Math 589 (W, 2022)

Math 589

(Winter/Spr. 2022)

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Affine Alg Sets (Shaf., I.2; Gathmann, Chapts 1,2)

k = field. Interested in alg subsets of

$$\mathbb{A}^n = \mathbb{A}^n(k) = \{(a_1, \ldots, a_n) \mid a_i \in k\}.$$

Fix finitely many polys:

$$f_1, \ldots, f_r \in k[x] = k[x_1, \ldots, x_n].$$

Def: Alg subset of $\mathbb{A}^n$ is set of form

$$X = \mathrm{Zeros}(f_1, \ldots, f_r) \underset{\text{def}}{=} \{a \mid f_i(a) = 0 \text{ all } i\} \subseteq \mathbb{A}^n.$$

Examples

(1) $M_{n\times m} = M_{n\times m}(k) = \left\{ \begin{matrix} \text{all } n\times m \text{ matrices} \\ \text{w. entries in } k \end{matrix} \right\} = \mathbb{A}^{nm} \ni A = (x_{ij})$.

Define:

$$M_{n\times m}^{\leq r} = \{A \mid \mathrm{rk}\, A \leq r\}.$$

Then

$$M_{n\times m}^{\leq r} \text{ is alg} : = \left\{ \begin{matrix} \text{dets of all } (r+1)\times(r+1) \\ \text{minors of } (x_{ij}) \end{matrix} = 0 \right\}$$

⊗ HW: $\mathrm{Nilp}(n), \mathrm{Diag}(n) \subseteq M_{n\times n}$

Ex (Vague). Consider a ("generic") sm map of mflds

$$f : X^m \longrightarrow Y^n, \quad n \geq m.$$

Under what conds on m & n do we expect that there might exist $x \in M$ st:

$$df_x : T_x X \longrightarrow T_{f(x)} Y \quad \text{drops rk?}$$

Idea: locally on X, can view $x \mapsto df_x$ as map

$$\begin{array}{ccc} X & \xrightarrow{df} & M_{n\times m}, \quad x \mapsto df_x \\ & & \cup\!| \\ & & M_{n\times m}^{<m} \end{array}$$

<u>Question is</u>: do we expect $\mathrm{Im}(df)$ to meet $M_{n\times m}^{<m}$?

$\mathbb{C}^m \quad \mathbb{C}^n$

Facts: $\dim M_{n\times m} = nm$

$\dim M_{n\times m}^{<m} = nm - (n-m+1)$

So: If $n-m+1 \geq m$, ie. if

$n \geq 2m-1$,

then on dimensional grounds can expect $\exists$ pts where df_x drops rk.

(2). Define:

$$P_d = P_d(k) = \left\{ \begin{array}{l} \text{all monic polys of deg } d \\ \text{in one variable } T \end{array} \right\}$$

So: $$P_d = \mathbb{A}^d \ni p(T) = T^d + c_1 T^{d-1} + \quad + c_d \leftrightarrow (c_1, \ldots, c_d).$$

Get various inter. subsets of $P_d = \mathbb{A}^d$ by imposing conds on poly.
Eg:

$$P_d(\mathbb{C}) \supseteq D = \left\{ \begin{array}{l} \text{polys w. a repeated} \\ \text{root.} \end{array} \right\}$$

($d=2$: T^2+bT+c has double root $\Leftrightarrow b^2-4c=0$)

<u>HW</u>: D = zeroes of single poly in $c_1, \ldots, c_d$.

Rmk: If $k = \bar{k}$, then $P_d = \left\{ \begin{array}{l} \text{unordered } d\text{-tuple} \\ \text{of pts of } k \end{array} \right\}$

(Food for thought: $\pi_1(P_d(\mathbb{C}) - D) \underset{\text{approx}}{\sim} Br(d)$)

(3). $X = \{x^2 + y^2 - 1 = 0\} \subseteq \mathbb{A}^2(k)$

Now there is strong dependence on k:

$\underline{k = \mathbb{Q}}$: For $a, b, c \in \mathbb{Z}$, $(\frac{a}{c}, \frac{b}{c}) \in X(\mathbb{Q}) \Leftrightarrow a^2 + b^2 = c^2$

$\underline{k = \mathbb{R}}$: $X(\mathbb{R})$ = unit circle in $\mathbb{R}^2$

$\underline{k = \mathbb{C}}$. $X(\mathbb{C}) \underset{\text{diffeo}}{\approx} S^2 - \{2 \text{ pts}\}$.

We won't be focusing on this, but there are deep connections bet arith. properties of diophantine eqns & the geometry of corresp cx variety.

Prop. Alg subsets satisfy axioms for closed subsets of top space, called <u>Zariski topology</u> on $\mathbb{A}^n$.

Pf: <u>Unions</u>: $X = \text{Zeroes}(f_i)$, $Y = \text{Zeroes}(g_j)$: $X \cup Y = \text{Zeroes}(f_i g_j)$

<u>Intersections</u>: Say $X_\alpha = \text{Zeroes}(f_{i,\alpha})$. Let

$$k[x] \supseteq I = \text{ideal gen by all the } \{f_{i,\alpha}\}_{i,\alpha}.$$

By Hilb basis thm, I is f.g. i.e. $I = (g_1, \ldots, g_r)$, $g_j \in k[X]$
Then

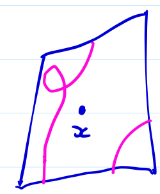
$$\cap X_\alpha = \text{Zeroes}(g_1, \ldots, g_r). \quad \text{(Check!)}$$

Sim, given $X \subseteq \mathbb{A}^n$ alg, Z-top on X is subspace top – closed sets are

$$X \cap F, \quad F \subseteq \mathbb{A}^n \text{ alg.}$$

Basis for open sets:

$$X_f = X - \{f = 0\}$$



"small Z nbd of x"

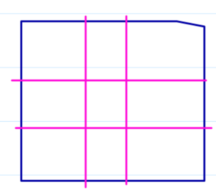
<u>NB</u>: Open nbd $x \in U \subseteq X$ "almost determines X." No std local models for alg vars.

<u>Products</u>: Have natural identification as sets:

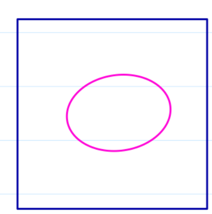
$$\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}.$$

Z top on $\mathbb{A}^n \times \mathbb{A}^m$ is defined to be Z-top on $\mathbb{A}^{n+m}$.

So: Z-top on $\mathbb{A}^n \times \mathbb{A}^m$ not prod top.



closed set in prod top



Z-closed set.

Sim:

$$X \subseteq \mathbb{A}^n, \ Y \subseteq \mathbb{A}^m \text{ alg}$$

$$\Downarrow$$

$$X \times Y \subseteq \mathbb{A}^{n+m} \text{ alg (Exerc!)}$$

Use to define Z-top on $X \times Y$.

Ideals & Coord Rings ~

Assume henceforth unless otherwise stated: $k = \bar{k}$.

- Let $X \subseteq \mathbb{A}^n$ be alg set. Ideal of X is

$$I_X = I(X) =_{\text{def}} \{ f \in k[x] \mid f(a) = 0 \text{ all } a \in X \}$$

- Given ideal $J \subseteq k[x]$, define

$$Z(J) = \text{Zeroes}(J) = \{ a \mid f(a) = 0, \text{ all } f \in J \} \subseteq \mathbb{A}^n$$

Exerc: $\quad Z(I(X)) = X$.

However: given J, it can happen that

$$I(Z(J)) \supsetneq J.$$

Ex. $J = (x^2) \subseteq k[x]$, $Z(J) = \{0\}$, $I_{\{0\}} = (x)$,

In general: given J, let $X = Z(J)$, say $f^m \in J$.

$$X \subseteq \{ f^m = 0 \} = \{ f = 0 \},$$

so

$$f \in I(X).$$

Recall: $\sqrt{J} =_{\text{def}} \{ f \mid f^N \in J \text{ some } N \} \quad (J \subseteq \sqrt{J})$

So: Given ideal J,

$$\sqrt{J} \subseteq I(Z(J))$$

↑ Tues 1/25

Thm (Version of Hilbert Nullstellensatz): Assume $k = \bar{k}$.

Then for any ideal J,

$$I(Z(J)) = \sqrt{J}.$$

Esp, there is 1-1 order-reversing corresp bet:

ideals I s.t. $I = \sqrt{I}$

$$\begin{matrix}\text{Alg subsets} \\ X \subseteq \mathbb{A}^n\end{matrix} \longleftrightarrow \begin{matrix}\text{radical ideals} \\ I \subseteq k[x_1, \ldots, x_n]\end{matrix}$$

Rmk: It can be very difficult to determine whether given ideal is radical. Eg consider

$$X = M_{n \times m}^{\leq r} \subseteq M_{n \times m}.$$

So

$$X = Z(J),$$

$J \subseteq k[x_{ij}]$ ideal gen by $(r+1) \times (r+1)$ minors,

Ask: is $J = I_X$, ie. is $J = \sqrt{J}$? (Yes, but non-trivial!)

Will discuss pf of Nullstellensatz / $\mathbb{C}$. Observe: if L any field, then

L alg closed $\iff$ any max ideal in $L[x]$ is $(x-a)$ for some $a \in L$

Thm ("Classical Nullstellensatz") Assume $k = \bar{k}$. Then any max ideal

$$M \subseteq k[x_1, \ldots, x_n] \quad \text{is}$$

$$M = (x_1 - a_1, \dots, x_n - a_n) \quad \text{some } a_i \in k,$$

ie.

$$M = \{ f \mid f(a_1, \dots, a_n) = 0 \}.$$

Will prove (1C) shortly. Grant for now:

<u>Cor</u>: Assume $k = \bar{k}$, let

$$J \subsetneqq k[x_1, \dots, x_n] \text{ be any non-triv ideal}$$

Then

$$\text{Zeroes}(J) \neq \phi.$$

(<u>Pf</u>. $J \subseteq M$ = max ideal)

<u>Thm</u>: Assume $k = \bar{k}$, let

$$X = \text{Zeroes}(J) \subseteq \mathbb{A}^n.$$

Let $f \in k[x_1, \ , x_n]$ be poly st f vanishes on X. Then

$$f^m \in J \quad \text{some } m,$$

ie. $I_X = \sqrt{J}$.

Pf ("Trick of Rabinowitz"). Pass to poly ring in one more var:

$$k[x, t] = k[x_1, \ , x_n, t].$$

Given $J = (g_1, \ , g_r)$, consider

$$J^\# = J + (tf - 1) = (g_1, \ , g_r, tf - 1).$$

ideal in $k[x, t]$.

Observe: $Z(J^{\#}) = \phi$.

(Say $(a,b) \in \mathbb{A}^n \times \mathbb{A}^1$ in $Z(J^{\#})$. Then $a \in Z(J) = X$. But $f \in I_X$, so $f(a) = 0 \Rightarrow bf(a) = 0 \neq 1$ #)

So by Cor, $1 \in Z(J^{\#})$, ie. $\exists\ h_i(x,t)$ s.t.

$$h_0(x,t)(tf-1) + h_1(x,t)\,g_1(x) + \cdots + h_r(x,t)g_r(x) = 1$$

Plug in $t = 1/f$:

$$h_1(x, \tfrac{1}{f})\, g_1(x) + \cdots + h_r(x, \tfrac{1}{f})\, g_r(x) = 1$$

Multiply thru by large power of f to clear denoms:

$$H_1(x) g_1(x) + \quad + H_r(x)\, g_r(x) = f^m,$$

ie. $f^m \in J$ for $m \gg 0$! $\square$

<u>Pf of Classical Nullst. when $k = \mathbb{C}$</u>. Fix max ideal

$$M \subseteq R = \mathbb{C}[x_1, \ldots, x_n], \quad \text{so} \quad L = M/R \text{ a field}$$

· Consider compos

$$\pi : \mathbb{C}[x_1] \hookrightarrow \mathbb{C}[x_1, \ , x_n] = R \longrightarrow R/M = L$$

· <u>Main claim</u>: $\ker(\pi) \neq (0)$.

Then $\ker\pi$ gen by irred poly since $\mathbb{C}[x_1]/\ker\pi \subseteq L$, so

$\ker(\pi) = (x_1 - a_1)$ some $a_1 \in \mathbb{C}$, $\Rightarrow$ $x_1 - a_1 \in M$.

Sim $x_i - a_i \in M$ all i, done

Pf of Main Claim: Suppose to contrary π injective. Then

$$\mathbb{C}(x_1) \hookrightarrow L$$

On other hand, the monomials in $\mathbb{C}[x_1, , x_n]$ gen L as a v.s $/\mathbb{C}$, i.e.

$$\dim_{\mathbb{C}} L \text{ is countable}$$

So get contr from observation:

$\mathbb{C}(x_1)$ is not gen by countably many elts as a v.s. $/\mathbb{C}$

(Pf: the fns $\left\{\frac{1}{x_1 - a}\right\}_{a \in \mathbb{C}}$ are lin indep $/\mathbb{C}$!) ∎

Coord Rings

Given $X \subseteq \mathbb{A}^n$, let $I = I_X$. Define

$$k[X] = k[x_1, , x_n]/I.$$

This is coord ring of X: elts are poly fns $f: X \longrightarrow k$. By constr, $k[X]$ is reduced (no nilpotents).

Prop: Any f.g k-alg R is $k[X]$ for some $X \subseteq \mathbb{A}^n$.

Pf. Choose gens $u_1, .., u_n \in R$. Define

$$\phi : k[x_1, \dots, x_n] \longrightarrow R, \quad x_i \mapsto u_i$$

Surj by def. Let $I = \ker(\phi)$,

$$X = Z(I) \subseteq \mathbb{A}^n.$$

I is radical since R reduced, so $I = I_X$.

<u>Ex.</u> Consider

$$\mathbb{C}[t] \supseteq R = \{ f(t) \mid f(1) = f(-1) \}$$

Let's realize R as coord ring. Gens:

$$u = t^2 - 1, \quad v = t(t^2-1)$$

Relations among u, v:

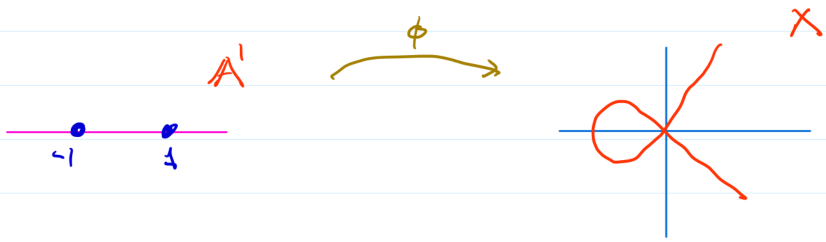
$$v^2 - u = u^3 \quad \text{ie} \quad v^2 = u(u^2-1).$$

In fact,

$$R \cong \mathbb{C}[x,y]/(y^2 - x(x^2-1)) = \mathbb{C}[X]$$

w/

$$X = \{ y^2 - x(x^2-1) = 0 \} \subseteq \mathbb{A}^2.$$



There is map $\phi : \mathbb{A}^1 \longrightarrow X$, $t \mapsto (t^2-1, t^3-t)$ that realizes X as quotient of $\mathbb{A}^1$ after identifying 1 & -1.

Rmk. Have 1-1 corresp:

$$\left\{\begin{matrix}\text{alg subsets} \\ Y \subseteq X\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{radical ideals} \\ \text{in } k[X]\end{matrix}\right\}$$

(always assuming $k = \bar{k}$)

$$\text{pts } x \in X \quad \leftrightarrow \quad \text{max ideals } \mathfrak{m} \subseteq k[X]$$

In gen:

$$\begin{matrix}\text{affine alg sets} & \leftrightarrow & \text{reduced f.g. } k\text{-algs} \\ \Psi & & \cup\!\!\mid \\ \text{pts} & \longleftrightarrow & \text{max ideals}\end{matrix}$$

Grothendieck's idea: find "geom" object corresp to arb comm ring.

Irreducibility:

Consider alg set $X \subseteq \mathbb{A}^n$

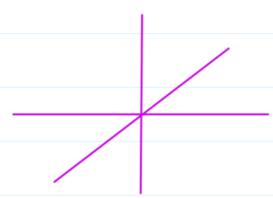
Def. X is reducible if

$$X = Y_1 \cup Y_2, \quad Y_1, Y_2 \subsetneq X \text{ proper alg subsets.}$$

X irreducible if not reducible.

Ex. $X = Z(xy, yz, zx) \subseteq \mathbb{A}^3$

(coord axes) – reducible.



Thm. Let $X \subseteq \mathbb{A}^n$ be alg set. Then X can be written as union

$$X = Y_1 \cup \dots \cup Y_s \text{ of irred alg subsets}$$

in such a way that there are no inclusions among the Y_i.
Decomp unique up to order.

Follows (as in the proof of unique factorizn in $\mathbb{Z}$) from two facts:

<u>Lemma 1</u>: Alg subsets of X satisfy the DCC: any chain

$$X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$$

of alg subsets stabilizes.

<u>Lemma 2</u>: Assume X irred, and

$$X \subseteq Y_1 \cup Y_2 \subseteq \mathbb{A}^n, \quad Y_i \text{ closed.}$$

Then

$$X \subseteq Y_1 \text{ or } X \subseteq Y_2.$$

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For details: see Gathmann, Chapt 2 or Shafarevich, Ch 1, §3.1.

<u>Additional Facts:</u>

(1) $U \subseteq X$ irred $\Rightarrow$ $\bar{U}$ irred.

U not union of proper closed sets.

(2). X, Y irred $\Rightarrow$ $X \times Y$ irred. (See Shaf, 1.3.1., Thm 3)

(3). X irred $\Rightarrow$ every non-empty Z-open $U \subseteq X$ is dense.

Ex. Consider

$$P_d = \{\text{monic polys deg } d \,/\, \mathbb{C}\}$$
$$\cup\!\!\mid$$
$$S = \{\text{polys w. } \leq d-2 \text{ distinct roots}\}$$

Then S is not irred: in fact

$$S = \left\{\begin{matrix}\text{polys w.}\\ \text{a 3-fold root}\end{matrix}\right\} \cup \left\{\begin{matrix}\text{polys w. two}\\ \text{double roots}\end{matrix}\right\}$$

Ex. $M^{\leq r}_{n \times m}$ is irred (HW)

Prop. $X \subseteq \mathbb{A}^n$ alg set. Then

$$X \text{ irred} \iff I_X \text{ a prime ideal (ie } k[X] \text{ an int. domain)}$$

Pf. Suppose f, g polys st. $fg \in I_X$. If $f, g \notin I_X$, then

$$X = (X \cap \{f=0\}) \cup (X \cap \{g=0\})$$

gives non-trivial decomp of X. Converse for you. $\square$

Def: Affine variety is irred affine alg set.

Morphisms ("Regular mappings"). [Shaf, §1.2.3]

Consider affine alg sets

$$X \subseteq \mathbb{A}^n, \quad Y \subseteq \mathbb{A}^m.$$

A morphism $f: X \longrightarrow Y$ is a mapping given by an m-tuple of reg fns

$$f = (f_1, \ldots, f_m), \quad f_i \in k[X].$$

Ex. $d: M_{n\times n} \to \mathbb{A}^1$, $d(A) = \det A$.

(In general, $f \in k[X]$ is same as $f: X \to \mathbb{A}^1$)

Similarly:

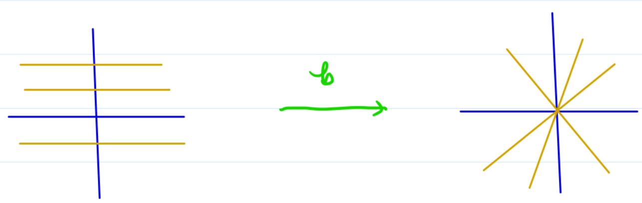
$Adj: M_{n\times n} \to M_{n\times n}$, $A \to$ matrix of cofactors

$m: M_{n\times m} \times M_{m\times \ell} \to M_{n\times \ell}$, $m(A,B) = A\cdot B$

$inv: SL(n) \to SL(n)$, $A \to A^{-1}$

Ex. $s: P_d \to P_{2d}$, $p(T) \to (p(T))^2$.

Ex. $b: \mathbb{A}^2 \to \mathbb{A}^2$, $(x,y) \to (x, xy)$



Note that b maps the lines (y = const) to lines of fixed slope thru 0.
In particular, see:

$$\mathrm{Im}(b) = \mathbb{A}^2 - \{y\text{-axis}\} \cup \{0\} \quad : \text{ not Zariski-closed}$$

Now get notion of isomorphism:

Def. Alg sets

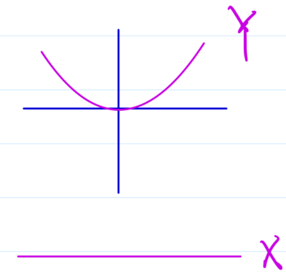
$$X \subseteq \mathbb{A}^n \ , \ Y \subseteq \mathbb{A}^m$$

are <u>isomorphic</u> if $\exists$ inverse morphisms

$$f: X \to Y \ , \ g: Y \to X.$$

<u>Ex.</u> $X = \mathbb{A}^1$, $Y = \{y - x^2 = 0\} \subseteq \mathbb{A}^2$

$$\begin{array}{cccc} t & \longmapsto & (t, t^2) \\ \cap & & \cap \\ X & & Y \end{array} \qquad \begin{array}{ccc} (x,y) & \longmapsto & x \\ \cap & & \cap \\ Y & & X \end{array}$$



More generally, for any $f : X \to Y$, form graph

$$\Gamma_f = \{(x, f(x))\} \subseteq X \times Y.$$

Then
$$\Gamma_f \cong X.$$

- Let $f : X \longrightarrow Y$ be morphism of aff alg sets. Then f induces k-alg homom

$$\begin{array}{ccc} f^* : k[Y] & \longrightarrow & k[X] \\ \psi & & \psi \\ \phi & & \phi \circ f \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ & & \downarrow \phi \\ & & k \end{array}$$

<u>Ex:</u> Say $f : \mathbb{A}^1 \longrightarrow \mathbb{A}^n$ is $t \longmapsto (g_1(t), \dots, g_n(t))$. Then

$$\begin{array}{ccc} f^* : k[x_1, \ \ , x_n] & \longmapsto & k[t] \\ \psi & & \psi \\ x_i & \longmapsto & g_i(t). \end{array}$$

<u>Prop:</u> Any k-alg homom $h : k[Y] \longrightarrow k[X]$ is of form $h = f^*$ for some $f : X \longrightarrow Y$

<u>Pf.</u> If $Y \subseteq \mathbb{A}^m$, then $k[Y]$ generated by images

$$u_1, \quad , u_m \in k[Y]$$

of coord fns on $\mathbb{A}^m$. Set

$$f_i = h(u_i) \in k[X].$$

Then f is $\quad f = (f_1, \quad , f_m) : X \longrightarrow Y \subseteq \mathbb{A}^m$.

<u>Cor.</u> $X \cong Y \quad \Longleftrightarrow \quad k[X] \cong k[Y]$ as k-algs.

<u>Def:</u> $f : X \longrightarrow Y$ is <u>dominant</u> if $f(X)$ is dense in Y.

