let $\mathcal{F}$ be a coherent sheaf on X and set

$$\mathcal{F}(x) \;=\; \mathcal{F}/\mathfrak{m}\cdot\mathcal{F}$$

This is called the *fibre* of $\mathcal{F}$ at x (but the notation $\mathcal{F}(x)$ isn't always standard). We can view $\mathcal{F}(x)$ as a finite dimensional vector space over $k = k(x)$.

(i). Show that if $\mathcal{F} = \mathcal{O}_X(\mathbf{F})$ is the sheaf of sections of a rank r vector bundle $\mathbf{F}$, then

$$\dim \mathcal{F}(x) = r \quad \text{for every } \; x \in X.$$

(In fact, one can identify $\mathcal{F}(x)$ as the fibre of $\mathbf{F}$ over x.)

(ii). Show that the function $x \mapsto \dim_k \mathcal{F}(x)$ is Zariski-upper semicontinuous on X. That is, for each $\ell \in \mathbf{N}$, the set

$$X_\ell = \{x \mid \dim \mathcal{F}(x) \geq \ell\}$$

is Zariski closed. (Hint: suppose that $\dim \mathcal{F}(x) = \ell$. Use Nakayama's Lemma to show that the stalk $\mathcal{F}_x$ is generated by ℓ elements as a module over $\mathcal{O}_x X$, and hence that $\dim \mathcal{F}(y) \leq \ell$ for all y in a neighborhood of x.)

(iii). Let $X = \mathbf{A}^{nm}$ be the affine space of all $n \times m$ matrices. Then multiplication by the matrix $A = (x_{ij})$ of variables defines a homomorphism of (locally free) sheaves

$$u : \mathcal{O}_X^m \longrightarrow \mathcal{O}_X^n.$$

Let $\mathcal{F} = \text{coker}(u)$. Describe the closed sets X_ℓ from (ii) in this example.

(iv). Continuing the line of thought of (iii), let $u : \mathbf{E} \longrightarrow \mathbf{F}$ be a homomorphism of vector bundles on a variety X, giving a homomorphism (that we also denote by u)

$$u : \mathcal{O}_X(\mathbf{E}) \longrightarrow \mathcal{O}_X(\mathbf{F})$$

of the corresponding sheaves. Describe the dimensions of the fibres of the sheaf $\text{coker}(u)$. Is there a corresponding statement for $\ker(u)$?