<u>Prop:</u> $f : X \longrightarrow Y$ dominant $\Leftrightarrow$ $f^* : k[Y] \longrightarrow k[X]$ injective.
In gen.,

$$\ker(f^*) = \begin{pmatrix} \text{ideal of Zariski closure} \\ \text{of image } f(X) \subseteq Y \end{pmatrix}$$

<u>Exerc</u> $(k = \bar{k})$. Consider

$$f : X \longrightarrow Y \quad , \quad f^* : k[Y] \longrightarrow k[X]$$

Given $x \in X$, let $m_x \subseteq k[X]$ be max ideal of x. Then

$$(f^*)^{-1}(m_x) \text{ is max ideal in } k[Y],$$

and

$$(f^*)^{-1}(m_x) = m_{f(x)} \subseteq k[Y]$$

Complements of Hypersurfaces:

- Consider $0 \neq f \in k[\mathbb{A}^n] = k[x_1, \ldots, x_n]$. Want to realize

$$\mathbb{A}^n - \{f = 0\} \quad \text{as affine var.}$$

- Consider:

$$\begin{array}{ccccc} \mathbb{A}^n \times \mathbb{A}^1 & \supseteq & Z_f & =_{\text{def}} & \{tf(x) - 1 = 0\} \\ \downarrow \scriptstyle pr_1 & & \downarrow \scriptstyle bij & & \\ \mathbb{A}^n & \supseteq & \mathbb{A}^n - \{f=0\} & & \end{array}$$

<u>Prop:</u> Z_f irred (check!) and

$$k[Z_f] \simeq k[\mathbb{A}^n]_f = k[x_1, \ldots, x_n, \tfrac{1}{f}].$$

We identify $\mathbb{A}^n - \{f=0\} \leftrightarrow Z_f$.

<u>Ex.</u> $\mathbb{A}^1 - \{0\} = \{xy = 1\}$

<u>Ex.</u> $GL(n) = \{A \in M_{n \times n} \mid \det A = 0\}$. It is an example of:

<u>Def.</u> Affine alg group is affine var G that is also a group in such a way that

$$\text{mult}: G \times G \longrightarrow G, \quad \text{inv}: G \longrightarrow G \quad \text{are morphisms of vars.}$$

<u>Variant:</u> Ditto for X irred, $0 \neq f \in k[X]$:

Realize

$$X_f = X - \{f=0\} \subseteq X \times \mathbb{A}^1,$$

&

$$k[X_f] = k[X]_f.$$

Rational Fns. (Shaf, I.3.2;

X = irred / $k = \bar{k}$.

Def: A rational fn on X is elt of

$$k(X) = \left\{ \begin{array}{l} \text{field of fractions} \\ \text{of } k[X] \end{array} \right\}.$$

So

$\varphi \in k(X)$ is represented as $\phi = f/g$ for some $f, g \in k[X]$.

Moreover

$$\frac{f_1}{g_1} = \frac{f_2}{g_2} \iff f_1 g_2 = f_2 g_1 \text{ in } k[X].$$

Else φ indeterminate at x.

φ is regular at a point $x \in X$ if can write $\varphi = \frac{f}{g}$, $g(x) \neq 0$.

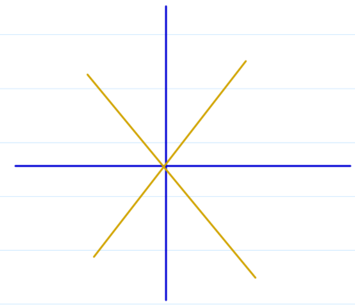
Warning: $k[X]$ typically not a UFD, so expression $\phi = f/g$ as ratio of reg fns usually not unique.

Rat fns that are reg at $x \in X$ are $k[X]_{\mathfrak{m}}$

Ex. $X = \{xy - zw = 0\} \subseteq \mathbb{A}^4$

$$k[X] = \frac{k[x,y,z,w]}{(xy - zw)}.$$

So $xy = zw$ in $k[X]$, so $\frac{x}{z} = \frac{w}{y} \in k(X)$. Regular off $z = y = 0$.



Ex. $\varphi = y/x$ on $\mathbb{A}^2$

Indeterminate on y-axis.

Rmk: φ is const on the lines $y/x = \text{const}$.
On y-axis $x=0$, φ becomes infinite.
Working over $\mathbb{C}$, can extend φ to

$$\mathbb{A}^2 - \{0\} \longrightarrow \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$$

However φ does not extend to contin map $\mathbb{A}^2 \longrightarrow \mathbb{P}^1$.

Rmk: Rat fn $\sim$ mero fn in complex geom.

Prop. Let X be affine var, and say $\varphi \in k(X)$ is rat fn that is regular at every pt. Then

$$\varphi \in k[X].$$

Pf. Given $x \in X$, can write

$$\varphi = \frac{f_x}{g_x} \qquad f_x, g_x \in k[X], \quad g_x(x) \neq 0.$$

Let $I \subseteq k[X]$ be ideal gen by all the g_x $(x \in X)$. Then

$$\text{Zeroes}(I) = \phi. \quad (\text{Given } x \in X, \; g_x(x) \neq 0).$$

So $I = k[X]$, esp $1 \in I$. So $\exists$ $x_1, \ldots, x_r \in X$, and $h_i \in k[X]$ s.t.

$$\Sigma h_i g_{x_i} = 1 \qquad (\text{"partition of unity."})$$

Now $g_{x_i} \phi = f_{x_i}$, so

$$\phi = \left(\sum h_i g_{x_i}\right)\phi = \sum h_i f_{x_i} \in k[X] \quad \text{QED}$$

Def. Let X = irred affine var, $U \subseteq X$ open set. A regular fn on U is k-valued fn

$$f : U \longrightarrow k$$

that is given by a rational fn on X which is regular at every pt of U.

Therefore get notion of morphism

$$\begin{array}{ccc} U & \longrightarrow & V \\ \cap\!\! \mid & & \cap\!\! \mid \\ X & & Y \end{array} \quad : \text{given by reg fns.}$$

hence also isom. (Such U & V are "quasi-affine" varieties.)

Rmk: In general, we will always define reg fns as rational fns that are everywhere regular. This has two consequences.

(a). By constr, regular fns will always extend to rational fns on closures. (I.e. "no essential sings in alg geom.")

(b). We will eventually define "quasi-proj vars:" open subsets of proj vars. Will give a class of vars where we have a natural concrete notion of rational fns.

Exerc. Say $U = \mathbb{A}^n - \{h = 0\}$. Then the regular fns on U in sense of this defn are exactly

$$f \in k[x_1, \ldots, x_n, \tfrac{1}{h}].$$

This justifies our constr of U as affine var.

Rational Mappings & Birat Isomorphisms:

Consider irred affine varieties

$$X \subseteq \mathbb{A}^n, \quad Y \subseteq \mathbb{A}^m.$$

Def: A rational mapping

$$\phi : X \dashrightarrow Y$$

is a "mapping" given by rational fns

$$\phi = (\phi_1, \ldots, \phi_m), \qquad \phi_i \in k(X)$$ (satisfying defining eqns of Y).

Ex. $\phi : \mathbb{A}^2 \dashrightarrow \mathbb{A}^2$, $(x, y) \longmapsto (x, y/x)$.

Note: Given $\phi : X \dashrightarrow Y$, $\exists$ non-empty (hence dense) open subsets $U \subseteq X$, $V \subseteq Y$ st ϕ restricts to a morphism

$$\phi : U \longrightarrow V \qquad (*)$$

Conversely, morphism $(*)$ is (by defn) restr of rational map.

I.e.

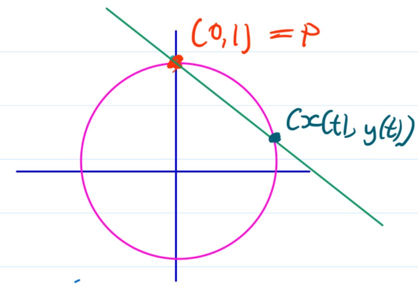
Rat. mapping $\phi : X \dashrightarrow Y$ $\Longleftrightarrow$ morph. defined on dense open set.

Def: $\phi : X \dashrightarrow Y$ is birational isom if it has a rational inverse (Say X, Y are birationally equiv.)

Ex. $\phi : \mathbb{A}^2 \dashrightarrow \mathbb{A}^2$ $(x, y) \longmapsto (x, y/x)$ is birat auto w inverse
$\psi : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$, $(u, v) \longmapsto (u, uv)$

Ex. Let $C = \{x^2+y^2=1\} \subseteq \mathbb{A}^2$. Then

$$C \sim_{\text{birat}} \mathbb{A}^1$$



How: fix pt $P = (0,1) \in C$. Consider line ℓ_t thru P with slope t. ℓ_t meets C at one additional pt $(x(t), y(t))$. Map

$$\mathbb{A}^1 \dashrightarrow C, \quad t \mapsto (x(t), y(t)) \quad \text{is birat isom}$$

Computations:

$$\ell_t: \quad (y-1) = tx, \quad \text{i.e. } y = tx+1$$

$$\ell_t \cap C: \quad x^2 + (tx+1)^2 = 1$$

$$x^2 + t^2x^2 + 2tx + 1 = 1$$

$$(1+t^2)x^2 + 2tx = 0$$

$$\boxed{x(t) = \frac{-2t}{1+t^2}, \quad y(t) = \frac{1-t^2}{1+t^2}} \qquad t = \frac{y-1}{x}$$

⊗ Sim: $\{x_1^2 + \cdots + x_{n+1}^2 = 1\} \sim_{\text{birat}} \mathbb{A}^n$

Rmk: For $d \geq 3$, consider

$$C_d = \{x^d + y^d = 1\} \subseteq \mathbb{A}^2$$

These are all birationally distinct.

(For sm proj curves / $\mathbb{C}$, birat isom $\Leftrightarrow$ bireg isom.)

Ex. Consider $P_d = \{\text{monic polys deg } d\} \supseteq \Delta = \{\text{polys w. a double rt}\}$

Then

$$\Delta \sim_{birat} P_{d-2} \times \mathbb{A}^1 \quad (= \mathbb{A}^{d-1}).$$

Idea: Consider

$$\Delta \supseteq U = \{\text{polys } p(T) \text{ w. exactly } d-1 \text{ distinct roots}\}$$

Then $p(T) \in U$ is of form $p(T) = (T-a)^2 \cdot p_{d-2}(T)$

some poly deg $d-2$

So Δ is (generically) parametrized by $a \in \mathbb{A}^1$ & P_{d-2}.

<u>Vocab</u>: X is <u>rational</u> if $X \sim_{bir} \mathbb{A}^n$.

Ask: what is alg meaning of rat maps?

Consider: <u>dominant</u> rat map $\phi : X \dashrightarrow Y$,

i.e. ϕ corresps to dominant morphism $\phi : U \longrightarrow V$

Then get

$$k(Y) \overset{\phi^*}{\hookrightarrow} k(X) \quad : \text{ inclusions of extensions of const field } k$$

$$\phi^*(f) = f \circ \phi.$$

Conversely: inclusion

$$k(Y) \hookrightarrow k(X) \quad \text{that is identity on } k$$

is

$$\phi^* \text{ for some } \phi : X \dashrightarrow Y. \quad (\text{Exerc!})$$

Hence:

<u>Thm.</u> $X \sim_{bir} Y \iff k(X) \simeq k(Y)$ as extensions of k.

<u>Warning:</u> If $X \overset{\phi}{\dashrightarrow} Y$ not dominant, don't get $\phi^* : k(Y) \hookrightarrow k(X)$
(Im ϕ might be contained in locus of indet of rat fn.)

<u>Ex.</u> (Cremona group):

$$Cr(n) =_{\text{df}} \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}(x_1, \ldots, x_n))$$

$$= \mathrm{Bir\,Aut}(\mathbb{A}^n(\mathbb{C}))$$

$n = 1$: $\quad \mathrm{Aut}(\mathbb{C}(x)) = PGL(2)$: $\quad x \mapsto \frac{ax+b}{cx+d}$

$n \geq 2$: Large and very subtle groups

<u>$n=2$:</u> $PGL(3) \hookrightarrow Cr(2)$ via:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} : \quad (x,y) \mapsto \left(\frac{ax+by+c}{gx+hy+i}, \frac{dx+ey+f}{gx+hy+i} \right)$$

$Cr(2)$ also contains involn. $\quad (x,y) \mapsto \left(\frac{1}{x}, \frac{1}{y}\right)$.

Noether: These generate

<u>$n \geq 3$:</u> Much more complicated

<u>Rmk:</u> For "most" alg vars X, $\quad \mathrm{Aut}_{Bir}(X) = \{id\}$.

Projective Varieties -

Want to compactify.

Proj. Space :

Let

$$V = k^{n+1} \qquad (n+1)\text{-dim v.s.}$$

Def 1: $\mathbb{P}^n = \mathbb{P}^n(k) = \{1\text{-dim subspaces of } V\}$

Pts described by homog coords:

Def 2: $$\mathbb{P}^n = \frac{\{(a_0, \dots, a_n) \mid \text{not all } a_i = 0\}}{\sim}$$

$$(a_0, \ , a_n) \sim (b_0, \ , b_n) \text{ if } b_i = \lambda a_i, \ \lambda \neq 0$$

Write: $[a_0, \dots, a_n]$.

Covering by affine spaces.

Say $a = [a_0, \ , a_n]$, $a_i \neq 0$. Then

$$a = \left[\frac{a_0}{a_i}, \dots, \frac{a_i}{a_i}, \ , \frac{a_n}{a_i}\right],$$ ratios $a_0/a_i, \ , a_n/a_i$ unambiguously def.

Set

$$\begin{array}{ccc} \mathbb{P}^n \supseteq U_i = \{[a_0, \dots, a_n] \mid a_i \neq 0\} \\ (*) \cong \downarrow \qquad \qquad \updownarrow \\ \mathbb{A}^n \ni \left(\frac{a_0}{a_i}, \ , \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \ \frac{a_n}{a_i}\right) \end{array}$$

Meaning of $(*)$:
- alg subsets of $\mathbb{P}^n$ will go over to alg subsets of $\mathbb{A}^n$
- U_i will be isom of alg var
- Over $\mathbb{C}$, gives mfld str on $\mathbb{P}^n$

<u>Ex</u> $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$

<u>$\mathbb{P}^n$ as $\mathbb{A}^n$ w hyperplane at ∞</u>:

Start w.

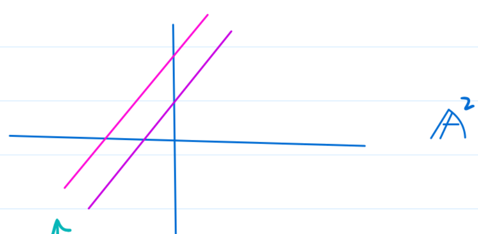
$$\mathbb{A}^n = U_0 = \{[1, x_1, \ \ , x_n]\} \subseteq \mathbb{P}^n$$

$$\mathbb{P}^n - U_0 = \{[0, x_1, \ \ , x_n]\} = \mathbb{P}^{n-1}$$

So

$$\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$$

view as parametrizing all directions in $\mathbb{A}^n$



these two lines meet at pt at ∞

<u>Proj Alg Sets</u> (Gathmann Ch 6, 7; Shaf Ch I.3, I.4)

<u>Note</u>: if $f \in k[x_0, \ldots, x_n]$ and $[a] \in \mathbb{P}^n$, neither the value $f(a)$, or even it's vanishing is well-defined.

However, say $F \in k[x_0, \ \ , x_n]$ is <u>homogeneous</u> of deg d. Then

$$F(\lambda a) = \lambda^d F(a),$$

so $\{F = 0\}$ well-defined subset of $\mathbb{P}^n$.

<u>Def</u>: Proj alg set is zero-locus of collection of homog polys.

These are closed subsets of Zariski topology.

Say $\mathbb{P}^n = \mathbb{P}V$, $\dim V = n+1$. If $W \subseteq V$ is subsp, then $\mathbb{P}W \subseteq \mathbb{P}V$ is linear subspace.

Dehomogenization – Suppose

$$X = Z(F_1, \ldots, F_r) \subseteq \mathbb{P}^n.$$

Set

$$f_j = F\left(1, \frac{X_1}{X_0}, \ \ , \frac{X_n}{X_0}\right) = F(1, x_1, \ \ , x_n) \ : \ x_i = X_i / X_0.$$

Then

$$X \cap U_0 = \{ f_1 = \ \ = f_n = 0 \} \subseteq U_0 = \mathbb{A}^n.$$

I.e. $X \cap U_0$ is affine alg subset of $\mathbb{A}^n$. In other words:

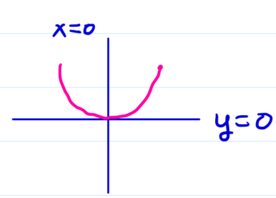
Zariski topology on $U_0 = \mathbb{A}^n$ (or $U_i = \{X_i \neq 0\}$) is subspace topology for Z-top on $\mathbb{P}^n$.

Cor: $X \subseteq \mathbb{P}^n$ alg $\Leftrightarrow$ $X \cap U_i \subseteq U_i = \mathbb{A}^n$ alg for each $i = 0, \ldots, n$.

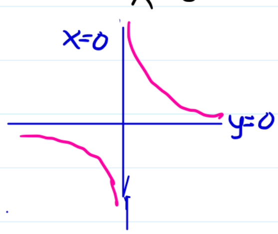
Ex. Consider $\mathbb{P}^2$ w coords X, Y, Z. We analyze the conic

$$C = \{ YZ - X^2 = 0 \}$$ by restricting to the 3 aff opens,

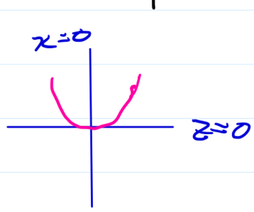
(1) $C \cap U_Z$: $y = x^2$

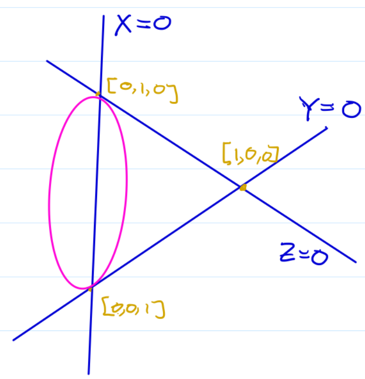


(3) $C \cap U_X$: $yz = 1$



(2) $C \cap U_Y$: $z = x^2$





curve meets line at ∞ $X=0$ at 2 pts,

<u>Homogenization</u>: Consider affine alg set

$$X_0 \subseteq \mathbb{A}^n = U_0.$$

<u>Ask</u>: How do we find Zariski closure

$$X = \overline{X}_0 \subseteq \mathbb{P}^n$$

of X_0 in $\mathbb{P}^n$? (So $\overline{X}_0$ is smallest proj alg set s.t. $\overline{X}_0 \cap U_0 = X_0$)

· Given $f = f(x_1, \ldots, x_n)$ of deg d, homogenization of f is

$$F = X_0^d \cdot f\left(\frac{1}{X_0}, \ldots, \frac{X_n}{X_0}\right)$$

(eg homog of $1 + x^2 + y^2x + y^4$ is $Z^4 + X^2Z^2 + Y^2XZ + Y^4$)

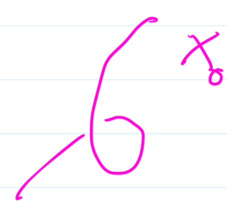
<u>Prop/Exer</u>: Zariski closure $\overline{X}_0 \subseteq \mathbb{P}^n$ is the proj alg set defined by the homogenizations of all polys

$$f \in I_{X_0} \subseteq k[x_1, \ldots, x_n].$$

<u>Warning</u>: Not in general enough to homog gens of I_{X_0}.

<u>Ex</u> Consider

$$X_0 = \{(t, t^2, t^3) \mid t \in k\} \subseteq \mathbb{A}^3$$



I_{X_0} gen by

$$y - x^2 = 0, \quad z - x^3 = 0.$$

Homogenize these:

$$YW - X^2, \quad ZW^2 - X^3$$

$$\text{Zeroes}(\uparrow\nearrow) = \overline{X}_0 \cup \{\text{line } W = X = 0\}$$

Ex. Consider

$$\Phi_n : \mathbb{P}^1 \longrightarrow \mathbb{P}^n \quad , \quad [S,T] \longmapsto [S^n, S^{n-1}T, \dots, ST^{n-1}, T^n]$$

Will show Φ_n 1-1, and $C_n \underset{\text{def}}{=} \operatorname{im}(\Phi_n) \subseteq \mathbb{P}^n$ alg.

- Note: $C_n \subseteq U_0 \cup U_n$.
- Show: $C_n \cap U_0 \subseteq U_0 = \mathbb{A}^n$ alg.

$$\Phi_n^{-1}(U_0) = \mathbb{P}^1 \cap \{S \neq 0\} \cong \mathbb{A}^1 \text{ w aff coord } t. \text{ Also:}$$

$$C_n \cap U_0 = \{(t, t^2, \dots, t^n) \mid t \in \mathbb{A}^1\} \text{ alg.}$$

Sim, $C_n \cap U_n$ alg. So $C_n \subseteq \mathbb{P}^n$ alg.

Rmk: What are actually all the homog polys defining $C_n \subseteq \mathbb{P}^n$? Check:

$$C_n = \left\{ \operatorname{rk} \begin{bmatrix} X_0 & X_1 & & X_{n-1} \\ X_1 & X_2 & & X_n \end{bmatrix} \leq 1 \right\}$$

(i.e. $C_n \subseteq \mathbb{P}^n$ defined by $\binom{n}{2}$ homog polys of deg 2: these generate homog ideal of C_n.)

Ideals: Write

$$S = k[X_0, \dots, X_n]$$

$$S = \bigoplus S_d, \quad S_d = \left\{ \begin{matrix} \text{homog polys} \\ \text{of deg } d \end{matrix} \right\} \qquad \left(\text{NB: } \dim S_d = \binom{n+d}{n} \right)$$

Lemma: Let $I \subseteq S$ be ideal. TFAE (and define homog ideal)

(a). I gen by homog polys

(b) $I_d \underset{\text{def}}{=} I \cap S_d \subseteq I$.

(c) $I = \bigoplus I_d$ (as v.s.)

· Given homog ideal $J \subsetneq S$, get

$$Z(J) \subseteq \mathbb{P}^n \quad : \text{proj alg set.}$$

· Conversely, if $X \subseteq \mathbb{P}^n$ is proj alg set, define homog ideal of X

$$I_X = I(X) = \bigoplus I(X)_d,$$

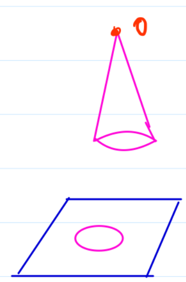
$$I(X)_d = \left\{ \begin{matrix} \text{homog polys deg } d \\ \text{vanishing on } X \end{matrix} \right\}.$$

Cones & Homog. Nullstellensatz –

By constr have natural map

$$\pi : \mathbb{A}^{n+1} - \{0\} \longrightarrow \mathbb{P}^n, \quad (a_0, \ldots, a_n) \longmapsto [a_0, \ldots, a_n].$$

Given $X \subseteq \mathbb{P}^n$ alg, define



$$C(X) = \pi^{-1}(X) \cup \{0\} \; : \; \underline{\text{affine cone over}} \; X.$$

If $X = Z(J)$ for $J \subseteq S$, then

$$C(X) = Z_{\mathbb{A}^{n+1}}(J), \text{ where } J \subseteq S = k[\mathbb{A}^{n+1}].$$

Sim, $I_{C(X)} = I_X$ viewed as ideal in $k[\mathbb{A}^{n+1}]$.

Now say $J \subsetneq S$ is homog ideal &

$$X = Z(J) \subseteq \mathbb{P}^n.$$

Then:

$$X = \phi \iff C(X) = \{0\} \subseteq \mathbb{A}^{n+1}$$

$$\underset{\text{Nullst.}}{\Longleftrightarrow} \quad (x_0, \dots, x_n)^M \subseteq J \quad \text{some } M > 0.$$

<u>Homog. Nullstell, I</u> Assume $k = \bar{k}$

$$\text{Zeroes}_{\mathbb{P}^n}(J) = \emptyset \iff (x_0, \dots, x_n)^M \subseteq J \quad \text{for } M \gg 0$$

$$\iff \sqrt{J} = (x_0, \ , x_n) \quad (\text{or } J = S)$$

<u>Ex.</u> Consider

$$J = (X_0^{d_0}, \ . \ . , X_n^{d_n}) \quad : \text{ this has empty zero-set.}$$

Then

$$(X_0, \ , X_n)^p \subseteq J \iff p \geqslant \Sigma d_i - n.$$

<u>Macaulay</u>: Ditto for any $F_0, \dots, F_n$ s.t. $\text{Zeroes}(F_0, \dots, F_n) = \emptyset$.

<u>Thm.</u> Assume $k = \bar{k}$. Consider:

$X \subseteq \mathbb{P}^n$ non-empty alg set

$J \subseteq S = k[X_0, \ , X_n]$ ideal s.t. $(x_0, \dots, x_n) \not\subseteq \sqrt{J}$.

Then

$$X = \text{Zeroes}_{\mathbb{P}^n}(I(X))$$

$$I(\text{zeroes}(J)) = \sqrt{J}$$

<u>Pf.</u> Apply classical Nullst. to $C(X)$.

Linear Autos of $\mathbb{P}^n$.

- $PGL(n+1) =_{\text{def}} GL(n+1)/(\text{scalar mxs})$ acts on $\mathbb{P}^n$:

$$g \cdot [a] = [ga].$$

(Rmk: turns out that $PGL = \mathrm{Aut}(\mathbb{P}^n)$.)

- Can use these autos to make judicious choices of coords.

Exerc: Consider $n+2$ pts

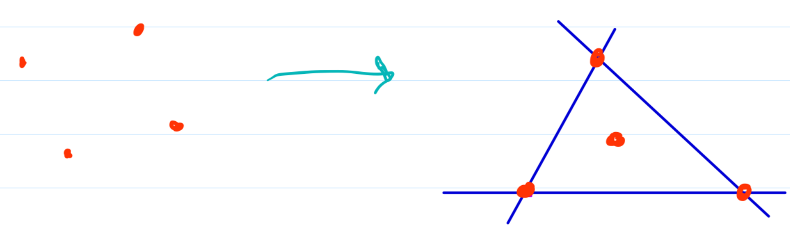
$$P_0, \ldots, P_{n+1} \in \mathbb{P}^n \text{ in "linear gen position"}$$

(no $n+1$ on hyperplane)

Then $\exists!$ $\sigma \in PGL(n+1)$ s.t.

$$\sigma(P_i) = [0, \ldots, \underset{\uparrow \, i\text{th spot}}{1}, \ldots, 0] \qquad 0 \le i \le n$$

$$\sigma(P_{n+1}) = [1, \ldots, 1]$$



Example (Proj Cones)

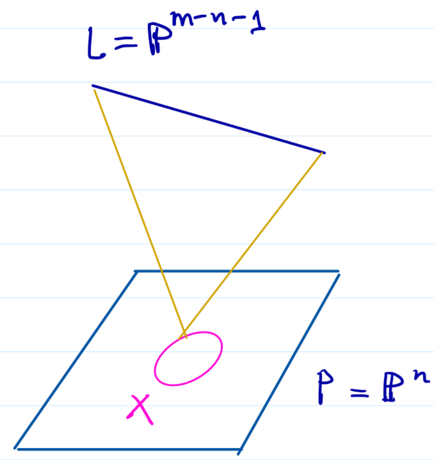
Fix $m > n$, and in $\mathbb{P}^m$ consider disjoint linear spaces

$$P = \mathbb{P}^n, \quad L = \mathbb{P}^{m-n-1} \qquad \text{of dims } n, m-n-1.$$

(After change of coords can assume $P = \{x_{n+1} = \quad = x_m = 0\}$, so $x_0, \quad , x_n$ coords on P.)

Given alg set $X \subseteq \mathbb{P}^n$, can consider

$$C_L(X) = \left\{ \begin{array}{l} \text{union of all pts lying on} \\ \text{line joining } x \in X \text{ w pt } a \in L \end{array} \right\}.$$



In coords as above, if

$$X = Z_{\mathbb{P}^n}(\ldots F_\alpha(X_0, \quad , X_n) \ldots),$$

then

$$C_L(X) = Z_{\mathbb{P}^m}(\ldots F_\alpha(X_0, \quad , X_n) \ldots)$$

Feb 8

Quasi-Proj Vars –

· Want to enlarge (one last time, for now) class of spaces we consider.

<u>Def.</u> A <u>quasi-projective variety</u> is a Zar-open subset of a proj alg set.
("locally closed" subset of proj space)

<u>Ex.</u> Affine var $X \subseteq \mathbb{A}^n$:

$$X = (\bar{X} \cap U_0) \subseteq U_0 \subseteq \mathbb{A}^n$$

<u>Issue:</u> what are rat or reg fns on QPV? They should be restrns of rat fns on $\mathbb{P}^n$.

<u>Def.</u> A rat fn on $\mathbb{P}^n$ is

$$\varphi = F/G, \quad \text{where } F, G \text{ homog polys of same degree.}$$

So if $G(a) \neq 0$, then

$$\varphi([a]) \in k \text{ well defined.}$$

(Have $F_1/G_1 = F_2/G_2 \Leftrightarrow F_1 G_2 = G_2 F_1$, but since $S = k[X_0, \ldots, X_n]$ a UFD can suppose F/G in lowest terms.)

<u>Def.</u> Let $X \subseteq \mathbb{P}^n$ be a locally closed set. A <u>regular fn</u> on X is a fn

$$\phi : X \longrightarrow k$$

w property that for every $x \in X$, $\exists$ homog polys

$$P_x, Q_x \in k[x_0, \ldots, x_n] \quad \text{of same deg,}$$

$Q_x(x) \neq 0$, s.t.

$$\varphi(y) = \frac{P_x(y)}{Q_x(y)} \text{ in nbd of } x.$$

$k[X] =$ ring of all such

<u>Ex</u> $k[\mathbb{P}^n] = k$: only global reg fns are constants.

<u>Pf.</u> $f \in k[\mathbb{P}^n]$ is globally P/Q, and $\text{Zeroes}(Q) = \phi$.

<u>Exerc:</u> For $X \subseteq \mathbb{A}^n$ affine alg set, recover previous notion of reg fn.

<u>Regular Mappings</u> - Consider QPV's

$$X \subseteq \mathbb{P}^n, \quad Y \subseteq \mathbb{P}^m.$$

<u>Ask:</u> How should we define a morphism $f : X \to Y$?

<u>Can't require</u>: $f = (f_1, \ , f_m)$ where f_i are reg fns, since often $k[X] = k$.

Right approach is to ask that f be <u>locally</u> given by reg fns.

<u>Def</u>: Given $X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$ locally closed,

$$f : X \longrightarrow Y$$

a morphism if following holds:

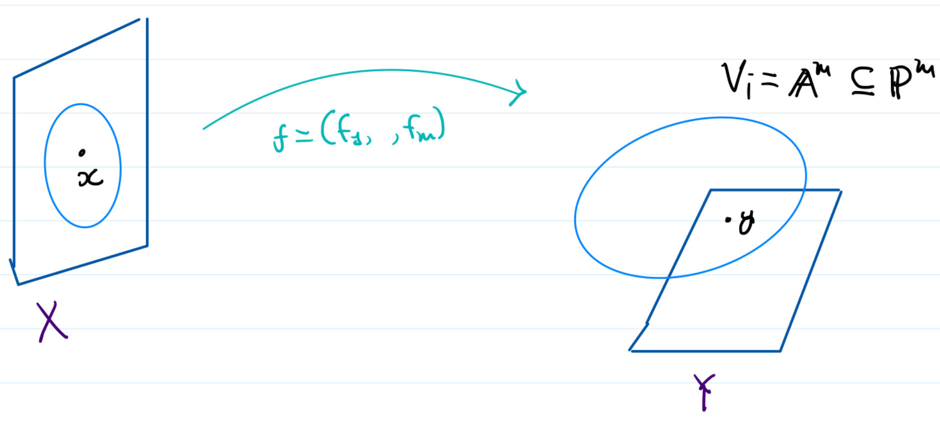
For every $x \in X$, $\exists$ affine chart of $\mathbb{P}^m$:

$$\mathbb{A}^m = V_i \ni y = f(x) \in \mathbb{P}^m$$

and nbd $\quad X \supseteq U \ni x \quad$ s.t.

$$f(U) \subseteq V_i,$$

and f given on U by m-tuple of reg fns $f = (f_1, \dots, f_m)$.



ith spot

<u>Informally</u>: f locally given in homog coords by $f = [f_0, \dots, 1, \ , f_m]$

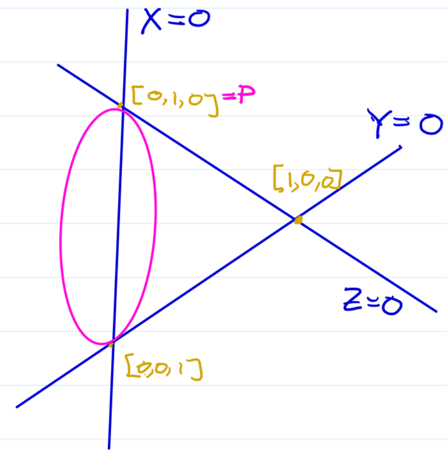
<u>Check</u>: Cond indep of choice of affine chart $V_i \subseteq \mathbb{P}^m$.

Ex. Consider the conic

$$C = \{YZ - X^2 = 0\} \subseteq \mathbb{P}^2$$

$\quad \Psi$

$P = [0,1,0]$



Look at "projection from P":

$$\begin{array}{cccc} \pi : & C & \longrightarrow & \{Y=0\} = \mathbb{P}^1 \\ & \Psi & & \Psi \\ & [X,Y,Z] & \longrightarrow & [X,Z] \end{array}$$

$$\pi(P) = [1,0]$$

Claim: π is everywhere regular (in fact, an isom)

Cover $\mathbb{P}^1$ (w coords X, Z) by V_X, V_Z

$\pi^{-1}(V_Z) = C - \{[0,1,0]\}$ given by $\varphi_Z = X/Z$ $\searrow \; \swarrow$ $V_Z = \mathbb{A}^1$.

$\pi^{-1}(V_X) = C - \{[0,0,1]\}$ given by $\varphi_X = \frac{Z}{X}$ $\searrow \; \swarrow$ $V_X = \mathbb{A}^1$: coord $\frac{Z}{X}$

Note: $\varphi_X = \frac{Z}{X} = \frac{X}{Y}$ as rat fn on C,

and this is regular at $[0,1,0]$, and

$\frac{X}{Y}([0,1,0]) = 0$ i.e. $[1,0]$

More useful characterization of morphisms of QPV:

<u>Prop.</u> $X = QPV$,

$$f : X \longrightarrow \mathbb{P}^m$$

a function. Then f a morphism $\Leftrightarrow$ $\forall p \in X$, $\exists$ nbd $U(p)$ of p on which:

$$f(x) = [F_0(x), \ldots, F_m(x)] \qquad (*)$$

w.

- F_i homog polys of same degree d
- F_i not simultaneously zero at any point of $U(p)$

NB $[F_0, , F_m]$, $[G_0, , G_m]$ give same morphism if

$$F_i G_j - F_j G_i \equiv 0 \text{ on } X.$$

<u>Pf.</u> Exercise! (Or see Shaf, p. 48)

<u>Ex.</u> $C = \{YZ - X^2 = 0\} \subseteq \mathbb{P}^2$ as above

$$\pi : C \longrightarrow \mathbb{P}^1_{X,Z} \quad :$$

$$\pi = [X, Z] \underset{\text{on } C}{=} [Y, X]$$

and at every pt of C, either X or $Z \neq 0$ or Y or $X \neq 0$.

Veronese Embedding –

Starting w $\mathbb{P}^n$, fix d and set

$$N = N(n, d) = \binom{n+d}{n} - 1 = \dim \left\{ \begin{matrix} \text{homog polys of} \\ \text{deg } d \text{ in } X_0, , X_n \end{matrix} \right\} - 1$$

· Let

$$F_0, \ldots, F_N \in k[X_0, \ldots, X_n]$$

be all monomials of deg d (in some order).

· Define:

$$u_d = u_{d,n} : \mathbb{P}^n \longrightarrow \mathbb{P}^N$$
$$\rotatebox{90}{$\in$} \qquad \rotatebox{90}{$\in$}$$
$$[x] \longmapsto [F_0(x), \ldots, F_N(x)]$$

<u>Exs:</u> $n=1$:

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^d$$
$$[S,T] \qquad [S^d, S^{d-1}T, \ldots, T^d]$$

$n=2$, $d=2$

$$\mathbb{P}^2 \longrightarrow \mathbb{P}^5$$
$$[X,Y,Z] \qquad [X^2, XY, Y^2, YZ, Z^2, XZ]$$

<u>Rmk:</u> Up to linear change of coords on target, could work w any basis of S_d.

<u>Thm:</u> $u_{n,d}$ is isom of $\mathbb{P}^n$ w subvar $V_{d,n} \subseteq \mathbb{P}^N$:

$$u_{d,n} : \mathbb{P}^n \xrightarrow{\cong} V_{d,n} = \mathrm{im}(u_{d,n}) \subseteq \mathbb{P}^N$$

called "d-fold Veronese variety" in $\mathbb{P}^N$.

Sketch of Pf of Thm: I'll do case $n=d=2$. Gen case only notationally more complicated

Consider:

$$u : \mathbb{P}^2 \longrightarrow V = \mathrm{im}(u) \subseteq \mathbb{P}^5$$

$$[X, Y, Z] \qquad [X^2, XY, Y^2, YZ, Z^2, XZ]$$

$$[T_{200}, T_{110}, T_{020}, T_{011}, T_{002}, T_{101}]$$

Need to show:

(1). $V \subseteq \mathbb{P}^5$ alg subvar.

(2) u is 1-1

(3). u^{-1} a morphism

Recall: enough to work locally for (1) & (3).

Note. $V \subseteq (T_{200} \neq 0) \cup (T_{020} \neq 0) \cup (T_{002} \neq 0)$

$(T_{200} \neq 0) = W_{200} \cong \mathbb{A}^5$, $\quad (T_{020} \neq 0) \overset{\text{def}}{=} W_{020}$, $\quad (T_{002} \neq 0) \overset{\text{def}}{=} W_{002}$

Consider $V \cap W_{002}$: have

$$u^{-1}(V \cap W_{002}) = \{Z^2 \neq 0\} = \{Z \neq 0\} = U_Z \cong \mathbb{A}^2 :$$

coord $x = \frac{X}{Z}$, $y = \frac{Y}{Z}$

On U_Z, u given by

$$u' : \mathbb{A}^2 \longrightarrow \mathbb{A}^5$$

$$(x, y) \longmapsto (x^2, xy, y^2, y, x)$$

This is the graph of the fn $(x,y) \mapsto (x^2, xy, y^2)$ so clearly

$$\operatorname{im}(u') \subseteq \mathbb{A}^5 \text{ alg, and } u' : \mathbb{A}^2 \longrightarrow \operatorname{Im} u' \text{ an isom.}$$

(In fact on $V \cap W_{002}$, inverse of u is

$$\varphi : V \cap W_{002} \longrightarrow \mathbb{P}^2$$

$$\varphi = [T_{002}, T_{101}, T_{011}] = [Z^2, ZX, ZY] = [X, Y, Z].$$

"QED."

Global Eqns of $V_{2,2} \subseteq \mathbb{P}^5$ –

V is cut out by the quadrics

$$T_{i_0 i_1 i_2} \cdot T_{j_0 j_1 j_2} - T_{l_0 l_1 l_2} \cdot T_{m_0 m_1 m_2}.$$

where

$$i_0 + j_0 = l_0 + m_0, \quad i_1 + j_1 = l_1 + m_1, \quad i_2 + j_2 = l_2 + m_2.$$

(Sim for $V_{2,d} \subseteq \mathbb{P}^{N(2,d)}$)

· Clearly these quadrics van on V. <u>Exer</u>: they vanish <u>only</u> on V.

<u>Special Descr of $V_{2,n}$</u>: Let

$$\mathbb{P}^N = \mathbb{P}^{\binom{n+2}{2}-1} = \mathbb{P}\begin{pmatrix} (n+1)\times(n+1) \\ \text{Symm matrices} \end{pmatrix}$$

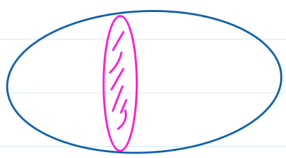
coords:
$$\begin{pmatrix} X_{00} & X_{01} & & X_{0n} \\ & X_{11} & & X_{1n} \\ & & \cdot & \\ & & & X_{nn} \end{pmatrix}$$

Then:

$$V_{2,n} \cong \left\{ \underset{A}{\text{symm}} \;\middle|\; \operatorname{rk} A \leq 1 \right\}$$ Exerc!

Hyperplane Sections of Veronese:

Def: Say $X \subseteq \mathbb{P}^r$ is an irred proj variety not contained in any linear space. Hyperplane section of X is intersection of X with hyperplane (hypsf of deg 1)



Now consider Veronese embedding:

$$u_d : \mathbb{P}^n \longrightarrow V_{d,n} \subseteq \mathbb{P}^{N(n,d)}$$
$$[x] \longmapsto [\cdots F_\alpha(x) \cdots]$$

$\{F_\alpha\}$ all monomials of deg d

Hyperplane $H \subseteq \mathbb{P}^N$ is cut out by k-linear comb of coords in $\mathbb{P}^N$
So

$$u^{-1}(H) = \left\{ \sum \lambda_\alpha F_\alpha(x) = 0 \right\}$$

ie.

This is hypsf of deg d, and all hypsfs of deg d are of this form.

Under isom $u: \mathbb{P}^n \longrightarrow V_{d,n} \subseteq \mathbb{P}^N$, hyperplane sections of $V_{d,n}$ are identified w hypsfs of deg d in $\mathbb{P}^n$.

ie, Veronese turns hyperplanes in big proj space into hypersurfaces in smaller proj space.

Intrinsic Meaning of Veronese (via lin alg).

$V = (n+1)$ dim vector space:

S^dV: d^{th} symm prod of V. Elt of S^dV is "homog poly" of deg d in vectors $v \in V$, ie

$\varphi \in S^dV$ is linear comb of expressions $v_1 \cdots v_d$ $(v_i \in V)$

Inside S^dV have

$$W = \{ v^d \mid v \in V \} \quad \text{(ie pure powers of single vector)}$$

Then

$$\text{"}\mathbb{P}(W)\text{"} \cong \text{vero var} \subseteq \mathbb{P}S^dV$$

Multitude of Proj Embeddings Veronese illustrates general fact:

Given proj var X, there typically exist many diff projective embeddings:

$$X \hookrightarrow \mathbb{P}^N \quad \text{(for various } N\text{)}$$

(Example: curves.) Interesting to study these "internally"

Existence of Affine Nbds:

Prop. X a QPV, $x \in X$ a pt. Then x has a (Zariski) nbd $V \subseteq X$ that is isom to an affine alg set.

So - for local questions can restr to affine vars.

Pf. Say

$$x \in X \overset{\text{loc. closed}}{\subseteq} \mathbb{P}^n.$$

Then $x \in U_i = \mathbb{A}^n$ for some i, so can suppose

$$x \in X \overset{\text{loc. cl}}{\subseteq} \mathbb{A}^n.$$

Now consider

$$\overline{X} = \text{closure of } X \subseteq \mathbb{A}^n.$$

Replacing X by suitable nbd, can assume:

$$x \in X \overset{\text{open}}{\subseteq} \overline{X} \overset{\text{closed}}{\subseteq} \mathbb{A}^n.$$

Let

$$Z = \underbrace{(\overline{X} - X)}_{\text{closed}} \subsetneq \overline{X}$$

Take

$$f \in k[\overline{X}], \quad f \in I_Z, \; f(x) \neq 0.$$

Then

$$x \in \underbrace{\overline{X} - \text{Zeroes}(f)}_{\text{affine.}} \subseteq X$$

Rational Fns on QPV. Consider

$X \subseteq \mathbb{P}^n$ irred, loc closed.

Define:

$$k(X) = \left\{ \frac{P}{Q} \;\middle|\; P, Q \text{ homog same deg} \right\} / \sim$$

where

$$P_1/Q_1 \sim P_2/Q_2 \quad \text{if} \quad P_1 Q_2 \equiv P_2 Q_1 \text{ on } X.$$

When X affine, agrees with old notion. Also note: if $U \subseteq X$ open, then $k(U) = k(X)$.

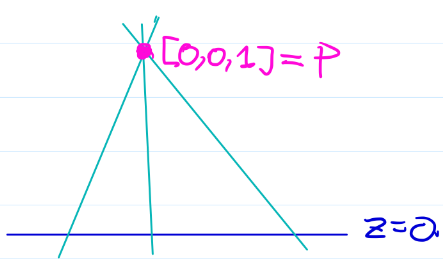
Def: $X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$ QPV. Rat map

$$\varphi : X \dashrightarrow Y \quad \text{is} \quad \varphi(x) = [F_0(x), \ldots, F_m(x)]$$

F_i homog same degree, not all vanishing on X. Gives notion of birat isom.

Ex. $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, $[X, Y, Z] \longmapsto [X, Y]$

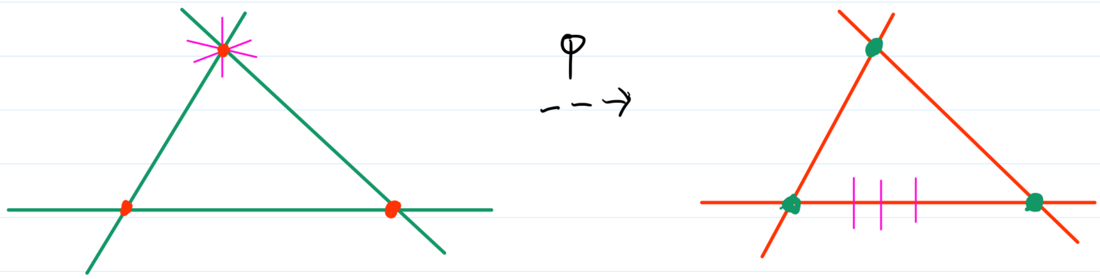
This is projection fr $P = [0,0,1]$ onto $Z = 0$



Ex $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, $[X, Y, Z] \longmapsto [YZ, XZ, XY]$.

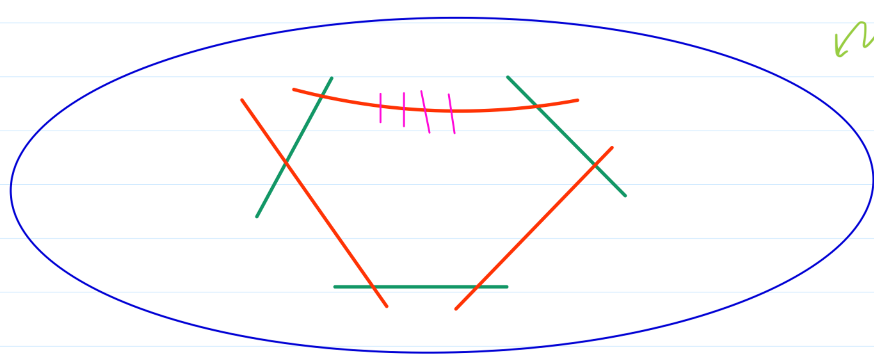
This is birat isom w points of indeterminacy at the three coord vertices

Exerc: Study geometry of φ.



Convince yourself that φ:

"Blows up the three points of indeterminacy and maps them to the three coord lines, and blows down the three coord lines to coord pts."



surface obtained by "blowing up" the three coord pts in $\mathbb{P}^2$.

As before: $X \sim_{birat} Y \iff k(X) \cong k(Y)$.

Products -

Question: How can we realize $\mathbb{P}^n \times \mathbb{P}^m$ as proj var?

NB: $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^m(\mathbb{C}) \not\cong \mathbb{P}^{n+m}(\mathbb{C})$ (Betti nos different!)

Answer is to use:

Segre Mapping -

Define

$$\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^{nm+n+m}$$

$$[a_0, \ldots, a_n] \times [b_0, \ldots, b_m] \longmapsto [\ldots, a_i b_j, \ldots]$$

Prop: Image of $\sigma_{n,m}$ is proj alg subset of $\mathbb{P}^{nm+n+m}$, and $\sigma_{n,m}$ is 1-1 onto its image.

Def. $\mathrm{Im}(\sigma_{n,m})$ is Segre variety $S_{n,m}$ (temp. notation), and we we have set-theoretic identification

$$\mathbb{P}^n \times \mathbb{P}^m \longleftrightarrow S_{n,m} \subseteq \mathbb{P}^{nm+n+m}$$

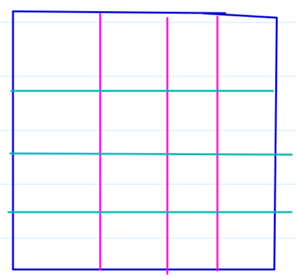
We'll see over the next few minutes that this is the "right" way to realize $\mathbb{P}^n \times \mathbb{P}^m$ as a proj var. (Ex: $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^m(\mathbb{C}) \cong S_{n,m}(\mathbb{C})$).

Ex. $n = m = 1$:

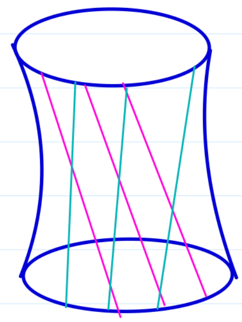
$$\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3, \quad [a_0, a_1] \times [b_0, b_1] \longmapsto [\underset{X}{a_0 b_0}, \underset{Y}{a_0 b_1}, \underset{Z}{a_1 b_0}, \underset{W}{a_1 b_1}]$$

Image is quadric $(XW - YZ = 0)$

Note that $\mathbb{P}^1 \times \{b\}$, $\{a\} \times \mathbb{P}^1$ map to lines in $\mathbb{P}^3$



$$\xrightarrow{\sigma}$$



Note: $\mathbb{P}^{nm+n+m} = \mathbb{P}((n+1) \times (m+1) \text{ mxs})$

$$S_{n,m} = \{\text{mxs of rk } 1\}$$

(Matrix has rk $\leq 1 \iff$ of form $\begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix}$ "$\otimes$" $(b_0 \; \cdots \; b_m) = (a_i b_j)$)

Why is $S_{n,m}$ the "correct" defn of $\mathbb{P}^n \times \mathbb{P}^m$?

(I). Consider the affine charts

$$\mathbb{A}^n = U_i \subseteq \mathbb{P}^n, \qquad \mathbb{A}^m \subseteq V_j \subseteq \mathbb{P}^m$$

Claim: Under the Segre embedding, $U_i \times V_j$ maps to a copy of $\mathbb{A}^{n+m}$.

(le Segre embedding compatible w prod structure on affine spaces.)

Pf. Say $i = j = 0$.

$$U_0 = \{[1, x_1, \ , x_n]\}, \quad V_0 = \{[1, y_1, \ldots, y_m]\}$$

$$\sigma(U_0 \times V_0) = \left\{ \begin{bmatrix} 1 & x_1 & x_2 & & x_n \\ y_1 & x_1 y_1 & x_2 y_1 & & x_n y_1 \\ & & \cdots & & \vdots \\ y_n & x_1 y_n & x_2 y_n & & x_n y_n \end{bmatrix} \right\} \cong \mathbb{A}^{n+m}$$

(graph of fn)

Moreover,

$$\sigma(U_0 \times V_0) = S_{n,m} \cap (T_{00} \neq 0)$$

(II). (Functorial Viewpt). Given any QPV T, and morphisms

$$u_1 : T \longrightarrow \mathbb{P}^n \quad , \quad u_2 : T \longrightarrow \mathbb{P}^m,$$

get

$$u_{12} : T \longrightarrow S_{n,m}$$ compatible w. set-theoretic identif of $S = \mathbb{P}^n \times \mathbb{P}^m$.

Pf: Say

$$u_1(t) = [F_0(t), \ldots, F_n(t)]$$
$$u_2(t) = [G_0(t), \ldots, G_m(t)]$$

Take

$$u_{12}(t) = [\ \cdots F_i(t) \cdot G_j(t) \ \ldots\ ]$$

Rmk: Proj $\mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^n$ is

$$\begin{bmatrix} z_{00} & & z_{0n} \\ & & \\ z_{m0} & & z_{mn} \end{bmatrix} \longmapsto [\ z_{i0}, \ \ , z_{in}] \quad (= [z_{j0}, \ \ z_{jn}] \text{ on } S),$$

Given QPV's

$$X \subseteq \mathbb{P}^n, \quad Y \subseteq \mathbb{P}^m$$

realize $X \times Y$ as QPV via

$$X \times Y \hookrightarrow \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{nm+n+m}$$

(Exerc: $X \times Y$ locally closed in $\mathbb{P}^{nm+n+m}$)

<u>Intrinsic Meaning of Segre:</u>

$$\begin{array}{ccc} \mathbb{P}(V) \times \mathbb{P}(W) & \longrightarrow & \mathbb{P}(V \otimes W) \\ & & \cup \\ [v] \times [w] & \longmapsto & [v \otimes w] \end{array}$$

(Recall that $V \otimes W$ is all <u>lin combs</u> $\sum v_i \otimes w_i$. The Segre var is the collection of all <u>pure</u> tensors.)

<u>Alg. subsets of $\mathbb{P}^n \times \mathbb{P}^m$:</u>

Say homog coords on

$$\begin{array}{lll} \mathbb{P}^n & \text{are} & X_0, \ldots, X_n \\ \mathbb{P}^m & \text{"} & Y_0, \ldots, Y_m \end{array}$$

By defn, closed subsets of $\mathbb{P}^n \times \mathbb{P}^m$ then cut out by polys of form

$$F(\ldots X_i Y_j \ldots), \quad F \text{ homog.}$$

These are homog of same deg in X's & Y's. But inconvenient. Turns out that one can work "internally" in $\mathbb{P}^n \times \mathbb{P}^m$.

<u>Def.</u> Poly $F(X,Y)$ is <u>bihomogeneous of bidegree</u> (d,e) if homog of deg d in X's, homog of deg e in Y's, i.e

$$F(\lambda X, \mu Y) = \lambda^d \mu^e F(X,Y) \qquad (*).$$

<u>Ex</u>: $X_0^2 Y_0^3 + X_0 X_1 Y_1 Y_2 Y_3$ bihom bideg $(2,3)$.

It follows fr $(*)$ that if F bihomog, then $\{F=0\}$ well-defined in $\mathbb{P}^n \times \mathbb{P}^m$.

<u>Prop</u>: $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m$ Zariski-closed iff it is zero locus of collection of bihomog polys.

<u>Variant</u> - Let

$$X \subseteq \mathbb{P}^n \times \mathbb{A}^m.$$

Then X is Zariski-closed $\Leftrightarrow$ X cut out by polys

$$F_\alpha(X;t) \in k[X_0, \ldots, X_n, t_1, \ldots, t_m]$$

homog in X_i, arb in t_j.

<u>Ex</u>. View $\mathbb{P}^1$ as set of lines thru 0 in $\mathbb{A}^2$. Consider:

$$\mathbb{A}^2 \times \mathbb{P}^1 \supseteq X = \{(p,[\ell]) \mid p \in \ell\}$$

Say

x, y : coords on $\mathbb{A}^2$

S, T " " $\mathbb{P}^1$

Eqn of X:

$$\det \begin{vmatrix} x & y \\ S & T \end{vmatrix} = 0,$$

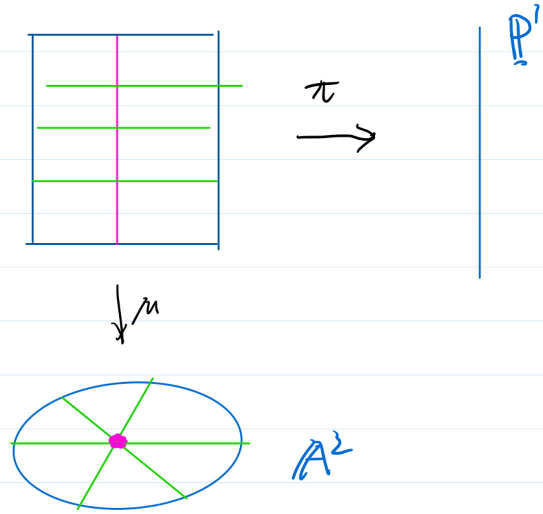
ie.

$$\boxed{xT - yS = 0}$$

Have interesting maps

$$\mu : X \longrightarrow \mathbb{A}^2$$
$$\pi : X \longrightarrow \mathbb{P}^1$$

$\pi^{-1}([\ell]) = $ line in $\mathbb{A}^2$



Completeness of $\mathbb{P}^n$

Exerc: Let X be a "reasonable" top space (eg a metric space). Then:

$$X \text{ is compact}$$
$$\Updownarrow$$
$$\forall T, \text{ the projection } pr_2 : X \times T \longrightarrow T \text{ is a closed mapping}$$

(ie pr_2(closed set) is closed.)

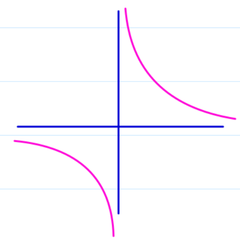
This turns out to be right analogue of compactness in alg geom.

Def: A QPV X is complete if following holds:

> For any QPV T, the projection
>
> $$pr_2 : X \times T \longrightarrow T$$
>
> is closed, ie $\forall$ Z-closed $Z \subseteq X \times T$, $pr_2(Z) \subseteq T$ is Z-closed

Non-Ex. $\mathbb{A}^1$ is not complete.

Reason: take $Z = \{xt = 1\} \subseteq \mathbb{A}^1 \times \mathbb{A}^1$.



Then

$$pr_2(Z) = \mathbb{A}^1 - \{0\}$$

not closed in $\mathbb{A}^1$.

THM. Let X be a proj alg set. Then X is complete.

Cor 1. Let $f : X \to Y$ be a morphism, w. X projective. Then

$$f(X) \subseteq Y \quad \text{is } Z\text{-closed.}$$

Pf. Consider the graph

$$\Gamma = \Gamma_f \subseteq X \times Y.$$

This is closed, and $im(f) = pr_2(\Gamma_f)$.

Cor 2. Assume X irred (or connected) and proj. Then

$k[X] = k$, ie only regular fns on X are const.

Pf. Say $f \in k[X]$. View f as defining

$$f : X \longrightarrow \mathbb{A}^1 \subseteq \mathbb{P}^1.$$

By Cor, $f(X)$ closed in $\mathbb{P}^1$ So:

$f(X) = \mathbb{P}^1$: impossible since $f(X) \subseteq \mathbb{A}^1 \subsetneq \mathbb{P}^1$

or

$f(X) \subseteq \mathbb{P}^1$ a finite set. Then X irred $\Rightarrow f(X) = $ pt.

Proof of Thm –

Step 1 – Reductions.

Claim: Suffices to prove Thm w. T replaced by each of the sets $\{T_\alpha\}$ is an open covering.

Pf. $F \subseteq T$ closed $\Leftrightarrow$ $F \cap T_\alpha$ closed in T_α for all α

So: can and do assume that T is affine.

Claim: Can assume $T = \mathbb{A}^m$.

Pf. Suppose know Thm for $\mathbb{A}^m$, let $T \subseteq \mathbb{A}^m$ be closed. Consider:

$$\begin{tikzcd} Z \overset{\text{closed}}{\subseteq} X \times T \arrow[d] & \overset{\text{closed}}{\subseteq} & X \times \mathbb{A}^m \arrow[d] \\ T & \subseteq & \mathbb{A}^m \end{tikzcd}$$

By Thm for $\mathbb{A}^m$, $pr_2(Z) \subseteq \mathbb{A}^m$ closed. So $pr_2(Z)$ closed in T.

Claim: Enough to prove Thm for $X = \mathbb{P}^n$.

Pf. As before; consider closed $X \subseteq \mathbb{P}^n$, and:

$$\begin{tikzcd} Z \overset{\text{closed}}{\subseteq} X \times \mathbb{A}^m \arrow[dr] & \overset{\text{closed}}{\subseteq} & \mathbb{P}^n \times \mathbb{A}^m \arrow[dl] \\ & \mathbb{A}^m & \end{tikzcd}$$

Z closed in $\mathbb{P}^n \times \mathbb{A}^m$, so Thm for $\mathbb{P}^n \Rightarrow pr_2(Z) \subseteq \mathbb{A}^m$ closed.

So finally we're reduced to:

THM*. ("Fund. Thm of Elimination Theory") Consider alg subset

$$Z \subseteq \mathbb{P}^n \times \mathbb{A}^m.$$

Then

$$pr_2(Z) \subseteq \mathbb{A}^m \text{ is Z-closed.}$$

What is this really saying?

$$\begin{tikzcd} Z \subseteq \mathbb{P}^n_X \times \mathbb{A}^m_t \arrow[d] \arrow[dr, "pr_2"'] & \\ & \mathbb{A}^m \end{tikzcd}$$

- Say $Z \subseteq \mathbb{P}^n \times \mathbb{A}^m$ is defined by polys

$$F_\alpha(X;t) = \sum a_{\alpha,I}(t)\, X^I$$

homog in coords X on $\mathbb{P}^n$, whose coeffs are polys in coords t on $\mathbb{A}^m$.

(View $F_\alpha(X;t)$ as family of homog. polys in X parameterized by t)

- Fix $b \in \mathbb{A}^m$. Then

$$b \in pr_2(Z) \iff \left\{ \begin{array}{l} \text{The polys } F_\alpha(X;b) = \sum a_{\alpha,I}(b)\, X^I \\ \text{have a common zero in } \mathbb{P}^n. \end{array} \right.$$

- So Thm* means:

$\exists$ collection of polys $g_\lambda(t)$ s.t.

$$\begin{array}{c}\text{All } g_\lambda(t) \text{ vanish at} \\ b \in \mathbb{A}^m\end{array} \iff \left\{ \sum a_{\alpha,I}(b)\, X^I = 0 \right\} \text{ have non-trivial common zero}$$

Ie. we've eliminated the X's in question of whether system of homog. polys has non-trivial soln.

Toy Example: Suppose the $F_\alpha(X,t)$ are linear in the X's, eg have

$$F_1 = a_{10}(t)X_0 + \cdots + a_{1n}(t)X_n$$
$$\vdots$$
$$F_r = a_{r0}(t)X_0 + \quad + a_{rn}(t)X_n$$

Then for fixed t, the F_i have a common soln in $\mathbb{P}^n$ iff coeff mx has rk $\leq n$:

$$\text{rk}\begin{bmatrix} a_{10}(t) & \cdots & a_{1n}(t) \\ a_{20}(t) & \cdots & a_{2n}(t) \\ & \vdots & \\ a_{r0}(t) & \cdots & a_{rn}(t) \end{bmatrix} \leq n \qquad (*)_t$$

And

$(*)_t \iff$ vanishing of $(n+1)\times(n+1)$ minors of this mx } ← this is alg. subset of $\mathbb{A}^m$.

Rmk: Pf in general won't be constructive; but there has been work trying to find explicit formulas in special cases.

Proof of Thm*. Given

$$Z \subseteq \mathbb{P}^n \times \mathbb{A}^m,$$

let

$$F_1(X;t), \ldots, F_r(X;t)$$

be polys defining Z, w.

$$F_i \text{ homog of deg } d_i \text{ in } X_0, \dots, X_n.$$

Step 1: Fix $t_0 \in \mathbb{A}^m$ (ie freeze t). Assume

$$(*) \qquad t_0 \in \mathbb{A}^m - pr_2(Z),$$

Need to show: $\exists$ Z-open nbd $V(t_0)$ of $t_0 \in \mathbb{A}^m$ s.t.

$$V(t_0) \subseteq \mathbb{A}^m - pr_2(Z),$$

Note:

$$(*) \text{ holds} \iff pr_2^{-1}(t_0) = \phi,$$

ie.

$$(**) \qquad \{F_i(X;t_0)\} \text{ have no common zeroes in } \mathbb{P}^n.$$

So suffices to show: $\exists$ nbd $V(t)$ of t_0 st for $t \in V(t_0)$:

$$\{F_i(X,t)\} \text{ have no common solns in } \mathbb{P}^n.$$

By Nullstellensatz:

$$(**) \iff (X_0, .., X_n)^l \subseteq (F_1(X,t_0), \dots, F_r(X;t_0))$$

some $l >> 0$

$$\iff \left.\begin{array}{l} \text{Any monomial monomial of deg } l \text{ can} \\ \text{be written as} \\ \qquad \sum G_i(X) \cdot F_i(X,t_0) \\ \text{for some homog polys } G_i \text{ of deg } l-d_i \end{array}\right\} (\circledast)_{t_0}$$

<u>Key Claim</u>: Assuming $(\star)_{t_0}$, $\exists$ nbd $V(t_0) \subseteq \mathbb{A}^m$ s.t.:

$(\star)_t$ $\begin{cases} \text{For } t \in V(t), \text{ polys of the form} \\ \qquad \sum G_i(X) \cdot F_i(X,t) \qquad (+)_t \\ (\deg G_i = l - d_i) \text{ consist of all polys of deg } l, \\ \text{i.e. every monomial of deg } l \text{ can be expressed in} \\ \text{form } (+)_t \text{ for suitable } G_i(X) \text{ (depending on } t). \end{cases}$

If $(\star)_t$ holds, then the $F_i(X;t)$ have no common solns, so

$$V(t) \subseteq \mathbb{A}^m - pr_2(Z),$$

as required. So we need to prove Key Claim.

<u>Step 2</u>. Let

$$S_e = \text{v.s of all homog polys of deg } e \text{ in } X_0, \ldots, X_n.$$

Consider for fixed t:

$$\begin{array}{ccc} S_{l-d_1} \oplus \cdots \oplus S_{l-d_r} & \xrightarrow{u_l(t)} & S_l \qquad (++) \\ \cup\!\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\!\shortmid \\ (G_1, \ldots, G_r) & \longmapsto & \sum G_i(X) \cdot F_i(X;t). \end{array}$$

If we pick bases for all the v.s. in question, can view:

$u_l(t)$ as being given by big mx whose entries are polys in t

($\Leftrightarrow^{ly}$: view u_l as morphism: $\mathbb{A}^m \longrightarrow \mathrm{Hom}_k(\text{LHS}, \text{RHS}) = M_{M \times N}(k)$)

Then for any fixed t:

$$\begin{matrix}\text{All monomials of} \\ \deg \ell\end{matrix} \subseteq \begin{pmatrix}\text{ideal generated by} \\ F_1(X;t), \dots, F_r(X;t)\end{pmatrix}$$

$$\Updownarrow$$

$$u_\ell(t) \text{ surj}$$

So:

$$(\star)_{t_0} \iff u(t_0) \text{ surj},$$

and we need to show $\exists$ $V(t)$ s.t. $u(t)$ surj for $t \in V(t_0)$.

So everything follows from

Lemma: Let A, B be vector spaces of dims $a \geq b$, and let

$$u(t) : A \longrightarrow B$$

be a family of linear transfs given by $a \times b$ mx of polys $u_{ij}(t)$.
Assume $u(t_0)$ is surj. Then $\exists$ Z-open nbd

$$t_0 \in V(t_0) \subseteq \mathbb{A}^m$$

surj for $t \in V(t_0)$.

Pf $u(t_0)$ surj $\Rightarrow$ $\mathrm{rk}(u(t_0)) = b$. Then $\exists$ $V(t_0)$ on which $u(t)$ also has rk b.

Rmk: Analogue of Thm in complex geom is Remmert's proper mapping thm:

If $f: X \to Y$ is proper mapping of analy vars, then
$f(X) \subseteq Y$ is analy subset.

Application - Spaces of Hypersurfaces

Consider $S = k[X_0, \ldots, X_n]$,

$$S_d = \{\text{homog polys deg } d\} : \dim \binom{d+n}{n}.$$

Then

$$\mathbb{P}(S_d) = \{\text{homog polys}\}/\text{scalars} = \mathbb{P}^{\binom{d+n}{n}-1}$$

View

$\mathbb{P}(S_d)$ as parametrizing all hypsf deg d in $\mathbb{P}^n$

So: coords on $\mathbb{P}(S_d)$ are coeffs of defining eqn of hypsf.

<u>Ex</u> $n=2$, $d=2$. (When $n=1$, $\mathbb{P}(S_d)$ = proj completion of $P_d = \{\text{monic polys}\}$)

$$\mathbb{P}(S_d) = \mathbb{P}^5 \ni [T_{200}, T_{110}, T_{020}, T_{011}, T_{002}, T_{101}] \longleftrightarrow \sum T_{ijk} X^i Y^j Z^k$$

<u>Idea:</u> put geom cond on hypsf, ask whether it's alg subset of $\mathbb{P}(S_d)$

Concretely: is geom cond described by poly eqns in coeffs of hypsfs?

<u>Ex</u> Fix pt $p \in \mathbb{P}^n$. Let

$$H_p = \{ F \in \mathbb{P}(S_d) \mid F(p) = 0 \} : \text{ hypsfs thru } p.$$

<u>Claim:</u> H_p is <u>hyperplane</u> in $\mathbb{P}(S_d)$

<u>Pf:</u> For fixed P, condition $F(P)=0$ is linear equation in coeffs of F.

<u>Ex.</u> $n=2$: Say $F = \sum T_{ijk} X^i Y^j Z^k$ so T_{ijk} coords on $\mathbb{P}(S_d)$.

If $P = [P_0, P_1, P_2]$, then

linear in T_{ijk}

$$H_p = \{ \sum (P_0^i P_1^j P_2^k) T_{ijk} = 0 \}$$

Rmk: Note that if $d \geq 2$, most hyperplanes are not of this form.

Can use Thm from last class to show many natural conds on hypsfs are alg.

Ex. Assume $n \geq 2$, $d \geq 2$. Let

$$\mathrm{Red}_d = \{ F \in \mathbb{P}S_d \mid F \text{ is reducible} \}.$$

Then $\mathrm{Red}_d \subseteq \mathbb{P}S_d$ is alg, If $d = 2$ or 3 it is irred, but for $d \geq 4$ it is reducible.

Pf. Fix $a, b > 0$ st $a + b = d$, let

$$\mathrm{Red}_{a,b} = \{ F \mid F = G \cdot H,\ \deg G = a,\ \deg H = b \}$$

Will show $\mathrm{Red}_{a,b}$ alg; $\mathrm{Red}_d = \bigcup_{a+b=d} \mathrm{Red}_{a,b}$.

Consider

$$\begin{aligned} \phi : \mathbb{P}S_a \times \mathbb{P}S_b &\longrightarrow \mathbb{P}S_{a+b} \\ (G, H) &\longmapsto G \cdot H \end{aligned}$$

Claim: ϕ is morph of vars.

Pf: Say eg $n = 2$, write

$$G = \sum T_{ijk} X^i Y^j Z^k, \quad H = \sum S_{ijk} X^i Y^j Z^k$$

Then

Φ given by collect of type $(1,1)$ bihomog polys in S's, T's.

So a morphism. (In fact, Φ a proj of Segre). But

$$R_{ab} = \operatorname{Im} \Phi_{a,b}$$, so alg.

Resultants -

- When $n=1$, consider

$$\mathbb{P}S_d \times \mathbb{P}S_e \supseteq \{(F,G) \mid F,G \text{ have common zero in } \mathbb{P}^1\}$$

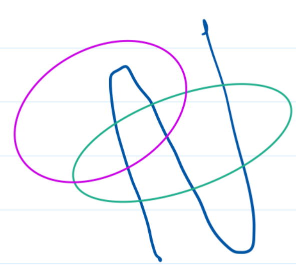
This is defined by a hypersf $\mathrm{Res} \subseteq \mathbb{P}^d \times \mathbb{P}^e$ given by specific hypsf (the "resultant") in coeffs of F, G. [HW]

- We can prove $\exists$nce of analogue in several vars (although at moment it won't be clear its a hypsf.)

- I'll do case $n=2$. Situation for $n \geq 3$ similar.

So: assume $n=2$.

Fact: Expect two general curves in $\mathbb{P}^2$ to meet in finitely many pts; three general curves should not meet.



Define:

$$\mathbb{P}S_d \times \mathbb{P}S_e \times \mathbb{P}S_f \supseteq R = \mathrm{Res}(d,e,f)$$
$$\|_{df}$$
$$\{(F,G,H) \mid \{F=G=H=0\} \neq \phi\}$$

<u>Prop.</u> R is alg subset of $\mathbb{P}S_d \times \mathbb{P}S_e \times \mathbb{P}S_f$ (in fact a hypsf.)

<u>Rmk:</u> There is active interest in finding explicit form of defining eqn: [GKZ].

<u>Pf of Prop.</u> Consider

$$\mathbb{P}S_d \times \mathbb{P}S_e \times \mathbb{P}S_f \times \mathbb{P}^2 \supseteq Z = \{(F,G,H,P) \mid F(P) = G(P) = H(P) = 0\}$$

- This is (obviously!) alg subset of LHS, (In fact of codim 3). (Exer!) and in partic is proj

- Now consider projection

$$\begin{array}{ccc} \mathbb{P}S_d \times \mathbb{P}S_e \times \mathbb{P}S_f \times \mathbb{P}^2 & \supseteq & Z \\ \searrow & & \swarrow \\ & \mathbb{P}S_d \times \mathbb{P}S_e \times \mathbb{P}S_f & \end{array}$$

- Im of Z is R, which is thefore alg.

Grassmannians –

These parameterize higher-dim linear subspaces of a vector space.

- Consider

$$V = k^{n+1} \text{ -- } (n+1)\text{-dim v.s. w fixed basis}$$

- Want to param. all

$$W = k^{\ell+1} \subseteq V \; : \; (\ell+1)\text{-dim sub sp.}$$

- The param space is called the Grassmannian $G = G(\ell+1, n+1)$.
- Can also view G as parameterizing all linear spaces

$$L = \mathbb{P}^\ell \subseteq \mathbb{P}^n :$$

Then write

$$G = G(\ell, n) = G(\mathbb{P}^\ell, \mathbb{P}^n).$$

<u>Construction</u>

- Fix $W = k^{\ell+1} \subseteq k^{n+1} = V$.
- Choose basis: $w_0, \dots, w_\ell \in W$. Write as row vectors:

$$\begin{bmatrix} w_0 \\ \\ w_\ell \end{bmatrix} = \begin{bmatrix} a_{00} & \cdots & a_{0n} \\ & & \\ a_{\ell 0} & \cdots & a_{\ell n} \end{bmatrix} =_{\text{def}} A_w = A$$

- Note:

$$\operatorname{rk} A = \ell + 1.$$

- Two matrices A, A' have same row-span iff

$$(*) \qquad A' = g \cdot A \quad \text{some } g \in GL(\ell+1).$$

- Want to find "coords" of A (modulo equiv. relation $(*)$)

- Idea: use maximal minors of A

Choose index set

$$I: \ 0 \le i_0 < i_1 < \dots < i_\ell \le (n+1).$$

Let

$$P_I = I^{th} \text{ minor of } A.$$

Note: ("Plücker coords of A")

$$P_I(g \cdot A) = (\det g) \cdot P_I(A).$$

So homog. vector

$$P(W) = [\dots P_I(A) \dots] \in \mathbb{P}^{\binom{\ell+1}{n+1}-1}$$

depends only on W. Get

$$\begin{array}{rccc} Pl: & \{\text{all } W\} & \longrightarrow & \mathbb{P}^{\binom{n+1}{\ell+1}-1} \\ & \Psi & & \Psi \\ & W & \longrightarrow & P(W) \end{array}$$

(Plücker embedding)

<u>Thm.</u> Plucker map is 1-1 onto its image, realizes $\mathbb{G}(\mathbb{P}^\ell, \mathbb{P}^n)$ as irred proj variety.

<u>Pf.</u> HW.

<u>Intrinsic meaning:</u>

Say $V = n+1$. Target of Plucker embedding in $\mathbb{P}(\Lambda^{\ell+1} V)$.

Given $W \subseteq V$ dim $\ell+1$, have

$$P(W) =: \underbrace{\Lambda^{\ell+1} W}_{\text{dim } = 1} \subseteq \mathbb{P}(\Lambda^{\ell+1} V).$$

$$G(\mathbb{P}^\ell, \mathbb{P}^n) = \{ v_0 \wedge \dots \wedge v_\ell \mid v_i \in V \text{ lin indep.} \} \subseteq \mathbb{P}(\Lambda^{\ell+1} V).$$

Local Description — Fix one I, say $I = \{0, \dots, \ell\}$. Let

$$U_I = \{ W \mid P_I(W) \neq 0 \} \subseteq G.$$

If $P_I(W) \neq 0$, can choose basis s.t. I-minor of A_W is Id, ie

$$W = \text{row span} \left[\begin{array}{c|c} Id_{\ell+1} & B \end{array} \right] \qquad B \in M_{\ell+1, (n-\ell)}$$

(ie. W = graph of $B : k^{\ell+1} \to k^{n-\ell}$. Moreover, different B's give different subspaces. So:

$$U_I \cong M_{(\ell+1)\times(n-\ell)} = \mathbb{A}^{(\ell+1)(n-\ell)}$$

Also

$$P\ell : U_I \hookrightarrow \left(\begin{array}{c} \text{affine} \\ \text{chart in } \mathbb{P} \end{array} \right)$$

i.e.

$Gr(\ell+1, n+1)$ is rat variety (of dim $(\ell+1)(n-\ell)$.)

Finite Mappings —

- In theory of Riemann sfs / alg curves it's very useful to study Riemann sf X by representing it as branched covering

$$f : X \longrightarrow \mathbb{P}^1$$

- In higher dims, analogous concept is finite map.
- I'll outline story, referring to Shaf, Ch. I.5.3, I.5.4 for details.
- This is situation where "algebra leads geom."

Algebra — consider

$$A \subseteq B \qquad \text{inclusion of comm rings}$$

Assume: B is fg as A-algebra

Prop. TFAE (say B integral, or module-finite $/A$).

(i). Every elt $b \in B$ is integral $/A$, ie. b satisfies a monic poly

$$b^n + a_1 b^{n-1} + \dots + a_{n-1} \cdot b + a_n = 0, \qquad a_i \in A.$$

(ii). B is fg as A-module.

Variant: ditto for $A \longrightarrow B$.

Finite Morphisms —

Consider

$$f : X \longrightarrow Y$$

morphism of affine alg sets, giving

$$f^* : k[Y] \longrightarrow k[X]$$

Say f <u>finite</u> if $k[X]$ integral over $k[Y]$.

<u>Rmk:</u> In book, it is assumed that f is dominant, so $k[Y] \subseteq k[X]$. We'll mainly stick to this case.

<u>Ex.</u> $X \subseteq Y$ closed subset.

<u>Prototype Example</u>

Consider

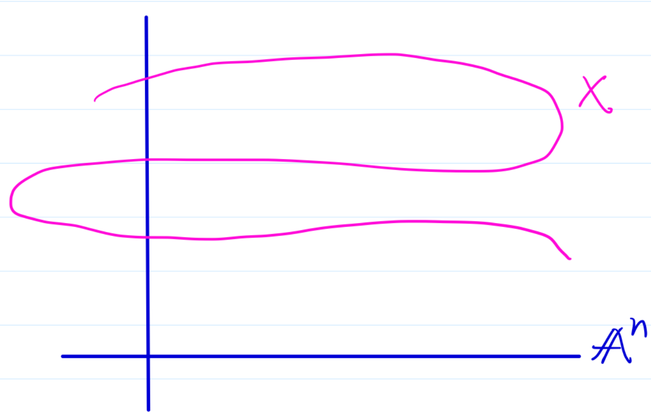
$$\mathbb{A}^{n+1}_{x,t} \supseteq X = \{t^d + a_1(x)t^{d-1} + \dots + a_d(x) = 0\}$$

hypsf given by <u>monic</u> poly $g(x,t)$, and

$$f : X \longrightarrow \mathbb{A}^n \qquad (x,t) \longmapsto x.$$

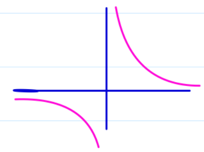
This is finite since $k[X]$ gen by $1, t, \dots, t^{d-1}$ as module over $k[\mathbb{A}^n]$

<u>Geom meaning:</u> for each $b \in \mathbb{A}^n$, $\pi^{-1}(b) =$ "d pts" (counting multipl) ie no solns go off to ∞.



<u>Intuition:</u> finite dominant map $\leftrightarrow$ branched covering

<u>Prop.</u> Finite mapping $f: X \to Y$ is finite-to-one

<u>Warning:</u> Converse false!! $\left(\text{Eg } tx = 1: \right.$  $\left. \right)$

(What's true: finite-to-one & proper = finite)

<u>Idea:</u> take $t_1, \dots, t_r \in k[X]$ generating $k[X]$ over $k[Y]$. Argue as in prototype example that for fixed $b \in Y$:

The fns t_i take on only finitely many values on $f^{-1}(b)$.

<u>Prop.</u> Consider $f: X \to Y$ dominant finite map of affine alg. sets. Then f is surjective.

(Shaf, Thm. 1.12, p.61)

<u>Cor.</u> Let $f: X \to Y$ be a finite mapping of affine alg. sets. Then f is closed, ie.

$$Z \subseteq X \quad \text{Zariski closed} \Rightarrow f(Z) \subseteq Y \quad \text{Z-closed}$$

<u>Pf.</u> Let

$$W = \overline{f(Z)} \subseteq Y.$$

Have:

$$\begin{tikzcd} k[X] \arrow[r, two heads] & k[Z] \\ k[Y] \arrow[u, "f^*"] \arrow[r, two heads] & k[W] \arrow[u, "\bar{f}^*"'] \end{tikzcd}$$

$k[X]$ is module-finite over $k[Y]$, hence $k[Z]$ is module-finite over $k[W]$ ie. $Z \to W$ is finite. By previous Prop, $Z \to W$ surj, ie. $\overline{f(Z)} = f(Z)$. QED

How should define finiteness for an arbitrary morphism

$$f : X \longrightarrow Y \ ?$$

Idea: take covering $Y = \bigcup_\alpha U_\alpha$ by affine open sets, and ask that

$$\left.\begin{array}{l} \cdot\ V_\alpha = f^{-1}(U_\alpha) \text{ is affine} \\ \&\ \cdot\ V_\alpha \longrightarrow U_\alpha \text{ finite.} \end{array}\right\} (*)$$

<u>Problem</u>: does this give same defn in case X, Y already affine?

<u>Prop.</u> Let

$$f : X \longrightarrow Y$$

be morphism of affine vars. Consider open cover $Y = \bigcup_\alpha U_\alpha$, where

$$U_\alpha = D(g_\alpha),$$

w.

$$D(g_\alpha) = Y - \text{Zeroes}(g_\alpha), \quad g_\alpha \in k[Y], \qquad (\ldots, g_\alpha, \ldots) = (1).$$

Let

$$V_\alpha = f^{-1}(U_\alpha) \subseteq X.$$

Then

$$f \text{ finite} \iff \text{each } V_\alpha \text{ affine, } \& \ V_\alpha \to U_\alpha \text{ finite}$$

So we take (*) a def of finiteness for general morphism.

Pf. Will prove $\Leftarrow$, assuming for simplicity of notation that f is dominant. Write

$$\begin{array}{c} B = k[X] \\ \cup\!| \\ A = k[Y] \ni g_\alpha \end{array}$$

Note

$$k[U_\alpha] = A_{g_\alpha} = k[Y][1/g_\alpha]$$

Situation: B_{g_α} fg as A_{g_α}-module $\forall \alpha$. WTS: B is f.g. as A-mod

· Let $w_{i,\alpha} \in B_{g_\alpha}$ be gens of B_{g_α} as A_{g_α}-module.

· Since $1/g_\alpha \in A_{g_\alpha}$, can suppose

$$w_{i,\alpha} \in B \text{ for all } \alpha.$$

<u>Claim</u>: The elts $\{w_{i,\alpha}\}_{i,\alpha}$ generate B as an A-module.

<u>Pf</u> Fix $b \in B$. Then for each α,

$$B_{g_\alpha} \ni b = \sum_i \frac{a_{i,\alpha}}{g_\alpha^m} w_{i,\alpha} \quad \text{suitable } a_{i,\alpha} \in A,\ m>0.$$

Recall the $g_\alpha \in A$ generate the unit ideal, so the same is true for the g_α^m (By Nullstellen.) So

$$1 = \sum h_\alpha g_\alpha^m \qquad \text{some } h_\alpha \in A.$$

So

$$b = \left(\sum h_\alpha g_\alpha^m\right) b$$

$$= \sum_i \sum_\alpha a_{i,\alpha} w_{i,\alpha} h_\alpha \quad \text{QED}$$

A important appln of these ideas is

<u>Thm</u> (Chevalley's Thm). Let

$$f : X \longrightarrow Y$$

be a dominant morphism of QPV's. Then

$$\operatorname{im}(f) \subseteq Y$$

contains a Zariski-open subset. (<u>Pf</u>. Shaf, Thm 1.14, pp 62–63)

Ex. Typical ex: $\mathbb{A}^2 \longrightarrow \mathbb{A}^2$, $(x,y) \longmapsto (x, xy)$.

Def. A finite union of locally closed sets is called constructible.

(So "constructible sets" $\leftrightarrow$ sets gen by unions, intersections & complements of Z-open sets)

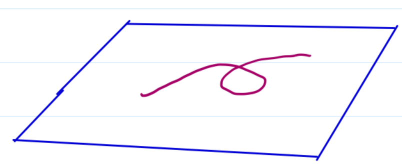
Cor of C's Thm: Image of morphism $f: X \longrightarrow Y$ is constructible

Noether Normalization —

Thm. Let
$$X \subseteq \mathbb{P}^n$$
be a proj alg set. Let $P \in \mathbb{P}^n$ be a point not lying on X. Then proj from P defines a finite mapping

$$\pi_P : X \longrightarrow \mathbb{P}^{n-1}$$

Cor. Let $L \subseteq \mathbb{P}^n$ be a linear space of dim ℓ disjoint from X. Then proj from L defines finite map



$$\pi : X \longrightarrow \mathbb{P}^{n-\ell-1}.$$

Pf of Cor: Proj from L is composition of projections from points.

For Thm: see Shaf Thm 1.15, p. 64.

Pf of Thm in toy (but illustrative) case.

- Say $X = \{F = 0\}$ hypsf of deg d. Take $P = [0, \ , 0, 1]$.
- Have $F(P) \neq 0$, so F must contain non-zero X_n^d term. So can write

$$F = X_n^d + A_1(X_0, \ , X_{n-1}) X_n^{d-1} + \quad + A_d(X_0, \ , X_{n-1}),$$

A_i homog of deg $= i$.

- Proj is then

$$\pi : X \longrightarrow \mathbb{P}^{n-1}, \qquad \pi([X_0, \ , X_n]) = [X_0, \dots, X_{n-1}].$$

Consider eg $U_0 = \{X_0 \neq 0\} = \mathbb{A}^{n-1}$. Then

$$V_0 = \pi^{-1}(U_0) = \{ x_n^d + A_1(1, x_1, \ , x_{n-1}) x_n^{d-1} + \quad + A_d(1, x_1, \ , x_{n-1}) = 0 \}$$

and this is finite over $\mathbb{A}^{n-1}$ by our prototypical ex.

Cor (Noether normalizn, I). Let $X \subseteq \mathbb{P}^n$ be irred proj var. Then $\exists$ finite surj mapping

$$f : X \longrightarrow \mathbb{P}^d \quad \text{for some } d \geq 0$$

Rmk: Of course, will turn out that $d = \dim X$.

Pf. Choose lin space $L \subsetneq \mathbb{P}^n$ of max dim disj from X and proj from L.

Cor (Noether normalizn, II). Let $X \subseteq \mathbb{A}^n$ be irred affine var. Then $\exists$ finite surj morphism

$$f : X \to \mathbb{A}^d$$

Pf.: For you — apply previous Cor to $\overline{X} \subseteq \mathbb{P}^n$.

Rmk: Alg interpr of N.N.II: say

$$k[X] = \text{coord ring of } X.$$

Then $\exists$

$$t_1, \ldots, t_d \in k[X]$$

s.t.

$$k[t_1, \ldots, t_d] \cong \text{poly ring in } d \text{ vars}$$

&

$$k[X] \supseteq k[t_1, \ldots, t_d] \text{ module-finite.}$$

Dimension —

· Want to define dimension of an irred var, and show it has good properties. I'll give defs and main pfs, but will refer to Shaf for some of pfs.

Preview

Want to dim(X) for any irred QPV X. Should have:

$$\bullet \dim \mathbb{A}^n = \dim \mathbb{P}^n = n$$

Note

$$k(\mathbb{A}^n) = k(\mathbb{P}^n) = k(x_1, \ldots, x_n), \text{ and}$$

$$\operatorname{tr deg}_k k(x_1, \ldots, x_n) = n$$

This suggests:

<u>Def 1</u>: If X is irred QPV, then

$$\dim X = \operatorname{tr\,deg}_k k(X)$$ (This will be the official defn.)

(Max no of alg indep rat fns on X)

Another point of view is to look at chains of subvars:

If X is irred, and $Y \subsetneqq X$ is irred subvar, expect that then

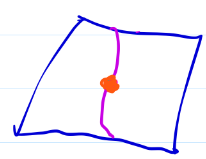
$$\dim Y < \dim X.$$

Also expect that can find subvar of codim 1. (More later)

This suggests:

<u>Def 2.</u> If X is irred QPV, then $\dim X$ is maximal length of irred subvars

$$\{pt\} = Z_0 \subsetneqq Z_1 \subsetneqq \cdots \subsetneqq Z_n = X$$



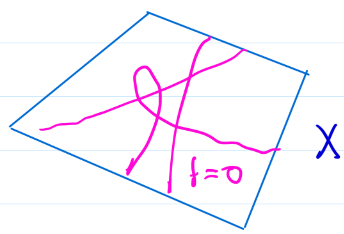
Will see this is equiv to Def 1. Note that this is essentially the def of dim of a nice local comm ring: one looks at chains of prime ideals.

There are then two crucial thms:

<u>Thm A.</u> If X irred of dim n, say affine, and

$$0 \neq f \in k[X] \quad \text{a reg fn,}$$

then every irred comp of Zeroes(f) has dim $= n-1$. (Krull's principal ideal thm)



(This will imply the equiv. of the two defs.)

<u>Thm B:</u> Let

$$f : X^n \longrightarrow Y^m$$

be a surj morph bet irred vars, w/

$$\dim X = n, \quad \dim Y = n.$$

Then for every $y \in Y$, every irred comp of $f^{-1}(y)$ has dim $\geq n-m$. Moreover:

$$\exists \text{ non-empty } Z\text{-open } U \subseteq Y \text{ s.t. for } y \in U:$$

$$\text{every irred comp of } f^{-1}(y) \text{ has dim} = n-m$$

o o

<u>Review of Transendence Degree</u> – Let

$$L \supseteq k$$

be a f.g field extn. Elts $u_1, \dots, u_n \in L$ are alg dep over k if $\exists$ poly

$$0 \neq P(x_1, , x_n) \in k[x_1, \dots, x_n]$$

s.t.

$$P(u_1, , u_n) = 0.$$

Else the u_i are alg independent.

Def. Transcendence basis of L/k is collection $u_1, \ldots, u_n \in L$ s.t.

$$u_1, \ldots, u_n \text{ alg indep} / k,$$

&

$$L \supseteq k(u_1, \ldots, u_n) \text{ is alg extn}$$

So

$$L \underset{\text{finite}}{\supseteq} k(u_1, \ldots, u_n) \underset{\text{purely transc}}{\supseteq} k$$

Thm / Def: Any two transc. bases have same no of elts: this is

$$\operatorname{tr\,deg}_k L$$

Ex. x = var, $L = k(x)$: $x^2 \in L$ is transc basis, i.e.

$$k(x) \supseteq \underset{\substack{\cong \\ k(y)}}{k(x^2)} \supseteq k$$

Dimension:

Def. X irred QPV $/ k = \bar{k}$. Define

$$\dim X = \operatorname{tr\,deg}_k k(X).$$

Ex Have $\dim \mathbb{A}^n = \dim \mathbb{P}^n = n$.

Ex. If $U \subseteq X$ is non-empty open set (i.e. dense), then

$$\dim U = \dim X.$$

Ex. Suppose X irred, and $\exists$ finite surj

$$f: X \to \mathbb{A}^n \qquad \text{or} \qquad f: X \to \mathbb{P}^n$$

Then $\dim X = n$.

Pf. In either case $k(X) \supseteq k(\mathbb{A}^n) = k(\mathbb{P}^n)$ a finite extn.

Prop. Fix X, Y irred. Then

$$\dim(X \times Y) = \dim X + \dim Y.$$

Sketch: Say X, Y affine

$$\dim X = n, \quad \dim Y = m.$$

Then $\exists$ finite surj

$$f: X \to \mathbb{A}^n, \quad g: Y \to \mathbb{A}^m.$$

(Noether normalizn). Then (Exerc!)

$$f \times g: X \times Y \longrightarrow \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$$

is finite (and of course surj.) So $\dim X \times Y = n+m$. (Or see Shaf).

Thm. Consider X, Y irred, $X \subseteq Y$ a subvar. Then

$$\dim X \leq \dim Y.$$

If $X \subsetneq Y$, then $\dim X < \dim Y$.

Sketch: Can assume X, Y affine, say

$$X \subseteq Y \subseteq \mathbb{A}^N, \quad \dim Y = n.$$

Choose coords $t_1, \dots, t_N$ s.t $t_1, \dots, t_n$ restr to transc basis for $k(Y)$. Then remaining coord fns are alg dep on these: ie for $j > n$, $\exists\ P_j \in k[x_1, \dots, x_n, x_j]$ s.t.

$$P_j(t_1, \dots, t_n, t_j) \equiv 0 \quad \text{in } k[Y].$$

This holds a fortiori in $k[X]$, ie. $\dim X \leq \dim Y$.

Now say $\dim X = \dim Y = n$: can assume that $t_1, \dots, t_n$ also form a transc. basis of $k(X)/k$. Let

$$0 \neq u \in k[Y] \text{ be fn van on } X.$$

u alg dep on $t_1, \dots, t_n$ in $k(Y)$, so $\exists\ a \in k[x_1, \dots, x_n][y]$ st

$$(*) \qquad a(t, u) = a_0(t)u^k + \dots + a_{k-1}(t)u + a_k(t) \equiv 0 \quad \text{in } k[Y].$$

Can suppose a irred, esp. $a_k(x_1, \dots, x_n) \not\equiv 0$. Now restrict $(*)$ to X. Since $u|X \equiv 0$, get

$$a_k(t_1, \dots, t_n) \equiv 0 \quad \text{on } X.$$

But this is # since $t_1, \dots, t_n$ alg indep on X. QED

Thm. Let

$$X \subseteq \mathbb{A}^n \quad \text{or} \quad X \subseteq \mathbb{P}^n$$

be hypsfs, i.e. the zero loci of a single poly. Then every irred comp of X has dim $n-1$.

Lemma: Let $0 \neq f \in k[\mathbb{A}^n]$ be poly. If f is irred, then

$$\text{Zeroes}(f) \subseteq \mathbb{A}^n \text{ irred}$$

Sim in $\mathbb{P}^n$.

(Pf for you)

Proof 1 of Thm: Assume

$$X = \text{Zeroes}(f) \subseteq \mathbb{A}^n.$$

By Lemma, can suppose f ($\therefore X$) irred. Say

$$f = f(x_1, x_n), \text{ and } x_n \text{ occurs non-triv. in } f.$$

Since $f \equiv 0$ on X, t_n alg dep on $t_1, , t_{n-1}$ on X, $t_i = x_i|X$

Claim: $t_1, , t_{n-1}$ alg indep on X.

Pf: Else $\exists$ $g \in k[x_1, \ldots, x_{n-1}]$ s.t. $g(t_1, , t_{n-1}) \equiv 0$ on X, i.e.

$$g(x) | f(x). \ \#$$

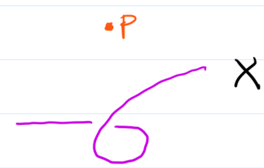
Proof 2 of Thm: Here we'll do the proj case. Say

$$X = \text{Zeroes}(F) \subseteq \mathbb{P}^n,$$

with F ($\therefore X$) irred. Need to show $\dim X = n-1$.

· Pick $P \in \mathbb{P}^n - X$, and consider projection

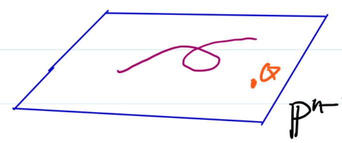
$$\pi_P : X \longrightarrow \mathbb{P}^{n-1}$$



By NN this is finite; suffices to show it's surj.

· If not, $\exists\ Q \in \mathbb{P}^{n-1}$, $Q \notin \mathrm{Im}(X)$. Then

$$(\text{line } PQ) \cap \{F = 0\} = \phi,$$



But a hypsf meets every line, #

<u>Cor</u>. Suppose

$$X \subseteq \mathbb{A}^n \qquad \text{or} \qquad X \subsetneq \mathbb{P}^n$$

has the property that every irred comp has codim 1. Then X is a hypsf, ie. cut out by a single poly.

<u>Pf</u>. Can suppose X irred, say $X \subseteq \mathbb{A}^n$. Let

$$f \in k[\mathbb{A}^n] \text{ be irred poly st. } X \subseteq \mathrm{Zeros}(f).$$

By Thm, $Z(f)$ irred of dim $n-1$. Since $\dim X = n-1$, earlier result $\Rightarrow X = \mathrm{Zeros}(f)$.

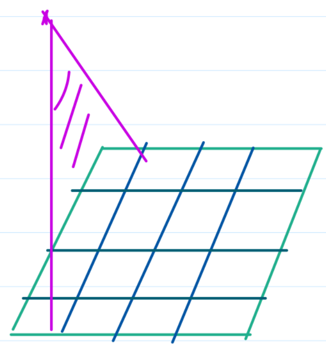
<u>Ditto</u>: $X \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ pure codim 1: defined by single multi-homog poly.

Warnings.

(1). For arbitrary ambient var V, it's very false that any $X \subseteq V$ of codim 1 is defined by a single equation.

Ex. $V = \{xy - zw = 0\} \subseteq \mathbb{A}^4$

dim 3



V is affine cone over $\mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$

$V \supseteq \Pi = \{(x, 0, z, 0) \mid x, z \in k\}$
$\cong \mathbb{A}^2$.

Can show not cut out by single eqn. (Eg $\{y=0\} \supseteq \{(x, 0, 0, w)\}$)

Rmk: Say V irred, affine, "normal". Then

Every irred $X \subseteq V$ of codim 1 cut out by sing fn $f \in k[V]$ $\iff$ $k[V]$ a UFD.

(2). Not true that subvar $X \subseteq \mathbb{A}^n$ or $X \subseteq \mathbb{P}^n$ of codim c is cut out by c fns.

Intersections w Hypsfs (around Krull's Thm)

Def. X QPV, possibly reducible. Define $\dim X =$ max of dims of comps of X. X has pure dim n if all comps dim n.

Consider irred proj var
$$X \subseteq \mathbb{P}^r.$$

Lemma. Let
$$F_0, \dots, F_m \text{ be homog. polys of deg } d$$
s.t

$$X \cap \{F_0 = \quad = F_m = 0\} = \phi$$

Then the mapping

$$\phi : X \longrightarrow \mathbb{P}^m, \quad x \mapsto [F_0(x), \ldots, F_m(x)]$$

is finite.

Pf: ϕ is linear proj of X under compos

$$X \subseteq \mathbb{P}^m \xhookrightarrow[\text{Veronese}]{\text{d-fold}} \mathbb{P}^{N_d}$$

from center that is disj from X. Thus ϕ a compos. of finite maps, hence finite.

Lemma 2. Say $X \subseteq \mathbb{P}^m$ irred of $\dim = n$. Let F be homog poly of of deg d. Then $\exists$ homog polys

$$F = F_0, \ldots, F_n \text{ of deg } d \text{ st}$$

$$X \cap \{F_0 = \quad = F_n = 0\} = \phi.$$

Sketch. Let Y = irred comp of $\{F=0\} \cap X$. So $\dim Y \leq n-1$

Check: general homog poly F_1 of deg d doesn't vanish on Y

Now conclude by induction on dim. (Can use Veronese to reduce to $d=1$)

Thm: Let $X \subseteq \mathbb{P}^m$ be irred of dim n, and

$$F = \text{homog poly deg } d \text{ not van on } X.$$

Then

$$X_F =_{\text{def}} X \cap \{F = 0\} \text{ has dim } n-1,$$

ie every irred comp of X_F has dim $\leq n-1$, and at least one has dim $= n-1$

(Rmk: In fact, they all have dim $n-1$, but I want to avoid this for moment.)

Pf. By Lemmas, $\exists$ $F = F_0, \ldots, F_n$ of deg d defining finite map

$$\phi : X \longrightarrow \mathbb{P}^n, \quad x \mapsto [F_0(x), \ldots, F_n(x)].$$

ϕ surj, else could proj from pt of $\mathbb{P}^n$ to get finite $X \to \mathbb{P}^{n-1}$. Let

$$H = \mathbb{P}^{n-1} = \{T_0 = 0\} \subseteq \mathbb{P}^n.$$

So

$$X_F = \phi^{-1}(H), \text{ and map } X_F \longrightarrow \mathbb{P}^{n-1} \text{ finite surj.}$$

So $\dim X_F = n-1$.

Cor 1. X irred QPV dim n. Then $\exists$ $Y \subset X$ irred of dim $n-1$

(Pf: Apply Thm to proj closure of X.)

Cor 2. X irred QPV. Then $n = \dim X$ is length of largest chain

$$X = Z_0 \supsetneqq Z_1 \supsetneqq \cdots \supsetneqq Z_n \supsetneqq \phi$$

of irred closed subvars ▥

Cor 3. Let $X \subseteq \mathbb{P}^m$ be irred proj var of dim $= n$. Given $r \leq n$ homog polys

$$F_1, \ , F_r \quad \text{(possibly of different degrees)},$$

have

$$X \cap \{F_1 = \ = F_r = 0\} \neq \phi.$$

Moreover,

$$\dim\left(X \cap \{F_1 = \ = F_r = 0\}\right) \geq n-r.$$

Rmk: Often happens "in nature" that

$$\dim\left(X \cap \{F_i = 0\}\right) > n-r.$$

Ex. Consider twisted cubic

$$C \subseteq \mathbb{P}^3 \quad \text{image of}$$

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^3, \quad [s,t] \hookrightarrow [\underset{X}{s^3}, \underset{Y}{s^2t}, \underset{Z}{st^2}, \underset{W}{t^3}]$$

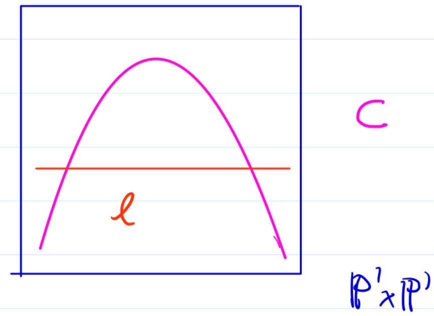
C is cut out by the three quadrics

$$\underset{Q_1}{\underbrace{XY - ZW}} = 0, \qquad \underset{Q_2}{\underbrace{Y^2 - XZ}} \qquad \underset{Q_3}{\underbrace{Z^2 - YW}}$$

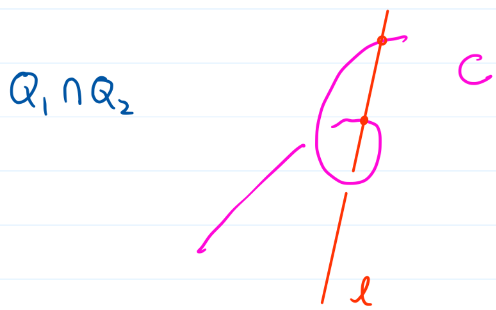
$Q_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$, and C is realized as curve of type $(2,1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$

Now consider

$$Q_1 \cap Q_2 :$$



This is a curve of type (2,2) on $\mathbb{P}^1 \times \mathbb{P}^1$, of form $C \cup$ line!



Intersection w Q_3 "removes" ℓ.

Thm. Consider

$$X \subseteq \mathbb{P}^m$$

irred proj var dim n, F = homog poly not vanishing ident. on X. Then:

<u>every</u> irred comp of $X \cap \{F = 0\}$ has dim $= n-1$.

Ditto X a QPV.

<u>Pf</u>: See Shaf, pp 73, 74. This is essentially a version of

Krull's Principal Ideal Thm: Let

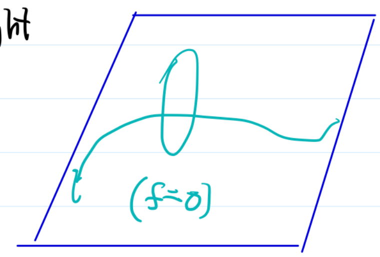
$$R = \text{Noeth domain}, \quad 0 \neq f \in R$$

Let P = minimal prime ideal containing f:

$$0 \subsetneq (f) \subsetneq P$$

Then P has height 1, ie no prime ideals lying in bet 0 & P

In geom sitn, apply to $R = k[X]$. If $0 \neq f$, minimal primes over $(f) \leftrightarrow$ irred comps of $(f=0)$, and to say they have height one means they have codim $= 1$.



A Global Appln

Thm Let

$$X^n, Y^m \subseteq \mathbb{P}^r$$

be irred QPV of dims n, m.

(1). Every irred comp of $X \cap Y$ has dim $\geq n+m-r$ (ie. $\operatorname{codim}(X \cap Y) \leq \operatorname{codim} X + \operatorname{codim} Y$).

(2) If X and Y are projective then

$$X \cap Y \neq \phi \quad \text{if } n+m \geq r.$$

Pf of (1). Assertion is local so can suppose X, Y (quasi-) affine, ie

$$X, Y \subseteq \mathbb{A}^r \quad \text{(loc closed).}$$

Key idea is "reduction to the diagonal". Viz:

Consider the diagonal

$$\mathbb{A}^r \cong \Delta \subseteq \mathbb{A}^r \times \mathbb{A}^r.$$

Have $\quad X \times Y \subseteq \mathbb{A}^r \times \mathbb{A}^r$, and

$$X \cap Y = \Delta \cap (X \times Y) \quad (\text{in } \mathbb{A}^r \times \mathbb{A}^r):$$

$$\begin{array}{ccc} \mathbb{A}^r = \Delta & \subseteq & \mathbb{A}^r \times \mathbb{A}^r \\ \cup\!| & & \cup\!| \\ X \cap Y & \subseteq & (X \times Y) \end{array}$$

Now suppose $t_1, .., t_r$, $s_1, , s_r$ are coords on the two copies of $\mathbb{A}^r$. Then

$$\Delta = \text{Zeroes}(t_i - s_i) \subseteq \mathbb{A}^r \times \mathbb{A}^r,$$

ie.

$$\Delta \text{ defined by } r \text{ eqns.}$$

Also, $\dim(X \times Y) = n+m$. So

$$\Delta \cap (X \times Y) = \text{Zeroes}(t_i - s_i \text{ in } k[X \times Y])$$

so has $\dim \geq n+m-r$.

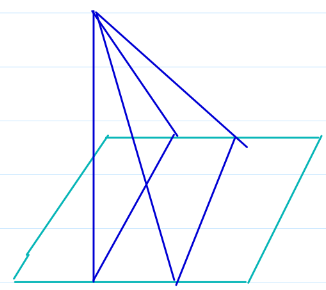
Rmk: Same argument works for intersections inside any non-singular var P:

If P is smooth of dim r, then

$\Delta_P \subseteq P \times P$ locally cut out by r eqns

However: lower bound on $\dim(X \cap Y)$ does not hold for subvars of arb var V:

Eg $V = \{xy - zw = 0\} \subseteq \mathbb{A}^4$



Take X, Y = cones over two lines in same ruling of $\mathbb{P}^1 \times \mathbb{P}^1$. Then

$\dim V = 3$, $\dim X = \dim Y = 2$,

but $\dim X \cap Y = 0$.

Pf of (2) Now want to show that if

$X^n, Y^m \subseteq \mathbb{P}^r$ are proj,

then $X \cap Y \neq \phi$ if $\dim X + \dim Y \geq r$. Idea: pass to affine cones i.e. consider

$C(X), C(Y) \subseteq \mathbb{A}^{r+1}$ affine cones over X, Y

Then

$\dim C(X) = n+1$, $\dim C(Y) = m+1$,

So by (1), every comp of $C(X) \cap C(Y)$ has dim $\geq$

$$(n+1)+(m+1)-(r+1) = (n+m-r)+1 \geq 1$$

But $C(X)\cap C(Y) \neq \phi$, since $0 \in C(X)\cap C(Y)$, So

$$\dim C(X)\cap C(Y) \geq 1.$$

But $C(X)\cap C(Y) = C(X\cap Y)$, so $X\cap Y \neq \phi$. QED

<u>Thm on dimension of fibres</u>.

<u>Thm</u>. Let

$$f: X^n \longrightarrow Y^m$$

be surj morph bet irred vars of dims n, m (so $n \geq m$).

(1) For every $y\in Y$, all irred comps of $f^{-1}(y)$ have dim $\geq n-m$.

(2). $\exists$ non-empty ($\therefore$ dense) open $U \subseteq Y$ st.

$$\dim f^{-1}(y) = n-m \quad \forall\, y \in U$$

(ie every irred comp of $f^{-1}(y)$ has dim $= n-m$.)

<u>Rmk</u>: Analogous statement holds assuming only that f is dominant: one replaces Y by open $V \subseteq \operatorname{im}(f) \subseteq Y$. (Chevalley's thm)

<u>Proof of Thm</u>:

(1°). Thm local on X & Y, so can suppose X, Y affine. (Check!)

· Take finite map $\eta: Y \to \mathbb{A}^m$ (Noether normalizn), and consider

$$h = n \circ f : X \longrightarrow \mathbb{A}^m.$$

Every fibre of h is union of finitely many fibres of f. So

$$\text{Thm for } h \Rightarrow \text{Thm for } f.$$

So we may assume f is surj

$$f : X \to \mathbb{A}^m.$$

(2°). Fix $b = (b_1, \ldots, b_m) \in \mathbb{A}^m$. Then

$$b = \text{Zeroes}(t_1 - b_1, \ldots, t_m - b_m).$$

So $f^{-1}(b)$ cut out by m eqns hence every comp of $f^{-1}(b)$ has $\dim \geqslant n-m$.
Remains to show equality holds on open set.

(3°). Alg. situation is as follows:

$$\begin{array}{ccc} k[X] & \subseteq & k(X) \\ \cup\!| & & \\ k[\mathbb{A}^m] = k[t_1, \ldots, t_m] & \subseteq & k(\mathbb{A}^m) \end{array}$$

By additivity of tr. deg in towers,

$$\text{tr. deg}_{k(\mathbb{A}^m)} k(X) = n - m$$

Choose

$$h_1, \ldots, h_N \in k[X]$$

that generate $k[X]$ as $k[\mathbb{A}^m]$-algebra
May suppose that

$$\begin{array}{c} k[X] = k[\mathbb{A}^m][h_1, \ldots, h_N] \\ \cup\!| \\ k[\mathbb{A}^m] = k[t_1, \ldots, t_m] \end{array}$$

$h_1, \ldots, h_{n-m}$ are a tr. basis for $k(X) / k(\mathbb{A}^m)$.

So for $j > n-m$, $\exists$ polys $P_j \in k[t_1, \ldots, t_m, x_1, \ldots, x_{n-m}, y]$ s.t.

$$P_j(t, h_1, \ldots, h_{n-m}, h_j) = 0 \qquad (*)$$

Now fix $b \in \mathbb{A}^m$. The (images of) the h_α generate $k[f^{-1}(b)]$, and $(*)$ gives a potential relation of alg dependence of h_j ($j > n-m$) on $h_1, \ldots, h_{n-m}$ along $f^{-1}(b)$. Issue is to show $\exists$ $U \subsetneq \mathbb{A}^m$ s.t. for $b \in U$:

$$P_j(b; h_1, \ldots, h_{n-m}, h_j) \text{ is non-trivial in } h_j.$$

This will show that each comp of $f^{-1}(b)$ has dim $\leq n-m$.

(4°). We view each $P_j(t, x, y)$ as poly in x & y whose coeffs are polys in t. The variable y must appear non-trivially in P_j since $t_1, \ldots, t_m, h_1, \ldots, h_{n-m}$ are alg indep. Let

$$Z_j \subsetneq \mathbb{A}^m$$

be the proper closed set defined by the t-coeffs of the y-leading term of $P_j(t, x, y)$, and set

$$U = \mathbb{A}^m - \left(\cup Z_j\right).$$

Then for $b \in U$, $P_j(b, h_1, \ldots, h_{n-m}, h_j) = 0$ gives a non-triv. relat of algebr. dependence of $h_j | f^{-1}(b)$ upon $h_1 | f^{-1}(b), \ldots, h_{n-m} | f^{-1}(b)$. QED.

<u>Cor.</u> Let

$$f: X^n \longrightarrow Y^m$$

be surj map of irred vars of dims n, m. Then

$$Y_k =_{def} \{ y \in Y \mid \dim f^{-1}(y) \geq k \}$$

is Z-closed i.e. the fn

$$y \longmapsto \dim f^{-1}(y)$$

is Z-upper semicontinuous.

<u>Pf.</u> By Thm, $Y_{n-k} = Y$, and $\exists$ closed $Z \subsetneq Y$ s.t. for $k > n-k$

$$Y_k \subseteq Z.$$

Proceed by ind on dim of target

<u>Cor.</u> Let $f: X \to Y$ be morph st Y irred and all fibres irred of same dim. Then X irred. (See Shaf p 76)

<u>Lines on Hypsfs:</u>

Consider

$$X_d \subseteq \mathbb{P}^{n+1} \quad : \text{ hypsf deg } d, \dim n.$$

<u>Ask:</u> when does X contain a line ie. a linearly embedded

$$\mathbb{P}^1 = \ell \subseteq X \; ?$$

Recall: have

$$\mathbb{G}(k, r) = \begin{array}{l}\text{Grassmannian parametrz}\\ \text{all linear } \mathbb{P}^k \subseteq \mathbb{P}^r.\end{array}$$

We saw:

$$\mathbb{G}(k,n) \sim_{\text{birat}} \mathbb{A}^{(k+1)(r-n)},$$

so

$$\mathbb{G}(k,n) \text{ irred, of } \dim = (k+1)(r-n).$$

In partic. set

$$\mathbb{G} = \mathbb{G}(1, n+1).$$

Then

$$\boxed{\dim \mathbb{G} = 2n.}$$

We also have

$$\mathbb{P}^{N_d} = \mathbb{P}\left(\begin{matrix}\text{homog polys deg } d \\ \text{in } (n+2)\text{-vars}\end{matrix}\right), \quad \text{so } N_d = \binom{d+n+1}{n+1} - 1$$

We view $\mathbb{P}^{N_d}$ as parametrizing all hypsfs deg d

We want to study:

$$\mathbb{P}^{N_d} \supseteq R_d = \{ [X] \mid X \text{ contains a line} \}$$

Key idea - study <u>correspondence</u>

$$\mathbb{P}^{N_d} \times \mathbb{G} \supseteq Z = \{ ([X], [\ell]) \mid \ell \subseteq X \}$$

By defn,

$$R_d = pr_1(Z).$$

To understand Z, we consider the second projection:

$$q : Z \longrightarrow \mathbb{G}, \quad ([X], [\ell]) \longmapsto [\ell]$$

Claim: For any ℓ,

$$q^{-1}([\ell]) = \mathbb{P}^{N_d-(d+1)} \overset{\text{linearly}}{\subseteq} \mathbb{P}^{N_d}$$

Cor: Z is irred, with

$$\dim Z = N_d + (2n-(d+1)).$$

Pf of Prop: Fix ℓ, say

$$\ell = (X_2 = \cdots = X_{n+1} = 0)$$

Given $F_d = F(X_0, \ldots, X_{n+1})$,

$$\ell \subseteq \{F=0\} \iff F(X_0, X_1, 0, \ldots, 0) \equiv 0 \quad (*)$$

Now consider map:

has dim $= d+1$

$$\begin{pmatrix} \text{Homog polys of} \\ \text{deg } d \text{ on } \mathbb{P}^{n+1} \end{pmatrix} \xrightarrow{\rho} \begin{pmatrix} \text{Homog polys of} \\ \text{deg } d \text{ on } \mathbb{P}^1 \end{pmatrix}$$

$$F \longmapsto F(X_0, X_1, 0, \ldots, 0)$$

ρ is surj, and so

$$\operatorname{codim} \ker \rho = d+1 \qquad \square$$

Consider:

$$q : Z \longrightarrow \mathbb{P}^{N_d}$$

irred, dim $N_d + (2n-d-1)$

dim N_d

Cor: If $d > 2n-1$, then gen hypsf of deg d in $\mathbb{P}^{n+1}$ does not contain a line, ie

$$R_d \subsetneq \mathbb{P}^{N_d}.$$

is a proper closed subvar

Ex. Suppose $d = 2n-1$. Then both source & target of

$$q : Z \longrightarrow \mathbb{P}^{N_d}$$

have same dim. So either:

- q surj, hence gen finite

or

- q not surjective: then gen hypsf of deg $d = 2n-1$ does not contain line and any hypsf that does contain line contains ∞'ly many.

Eg: $n=2$, $d=3$: $X \subseteq \mathbb{P}^3$ a cubic sf.

Check: Fermat sf contains finitely many lines

So in this case, gen cubic sf contains finitely many lines.

Def Given hypsf $X \subseteq \mathbb{P}^{n+1}$, write

$$F(X) = q^{-1}([X]) \subseteq \mathbb{G}(1,n) :$$

so

$$F(X) = \{ \text{lines } \ell \subseteq X \}.$$

Guess: q is surjective whenever $d \leq 2n-1$, and hence

$$\dim F(X) = 2n-d-1$$

for general hypsf $X_d \subseteq \mathbb{P}^{n+1}$.

Thm (Altman-Kleiman). Guess is true.

Conj: (Debarre-deJong, char 0). For every smooth X,

$$\dim F(X) = 2n-d-1$$

(Known when $d \lessapprox \frac{n}{2}$.)

Chow Forms

Ask: Is there a way of parametrizing all curves in $\mathbb{P}^3$?

Consider:

$$C \subseteq \mathbb{P}^3 \quad \text{irred curve, not in } \mathbb{P}^2.$$

Note: if $\ell \subseteq \mathbb{P}^3$ is general line, then

$$\ell \cap C = \phi.$$

Idea: look at "special lines" that do meet C.

Def. Define Chow form of C to be

$$G(1,3) \supseteq \Sigma_C =_{def} \{ [\ell] \mid \ell \cap C \neq \phi \}.$$

<u>Prop.</u> Σ_C is hypsf in $\mathbb{G}$, and one can recover C from Σ_C.

Then Σ_C corresponds to a point $[\Sigma_C] \in \mathbb{P}\left(\begin{smallmatrix}\text{Hypsfs of suitable}\\ \text{degree in } \mathbb{G}\end{smallmatrix}\right)$

So:

$$\left\{ [C] \;\middle|\; \begin{matrix} C \text{ curve of} \\ \text{given deg in } \mathbb{P}^3 \end{matrix} \right\} \hookrightarrow \left\{ \begin{matrix} \text{Hypsfs of suitable} \\ \text{degree in } \mathbb{G} \end{matrix} \right\} = \mathbb{P}^N$$

<u>Idea of Prop:</u> Consider

$$C \times \mathbb{G} \supseteq Z_C = \{ (p, [\ell]) \mid p \in \ell \cap C \}$$

Show fibres of $pr_1 : Z_C \to C$ are all $\mathbb{P}^2$'s. So $\dim Z = 3$, maps to hypsf in $\mathbb{G}$.

Singularities

<u>General Rmk:</u> Given poly

$$f \in k[x_1, \ , x_n],$$

can compute deriv $\partial f / \partial x_i$ formally, and usual properties (eg. Taylor expansion) hold

If char $k = 0$, nothing funny happens. In char $p > 0$, main non-classical phenomenon is exist of non-const polys whose derivs vanish identically.

$$f(x) = x^p, \qquad f'(x) = p x^{p-1} \equiv 0.$$

Hypersurfaces -

Let

$$X = \{f = 0\} \subseteq \mathbb{A}^n, \quad f \in k[x_1, \ldots, x_n] \text{ reduced.}$$

$\ni a$, so $f(a) = 0$

Def. X is non-sing (or smooth) at a if

$$\frac{\partial f}{\partial x_i}(a) \neq 0 \quad \text{some } i.$$

Else X sing at a.

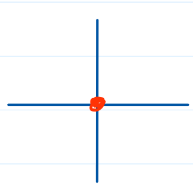
Ex. $f = x_n - h(x_1, \ldots, x_{n-1})$: non-sing at all pts.

Ex. Say $k = \mathbb{C}$, $X = \{f = 0\} \subseteq \mathbb{C}^n$ non-sing at a. Then in (classical) nbd of a, X is cx mfld of cx dim $n-1$.

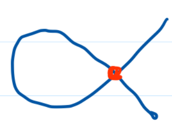
(Pf. Implicit fn thm true for holo fns.)

Sing pts are more interesting:

$xy = 0$

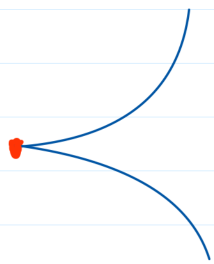


$y^2 = x^2(x+1)$

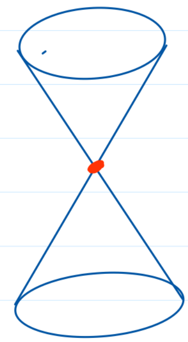


sing at origin

$y^2 - x^3 = 0$



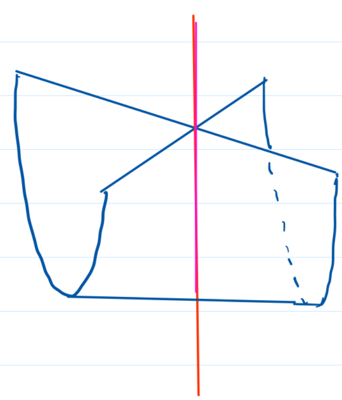
$x^2 + y^2 - z^2 = 0$



sing at 0

$x^2 - zy^2 = 0$ ("Whitney's umbrella")

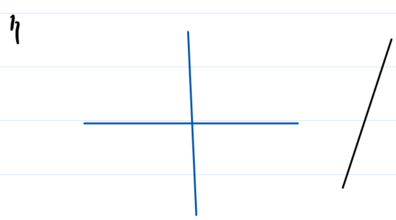
singular along whole z-axis.



NB: Can realize this as

$\mathbb{A}^2$ w pos & neg u-axis identified

ie.



$/ \; (u,0) \sim (-u,0)$ //

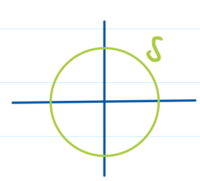
invariant fns:

u^2	v	uv
$\parallel$	$\parallel$	$\parallel$
z	y	x

<u>Ex</u> Consider simplest sing

$X = (xy = 0) \subseteq \mathbb{C}$

Topology already very interesting.



- Consider sphere $S = S(\varepsilon)$ of radius $\varepsilon > 0$ about 0:

$$S = S(\varepsilon) = \left(|x|^2 + |y|^2 = \varepsilon^2\right):$$

So S is 3-sphere $= \mathbb{R}^3 \cup \{\infty\}$.

- Consider

$$L_\varepsilon = X \cap S(\varepsilon) \subseteq S(\varepsilon) \qquad \left(\text{"link" of singularity}\right)$$

- For $0 < \varepsilon << 1$, topology of L_ε indep of ε.

$$L = \text{compact } 1\text{-manifold}$$

(ie union of circles)

- For $xy = 0$, get two <u>linked</u> circles in S^3:

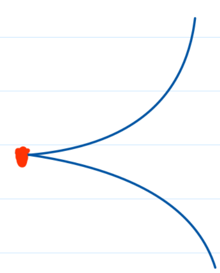
Ex. Now consider cusp

$$X = \{y^2 - x^3 = 0\}$$

Map

$$\mathbb{C} \longrightarrow X, \quad t \longmapsto (t^2, t^3)$$

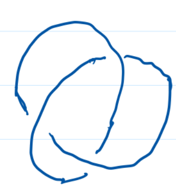
is homeom, so X a top mfld, but not sm submfld of $\mathbb{C}^2$.



If we do same constr as before, find that

$$L = X \cap S(\varepsilon) \subseteq S(\varepsilon)$$

is trefoil knot!



Ex. Take

$$X = \{ x_1^2 + x_2^2 + x_3^2 + x_4^3 + x_5^{6k-1} = 0 \} \subseteq \mathbb{C}^5, \quad 1 \leq k \leq 28$$

So

$$L = X \cap S(\varepsilon) \subseteq S(\varepsilon) = S^9$$

a 7-mfld.

Brieskorn: For $k = 1, \dots, 28$ get all exotic 7-spheres!

Classical Ref: Milnor, Sing pts of complex hypersurfaces

Singular locus of Hypsf.

Say

$$X = \{f = 0\} \subseteq \mathbb{A}^{n+1}.$$

Then

$$\mathrm{Sing}(X) = \{ f = \frac{\partial f}{\partial x_1} = \quad = \frac{\partial f}{\partial x_n} = 0 \}$$

Esp

$$\mathrm{Sing}(X) \subseteq X \text{ is } Z\text{-closed.}$$

Prop. If f is reduced, then $\mathrm{Sing}(X)$ is a proper closed subset.

Pf. Say $X = \{f = 0\}$, f no repeated factors. Then $\sqrt{(f)} = (f)$.
Now suppose every $a \in X$ a sing. pt. Then for each i:

$$\frac{\partial f}{\partial x_i} \equiv 0 \text{ on } X,$$

so

$$\frac{\partial f}{\partial x_i} \in \sqrt{f} = (f), \text{ i.e. } f \mid \frac{\partial f}{\partial x_i} \quad \forall i.$$

This is impossible for reasons of degree unless

$$\frac{\partial f}{\partial x_i} \equiv 0.$$

So $\partial f / \partial x_i \equiv 0$ $\forall i$. This is impossible in char 0. In char 0 it means that each monomial is a p^{th} power, so f itself is a p^{th} power

Multiplicity of a hypsf:

Given $X = \{f = 0\} \subseteq \mathbb{A}^n$, and $a \in X$, define:

$$\mathrm{mult}_a(X) = \mathrm{mult}_a(f) = \left\{ \begin{array}{l} \text{least } e \geq 1 \text{ st. all partials} \\ \text{of } f \text{ of order} < e \text{ van at } a, \\ \text{some } e^{th}\text{-partial non-vanishing} \end{array} \right\}$$

Ex $\mathrm{mult}_a(X) = 1 \Leftrightarrow X$ non-sing at a.

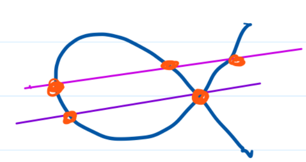
Ex. $X = \{x^2 - zy^2 = 0\}$: mult $= 2$ at every pt of z-axis.
(Whitney umbrella)

Consider

$$X = \{f = 0\} \subseteq \mathbb{A}^n; \quad f \text{ reduced, deg } d.$$

Prop: (i) A general line

$$\mathbb{A}^1 = \ell \subseteq \mathbb{A}^n$$



meets X at d distinct pts.

(ii) If $\mathrm{mult}_a(X) = e$ then a general line thru a meets X at $(d-e)$ pts other than a.

Projective hypersurfaces

Say

$$X = \{F=0\} \subseteq \mathbb{P}^n, \qquad F \text{ homog of degree } d$$

Lemma (Euler formula)

$$\sum X_i \frac{\partial F}{\partial X_i} = \deg(F)\cdot F. \qquad \square$$

So: In char $=0$, $\operatorname{Sing}(X) = \{ \frac{\partial F}{\partial X_0} = \dots = \frac{\partial F}{\partial X_n} = 0\}$.

Prop: Assume F has no repeated factors, and fix

$$a \in X = \{F=0\}.$$

Then TFAE:

(i) If $a \in X \cap U_i \subseteq U_i = \mathbb{A}^n$, then a is a sm pt of $X \cap U_i$.

(ii) Cond in (i) holds for every $U_j \ni a$.

(iii) $\frac{\partial F}{\partial X_\alpha}(a) \neq 0$ for some $0 \leq \alpha \leq n$.

Pf. Exer. So: we usually reduce questions of smoothness to affine setting.

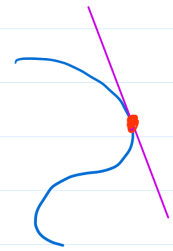
Zariski tangent space

Given

$$X = \{f=0\} \subseteq \mathbb{A}^n,$$

embedded tang space to X at a is:

$$\sum \frac{\partial f}{\partial x_i}(a)(x_i - a_i) = 0.$$



"Abstract" tang space :

$$\sum \frac{\partial f}{\partial x_i}(a) \cdot \xi_i = 0 \qquad T_a \mathbb{A}^n = n\text{-dim vect. space w coords } \xi_i.$$

$$(= \text{"} df = 0 \text{"})$$

Note: If $a \in X$ is non-sing pt, these are lin spaces of $\dim = n-1 = \dim X$.

If X sing at a, then $\dim T_a X > \dim X$.

Higher Codimension:

- How to proceed when X not a hypsf?
- Start by working over $\mathbb{C}$, where we have a priori notion what the def should be.
- Say

  $X \subseteq \mathbb{C}^r$ is affine var of dim n, codim $e = r - n$

  $f_1, \dots, f_p$ = gens of ideal of X.

- Fix $a \in X$.

Ask: When is $X \subseteq \mathbb{C}^r$ a cx submfld near a?

Implicit fn thm $\Rightarrow$

X a cx mfld near a (in classical topology) if $\exists$

$$\tilde{f}_1, \dots, \tilde{f}_e \in \mathbb{C}\text{-span}\{f_1, \dots, f_p\}$$

s.t.

$$\mathrm{rk}\begin{bmatrix} \frac{\partial \tilde{f}_1}{\partial x_1}(a) & \cdots & \frac{\partial \tilde{f}_1}{\partial x_r}(a) \\ & \vdots & \\ \frac{\partial \tilde{f}_e}{\partial x_1}(a) & & \frac{\partial \tilde{f}_e}{\partial x_r}(a) \end{bmatrix} = e \qquad (*).$$

<u>NB</u> This is equiv to asking that $\exists$ $\tilde{f}_1, \dots, \tilde{f}_e \in I_X$ (ie poly-lin combs of the f_i) st. $(*)$ holds

Also, $(*) \Leftrightarrow$ to

If $f_1, \dots, f_p \in I_X$ are generators, then

$$\mathrm{rk}\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_r}(a) \\ & \vdots & \\ \frac{\partial f_p}{\partial x_1}(a) & \cdots & \frac{\partial f_p}{\partial x_r}(a) \end{bmatrix} = r - n = e \qquad (**)$$

<u>Provisional Def:</u> $X \subseteq \mathbb{A}^r$ non-sing at $a \in X$ if $(**)$ holds.

<u>Problem</u>: not clear that it depends intrinsically on X, and not the particular embedding.

So: want to develop alternative interpr.

<u>Zariski Tang Space:</u>

Consider $X \subseteq \mathbb{A}^n$ as above, w. I_X gen by polys $f_1, \dots, f_p \in k[\mathbb{A}^n]$

<u>Def</u>: Zariski tang space to X at a is

$$T_a\mathbb{A}^n \supseteq T_aX \underset{\text{def}}{=} \left\{ \begin{array}{l} \text{subspace cut out by} \\ df_1(a) = \dots = df_p(a) = 0 \end{array} \right\}$$

$$= \left\{ (\xi_1, \dots, \xi_n) \,\middle|\, \sum \frac{\partial f_\alpha}{\partial x_j}(a) \cdot \xi_j = 0, \quad 1 \le \alpha \le p \right\}$$

So: $a \in X$ non-singular $\iff \dim T_aX = n = \dim X$. (Else "$>$").

<u>Thm</u>. Assume X affine, let

$$\mathfrak{m} = \mathfrak{m}_a \subseteq k[X] \text{ be max ideal of } a \in X.$$

Then

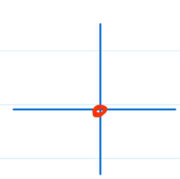
$$\mathfrak{m}/\mathfrak{m}^2 = (T_aX)^\vee = \mathrm{Hom}_k(T_aX, k).$$

Cor: $\dim T_aX$ depends intrinsic. on X at a, hence def of sing/sm pt intrinsic.

<u>Note</u>: $\mathfrak{m}/\mathfrak{m}^2$ is $k[X]/\mathfrak{m} = k$-module.

$\circledast$

<u>Ex</u> $X = \{xy = 0\} \subseteq \mathbb{A}^2$



$k[X] = k[x,y]/(xy)$.

$\underline{a=0}$ $\mathfrak{m} = (\bar{x}, \bar{y})$, $\bar{x}, \bar{y} \in k[X]$ image of coords x, y.

$$\mathfrak{m}^2 = (\bar{x}^2, \underset{\overset{\shortparallel}{0}}{\bar{x}\bar{y}}, \bar{y}^2) \quad : \quad \text{so} \quad \dim_k \mathfrak{m}/\mathfrak{m}^2 = 2.$$

On the other hand, consider eg $a = (1,0)$. Then

$$\mathfrak{m} = (\bar{x}-1, \bar{y})$$

$$\mathfrak{m}^2 = ((\bar{x}-1)^2, (\bar{x}-1)\bar{y}, \bar{y}^2)$$

But $(\bar{x}-1)\bar{y} = \bar{y}$, so $\mathfrak{m}^2 = ((\bar{x}-1)^2, \bar{y})$, $\dim \mathfrak{m}/\mathfrak{m}^2 = 1$.

Special Case of Thm: Consider

$$X = \mathbb{A}^n, \quad \mathfrak{m} = (x-a_1, \dots, x-a_n), \quad f \in k[x_1, \dots, x_n]$$

· Whatever $T_a^* X$ is, it should receive "$d_a f$." Note:

$$f(x) = f(a) + \sum \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + (\text{stuff in } \mathfrak{m}^2)$$

so

$$f(x) - f(a) \equiv \sum \frac{\partial f}{\partial x_i}(a)(x_i - a_i) \quad (\text{mod } \mathfrak{m}^2).$$

Let

$$\xi_i = (x_i - a_i)_{\text{mod } \mathfrak{m}^2} \in \mathfrak{m}/\mathfrak{m}^2.$$

Then

$$f(x) - f(a) = \sum \frac{\partial f}{\partial x_i}(a) \cdot \xi_i \in \mathfrak{m}/\mathfrak{m}^2.$$

This suggests that we define:

$$d_a f = f(x) - f(a) \in \mathfrak{m}/\mathfrak{m}^2.$$

Then relation

$$(f(x)-f(a))\cdot(g(x)-g(a)) \equiv 0 \pmod{m^2}$$

yields

$$d_a(fg) = f_a(a)\, d_a g + g(a)\, d_a(f).$$

So when $X = \mathbb{A}^n$, get "right" defn of $T_a^* X$.

<u>Proof of Thm</u>: Consider $X \subseteq \mathbb{A}^r$ dim n, fix $a \in X$, let

$$I = I_X = (f_1, \ldots, f_p).$$

Write

$$M = (x_1 - a_1, \ldots, x_r - a_r) \subseteq k[x_1, \ldots, x_r]$$

$$m = (\bar{x}_1 - a_1, \ldots, \bar{x}_r - a_r) \subseteq k[X].$$

Note:

$$a \in X \implies I \subseteq M.$$

Also

$$m/m^2 \cong M/M^2 + I.$$

Define:

$$M \xrightarrow{\Phi} (T_a X)^* \qquad (T_a X \subseteq T_a \mathbb{A}^r),$$

$$\Phi(g) \longmapsto (d_a g)\,|\,T_a X$$

<u>Claim</u>: Φ gives isom $M/M^2 + I \cong T_a^* X$.

<u>Pf.</u> Clearly Φ surj, since $T_a^* \mathbb{A}^n \twoheadrightarrow T_a^* X$.

(1°). Claim: $M^2 + I \subseteq \ker(\Phi)$.

Pf. Say $g \in M^2$. Then $d_a g = 0 \in T_a^* \mathbb{A}^n$, so $\Phi(g) = 0$

Now say $g \in I$, so $g = \sum h_i f_i$ ($f_i \in I$ gens). Then

$$d_a g = \sum h_i(a) \cdot d_a f_i + \sum f_i(a)\, dh_i.$$

Now $f_i(a) = 0$ since $a \in X$, and by defn $d_a f_i = 0$ on $T_a X$. So

$$\Phi(g) = 0$$

(2°). Now say $g \in M$ is fn s.t. $\Phi(g) = 0$, i.e.

$$d_a g \mid T_a X = 0.$$

Then

$$d_a g = \sum \lambda_i d_a f_i \text{ in } T_a^* \mathbb{A}^n,$$

so

$$d_a(g - \sum \lambda_i f_i) = 0 \in T_a^* \mathbb{A}^n$$

But this means

$$g - \sum \lambda_i f_i \in M^2, \text{ so}$$

$$g \in M^2 + I. \text{ QED.}$$

Cor. $a \in X$ a smooth pt $\Leftrightarrow \dim_k(m/m^2) = \dim X$.

Thm. $\mathrm{Sing}(X) \subsetneq X$ a proper closed subvar.

Sketch (in char$=0$). Evidently $\mathrm{Sing}(X) \subseteq X$ closed. Issue is to show $\exists$ sm pts. Will show they exist on every irred comp. Can suppose X irred, $\dim X = n$. By Thm on prim elt, $\exists$

$$\phi_1, \ldots, \phi_n, \phi_{n+1} \in k(X)$$

st

$$\phi_1, \ldots, \phi_n \text{ a tr basis for } k(X)/k, \text{ and}$$

$$k(X) = k(\phi_1, \ldots, \phi_n)(\phi_{n+1}).$$

$$\phi_i \text{ separable} / k(\phi_1, \ldots, \phi_n).$$

The ϕ_i define a birat map

$$X \overset{\sim}{\dashrightarrow} Y \subseteq \mathbb{A}^{n+1}, \text{ ie.}$$

$$\boxed{X \text{ is birational to a hypsf in } \mathbb{A}^{n+1}}$$

Esp $\exists$

$$X \supseteq U \xrightarrow{\approx} V \subseteq Y.$$

But we've seen that on hypsf, sm pts are dense QED

Local Rings:

Consider

$$X = \text{affine alg set}$$
$$x \in X: \ m_x \subseteq k[X]: \text{ max ideal of } x.$$

Recall:

$$k[X]_m = \left\{ \frac{f}{g} \,\middle|\, f, g \in k[X], \ g \notin m \right\}$$

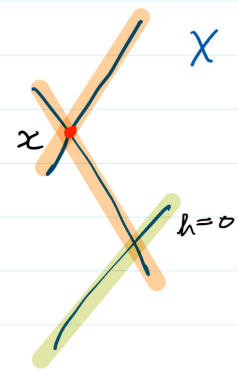
w

$$\frac{f}{g} \sim \frac{f'}{g'} \text{ if } \exists\, h \in k[X] - m \text{ st.}$$

$$h \cdot (gf' - fg') = 0 \qquad (*)$$

NB: (*) means

$(fg' - gf')$ vanishes in Z-nbd of x



Def: $k[X]_m = \mathcal{O}_x X = \mathcal{O}_{X,x}$: local ring of X at x

$\mathcal{O}_x X$ is a local ring, i.e. ring w unique max ideal, viz $m \cdot \mathcal{O}_x X = \{\phi \mid \phi(x) = 0\}$

Alternative viewpoint:

$\mathcal{O}_x X$ = germs of regular fns at x

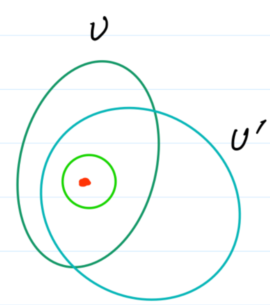
$= \{(U, \phi) \mid U \ni x,\ \phi \in k[U]\} / \sim$

where

$(U, \phi) \sim (U', \phi')$ if $\exists$

$x \in V \subseteq U \cap U'$ s.t.

$\phi|V = \phi'|V$.



(Exercise: prove the equivalence)

Now say

X = arb QP alg set, $x \in X$

Take affine $V \ni x$, put

$\mathcal{O}_x X = \mathcal{O}_x V$

(Check: indep of $V \ni x$)

<u>NB</u>: If $\mathfrak{m} \subseteq \mathcal{O}_x X$ is the max ideal, and if $\tilde{\mathfrak{m}} \subseteq k[X]$ is corresp max ideal of x in X, then

$$\mathfrak{m}/\mathfrak{m}^2 \cong \tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^2 \quad \text{as } \mathcal{O}/\mathfrak{m} = k\text{-modules}$$

<u>So</u>: if X has pure dim $= n$, then:

$$X \text{ non-sing at } x$$
$$\Updownarrow$$
$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = n. \qquad (*)$$

Assume $(*)$ holds. Choose

$$f_1, \dots, f_n \in \mathfrak{m} \subseteq \mathcal{O}_x X$$

s.t.

$$\bar{f}_1, \dots, \bar{f}_n \text{ a basis of } \mathfrak{m}/\mathfrak{m}^2.$$

<u>Nakayama's Lemma</u>: Say $(A, \mathfrak{m})$ a Noeth local ring, and V a f.g. A-module. If

$$v_1, \dots, v_n \in V$$

are elts that generate $V/\mathfrak{m}V$, then the v_i generate V.

So: in our situatuation, the f_i generate $\mathfrak{m}$. Ie.

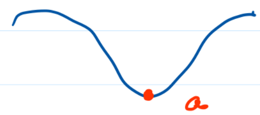
> If X has pure dim n, and $x \in X$ is a point, then
>
> $$X \text{ non-sing at } x$$
> $$\Updownarrow$$
> $$\mathfrak{m} \text{ can be generated by } n = \dim X \text{ elts.}$$

(Comm alg: $\mathcal{O}_x X$ is a "regular local ring.") The f_i are called <u>local params</u> at x. Over $\mathbb{C}$, they give local holo coords.

<u>Exerc:</u> $\exists$ affine nbd $x \in V \subseteq X$ s.t. the f_i are regular on V, and
$$\mathfrak{m}_x = (f_1, \dots, f_n) \subseteq k[V].$$

<u>Ex</u> $X = \{ h(x_1, \dots, x_n) = 0 \} \subseteq \mathbb{A}^n$, say

$a \in X$, $\frac{\partial h}{\partial x_n}(a) \neq 0.$



Then $x_1, \dots, x_{n-1}$ local coords at a.

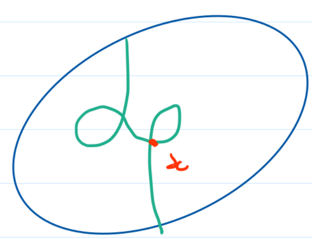
Unique Factorization

<u>Basic Thm in Comm Alg:</u> A regular local ring is a UFD

(This is goal of Ch. 2.2 of Shaf.) So

If $x \in X$ is a non-singular pt, then
$$\mathcal{O}_x X \text{ a UFD.}$$

<u>Cor:</u> Say $x \in X$ a smooth pt,



$x \in Z \subseteq X$: codim 1 subvar.

Then $I_Z \subseteq k[X]$ locally principal at x, ie.
$$I_Z \cdot \mathcal{O}_x X = (g), \text{ some } g \in \mathcal{O}_x X$$

Equivalently,

$\exists$ affine nbd $x \in V \subseteq X$ and $g \in k[V]$ s.t.

$I_{Z \cap V} \subseteq k[V]$ gen by g.

(Exerc: prove this using Krull's Thm)

Blowing Up a Point

View $\mathbb{P}^{n-1}$ as lines thru 0 in $\mathbb{A}^n$. Consider

$$\mathbb{A}^n \times \mathbb{P}^{n-1} \supseteq B = \{(x,[\ell]) \mid x \in \ell\}$$

coords: x, T

$$= \left\{ \operatorname{rk} \begin{bmatrix} x_1 & & x_n \\ T_1 & & T_n \end{bmatrix} \leq 1 \right\}$$

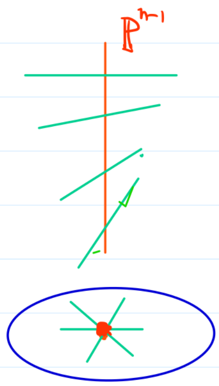
Picture:

$$\begin{array}{ccc} & B & \\ {}^{\mu}\swarrow & & \searrow^{p} \\ \mathbb{A}^n & & \mathbb{P}^{n-1} \end{array}$$

- All fibres of p are copies of $\mathbb{A}^1$

$$\mu^{-1}(x) = \begin{cases} 1 \text{ pt} & \text{if } x \neq 0 \\ \mathbb{P}^{n-1} & \text{if } x = 0 \end{cases}$$

(esp, μ birat)



Canonically: $\mathbb{P}^{n-1} = \mathbb{P}\, T_0 \mathbb{A}^n$

$B = Bl_0(\mathbb{A}^n)$: blowing up of $\mathbb{A}^n$ at 0

Local description-

$\mathbb{A}^n \times \mathbb{P}^{n-1}$ is covered by the affine spaces

$$\mathbb{A}^n \times U_i = \mathbb{A}^n \times \mathbb{A}^{n-1} \quad \text{w } U_i = \{T_i \neq 0\}$$

On eg $\mathbb{A}^n \times U_1$, setting $T_1 = 1$, the defining eqns

$$\text{rk}\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ 1 & t_2 & t_3 & & t_n \end{bmatrix}$$

give

$$x_2 = x_1 t_2, \quad x_3 = x_1 t_3 \quad \cdots \quad x_n = x_1 t_n$$

so

$$B \cap (\mathbb{A}^n \times U_i) \cong \mathbb{A}^n \text{ w coords } x_1, t_2, \ , t_n,$$

and blow-down map

$$B = \mathbb{A}^n \longrightarrow \mathbb{A}^n \text{ is } (x_1, t_2, \ , t_n) \mapsto (x_1, x_1 t_2, \ldots, x_1 t_n).$$

Sim for other affine charts.

Ex. $Bl_0(\mathbb{A}^2) \longrightarrow \mathbb{A}^2$ has two charts:

$$(u, v) \mapsto (u, uv) \quad \& \quad (u, v) \longrightarrow (uv, v).$$

Now consider:

$$X = \text{irred var, dim } n, \text{ say affine}$$
$$\cup$$
$$x: \text{ a non-sing pt of } X.$$

Say

$$f_1, \ldots, f_n \in \mathfrak{m} \text{ local params}$$

- Assume for simplicity:

$$f_i \in k[X], \text{ and that the } f_i \text{ vanish only at } x.$$

(In gen, can arrange for this by replacing X w suitable affine nbd of x.)

<u>Def.</u> Consider:

$$X \times \mathbb{P}^{n-1} \supseteq B = \left\{ (x, T) \;\middle|\; \operatorname{rk}\begin{bmatrix} f_1(x) & \cdots & f_n(x) \\ T_1 & & T_n \end{bmatrix} \leq 1 \right\}$$

Have:

$$\begin{tikzcd} & B = Bl_x(X) \arrow[dl, "\mu"'] \arrow[dr, "q"] & \\ X & & \mathbb{P}^{n-1} = \mathbb{P}T_xX \end{tikzcd}$$

See:

$$\mu^{-1}(p) = \begin{cases} 1 \text{ pt } [f_1(p), \dots, f_n(p)] & \text{if } p \neq x \\ \mathbb{P}^{n-1} & \text{if } p = x. \end{cases}$$

Write

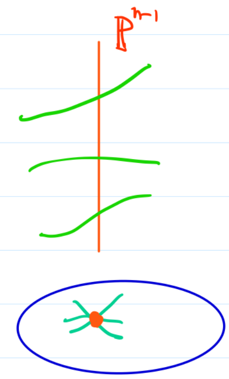
$$\mu^{-1}(x) = E: \text{ the "exceptional divisor"}$$

Can check: under our assumptns, μ restricts to isom

$$(B - E) \xrightarrow{\cong} (X - \{x\})$$

Given $A = [A_1, \dots, A_n] \in \mathbb{P}^{n-1}$, fibre of q is: |

$$q^{-1}(A) = \text{curve } C_A = \left\{ \operatorname{rk}\begin{bmatrix} f_1 & \cdots & f_n \\ A_1 & & A_n \end{bmatrix} \leq 1 \right\}$$



As A varies, the tangent spaces of these curves sweep out $T_x X$.

<u>Fact</u>: Up to isom, $Bl_x(X)$ indep of choice of local coords (Shaf, II.4.2)

<u>Exerc</u>: $Bl_x(X)$ is sm irred var of dim n, birat to X.

<u>Exerc</u>. Can also view $Bl_x(X)$ as closure of graph of rat map

$$\begin{array}{ccc} X & \dashrightarrow & \mathbb{P}^{n-1} \\ \ni & & \ni \\ p & \longmapsto & [f_1(p), \ldots, f_n(p)] \end{array}$$

<u>Variant / Generalization</u> –

- Consider X affine of $\dim X = n$, and any ideal

$$\mathfrak{a} \subseteq k[X].$$

- Choose generators

$$f_1, \ldots, f_r \in \mathfrak{a},$$

so

$$Z = Z(f_1, \ldots, f_r) = Z(\mathfrak{a}) \subsetneq X.$$

- Consider

$$\begin{array}{ccccc} \phi : & U = X - Z(\mathfrak{a}) & \longrightarrow & \mathbb{P}^{r-1} \\ & \ni & & \ni \\ & x & \longmapsto & [f_1(x), \ldots, f_r(x)] \end{array}$$

Define

$$Bl_{\mathfrak{a}}(X) = \text{closure of graph of } \phi \text{ in } X \times \mathbb{P}^{r-1}.$$

Then

$Bl_{\mathfrak{a}}(X)$ is irred var of dim n, and proj

$$\mu : Bl_{\mathfrak{a}}(X) \longrightarrow X$$

is birat morphism that is isom over $X - Z(\mathfrak{a})$

Ex. $\mathfrak{a} = (x^2, y^2) \subseteq k[x,y] = k[\mathbb{A}^2]$

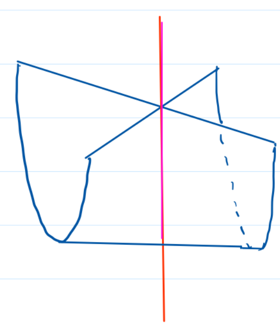
$$\mathbb{A}^2 \times \mathbb{P}^1 \supseteq Bl_{\mathfrak{a}}(\mathbb{A}^2) = \left\{ \operatorname{rk} \begin{bmatrix} x^2 & y^2 \\ S & T \end{bmatrix} \leq 1 \right\}$$

$$= \left\{ x^2 T - y^2 S = 0 \right\}$$

On affine piece $\mathbb{A}^2 \times \mathbb{A}^1_{\{T \neq 0\}}$ this is

$$x^2 - sy^2 = 0$$

i.e. Whitney's umbrella!



"Exerc": Show that this constr agrees w the one in Hartshorne:

$$Bl_{\mathfrak{a}}(X) = \operatorname{Proj}(k[X] \oplus \mathfrak{a} \oplus \mathfrak{a}^2 \oplus \cdots),$$

March 29

Blow-ups and Desingularization.

[Cultural enrichment]

Return to

$$\mu : B = Bl_0(\mathbb{A}^n) \longrightarrow \mathbb{A}^n.$$

Recall that B covered by affine opens $U_\alpha \cong \mathbb{A}^n$ on which μ is given by

$$\mu(z_1, \dots, z_n) = (z_1 z_\alpha, z_2 z_\alpha, \dots, z_\alpha, \dots, z_n z_\alpha)$$

On this chart

$(z_\alpha = 0)$ defines exceptnl divisor.

Now consider subvar:

$$x \in X \subseteq \mathbb{A}^n$$

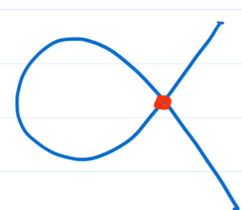
$$\begin{tikzcd} X' \arrow[d] & \subseteq & Bl_x(\mathbb{A}^n) \arrow[d, "\mu"] \\ X & \subseteq & \mathbb{A}^n \end{tikzcd}$$

<u>Def.</u> <u>Proper transform</u> of X in $Bl_x(\mathbb{A}^n)$ is

$$X' = \text{closure of } \mu^{-1}(X - \{x\}) \subseteq Bl_x(\mathbb{A}^n).$$

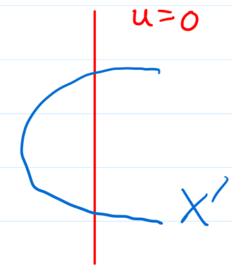
(Rmk: $X' = Bl_x(X)$)

<u>Ex</u> $X = \{y^2 - x^2(x+1) = 0\} \subseteq \mathbb{A}^2$



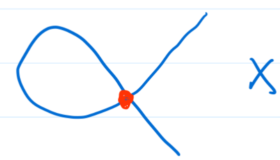
Pull back defining eqn under

$$(u,v) \mapsto (u, uv)$$



$x = u, \quad y = uv$

$$y^2 - x^2(x+1) = u^2v^2 - u^2(u+1)$$
$$= u^2(v^2 - u - 1)$$



$$X' = \{ v^2 - (u+1) = 0 \}$$

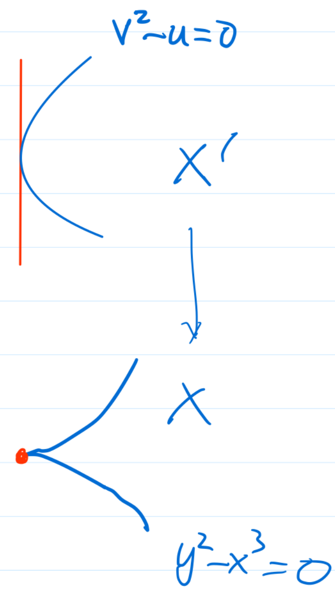
In fact, note that when we pull back eqn defining X, the exceptional divisor $E = (u=0)$ appears w. "multiplicity two." One writes:

$$\text{"}\mu^*(X) = X' + 2E\text{"}.$$

<u>Ex:</u> $\quad X = \{ y^2 - x^3 = 0 \}$ (cusp)

Pull back under $(u,v) \mapsto (u,uv)$

$$u^2v^2 - u^3 = 0, \quad u^2(v^2 - u) = 0$$



Note in these examples, the proper transform of our singular curve becomes smooth, and we end up w

$$\mu : X' \longrightarrow X \quad \text{birational}$$

non-sing

<u>Desingularization Problem</u>: Given irred proj var X, can one find non-sing proj var X', and

$$\mu : X' \longrightarrow X \quad \text{birat morphism.}$$

<u>NB</u>: Don't really care that X proj, but then should require that μ be proper or projective.

$\dim X = 1$: elem (normalization)
$\dim X = 2, 3$ Classical (Zariski)

<u>Hironaka</u> (1964) : in char 0, resolns exist in all dims. Moreover, can be constructed in nice way.

Char $p > 0$: unknown (but recently claimed by Yi Hu)

<u>Early 2000's</u>: De Jong, Abramovich, Bogomolov, Pantev: simpler approach to weak thm

Wlodarcyzk et al: simpler approach to full thm

<u>Embedded Resoln</u>:

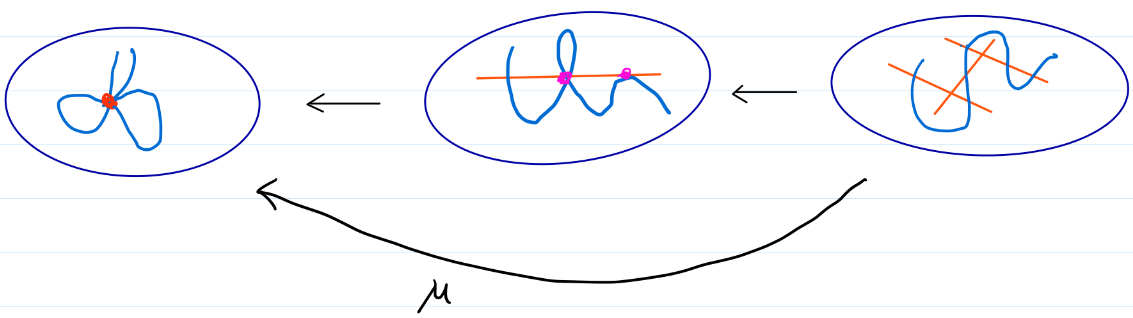
<u>Thm.</u> Consider reduced curve

$$X \subseteq \mathbb{A}^2 \qquad (\text{or } X \subseteq \text{smooth sf}).$$

Then can perform seq of blowings up at points in such a way that proper transf of X is non-sing.

(Even better: $\exists\ \mu : A' \longrightarrow \mathbb{A}^2$ s.t. $\mu^*(X)$ has "normal crossing support.")

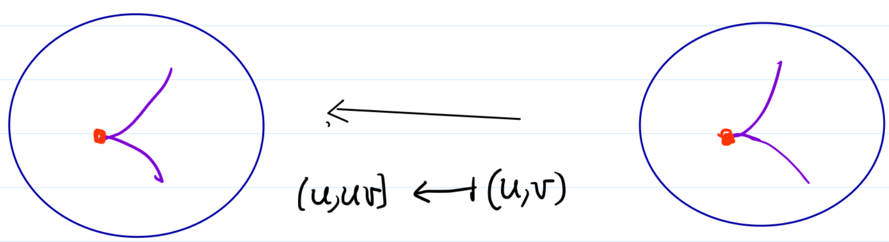
Picture:



<u>Idea</u>: At each stage, proper transform has only finitely many sing pts. Blow up at these.

<u>Problem</u>: to show process terminates, need invariant that "improves" upon blowing up.

<u>Ex</u>: Multiplicity needn't decrease on blowing up:



X: $y^2 - x^5 = 0$ (mult $=2$)

X': $v^2 - u^3 = 0$ (mult $=2$)

$u^2v^2 - u^5 = 0$

$u^2(v^2 - u^3) = 0$

Correct Invariant:

$$\delta = \dim_k\left(k[X']/k[X]\right)$$

Higher dims – Analogous statement true (Hironaka), but situation much more complicated

Ex: Hypsf
$$X = (x^2 + y^4 + z^4 = 0) \subseteq \mathbb{A}^3$$
has isolated sing at 0, but when you blow up $0 \in \mathbb{A}^3$ proper transf sing along a curve!

Line Bundles, Sheaves & Cohomology

Question. How could we define "abstract variety" X, w.o. asking that X be realized as locally closed subset of $\mathbb{P}^N$?

- Given locally closed $X \subseteq \mathbb{P}^N$ we used ratios of homog. polys in order to specify:

  For each open set $U \subseteq X$, the ring $k[U]$ of regular fns on X.

- Knowing $k[U]$ for each open $U \subseteq X$ also let us define morphisms of QPV's.

- If we want to define "abstract" vars X, we will need to specify $k[U]$ for each $U \subseteq X$ as part of defn.

- These will need to satisfy some compatibility conds. This leads to notion of a sheaf.

Tentative plan for next few classes:

1. A little sheaf theory
2. Abstr vars
3. Line bundles
4. More sheaf theory

Presheaves & Sheaves:

X a top space.

Concise Defn. A presheaf of (additive) abelian groups on X is a contravariant functor

$$\mathcal{F} : \begin{pmatrix} \text{category of open} \\ \text{subsets of } X \end{pmatrix} \longrightarrow \begin{pmatrix} \text{additive abelian} \\ \text{groups} \end{pmatrix}$$

In more detail: A preseaf $\mathcal{F}$ is a rule that assigns:

- to each open $U \subseteq X$ a group $\mathcal{F}(U)$
- to each $V \subseteq U$ a homom $\rho^U_V : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ or $\rho_{U,V}$ in such a way that:
  - $\rho^U_U = \mathrm{id}$
  - For $W \subseteq V \subseteq U$

$$\rho^V_W \circ \rho^U_V = \rho^U_W$$

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\rho^U_V"] \arrow[dr, "\rho^U_W"'] & \mathcal{F}(V) \arrow[d, "\rho^V_W"] \\ & \mathcal{F}(W) \end{tikzcd}$$

- Agree that $\mathcal{F}(\phi) = (0)$

Similarly: Presheaf of rings, vect. spaces, etc.

Examples: (1). $\mathcal{C}_X$: presheaf of cont fns on X, ie.

$$\mathcal{C}_X(U) = \{\text{cont fns } f : U \to \mathbb{R}\}$$

ρ = restrictions

(2). X a real mfld, $\mathcal{A}_X^p$ = real p-forms on X.

(3). X a QPV, $\mathcal{O}_X$ = reg fns: $\mathcal{O}_X(U)$ = reg fns on U

(4) $\mathbb{R}_X^{const}$ = const presheaf: $\mathbb{R}_X^{const}(U) = \mathbb{R}$

Vocab: $\mathcal{F}(U)$ = sections of $\mathcal{F}$ on U.

A sheaf is a presheaf whose sections are determined locally:

Def. $\mathcal{F}$ a presheaf on X. $\mathcal{F}$ is a sheaf if following condition holds:

Let $U \subseteq X$ be open, and let

$$U = \bigcup_{\alpha \in I} V_\alpha$$

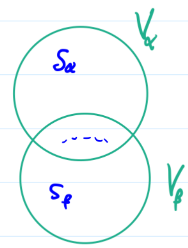
be a covering of U by open subsets $V_\alpha \subseteq U$. Suppose we are given sections

$$s_\alpha \in \mathcal{F}(V_\alpha)$$

that "agree on all the intersections $V_\alpha \cap V_\beta$" in the sense that

$$\rho^{V_\alpha}_{V_\alpha \cap V_\beta}(s_\alpha) = \rho^{V_\beta}_{V_\alpha \cap V_\beta}(s_\beta)$$

$\forall\, \alpha, \beta.$



Then $\exists!\ s \in \mathcal{F}(U)$ s.t.

$$\rho^{U}_{V_\alpha}(s) = s_\alpha \quad \forall \alpha.$$

Exs. (a). Exs (1)-(3) above are sheaves

(b). $\mathbb{R}^{\text{const}}_X$ is not a sheaf: eg take $X = \mathbb{R} - \{0\}$: $U_+ = \mathbb{R}^{>0}$, $U_- = \mathbb{R}^{<0}$, and

$$s_+ = 1 \in \mathbb{R}^{\text{const}}(U_+), \quad s_- = -1 \in \mathbb{R}^{\text{const}}(U_-)$$

(c) What is a sheaf is $\mathbb{R}_X =$ (locally) const sheaf ie

$$\mathbb{R}_X(U) = \left\{ \begin{array}{c} \text{locally const fns} \\ U \to \mathbb{R} \end{array} \right\}$$

(This is usually called const sheaf)

Def. $\mathcal{F}$ = sheaf or presheaf on X, $V \subseteq X$ open. Then define $\mathcal{F}|V$ to be sheaf or presheaf given by:

$$(\mathcal{F}|V)(U) = \mathcal{F}(V \cap U).$$

Ringed spaces (see Gathmann, Ch 4)

Goal: introduce formalism for keeping track of "nice" fns on space X.

<u>Def:</u> A <u>ringed space</u> is a top space X equipped with a sheaf of rings $\mathcal{O}_X$ on X (called the "structure sheaf" of X):

Notation: $(X, \mathcal{O}_X)$.

<u>Convention:</u> We will always assume that $\mathcal{O}_X$ is a sheaf of k-valued fns for some field k, so

$$\mathcal{O}_X(U) \ni g \Rightarrow g : U \longrightarrow k.$$

(Not the case for arb schemes)

(so will have natural notion of pull-back.)

<u>Ex</u> (1) $X = C^\infty$ mfld, $\mathcal{O}_X = \{\text{sm fns on } X\}$

(2) $X = \text{QPV}$: $\mathcal{O}_X =$ sheaf of reg fns on X.

<u>Def.</u> $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ ringed spaces w/

$$\mathcal{O}_X, \mathcal{O}_Y \text{ sheaves of } k\text{-valued fns.}$$

<u>Morphism</u> $f : X \longrightarrow Y$ of ringed spaces is given by cont map from X, Y w. the property:

$\forall\, U \subseteq Y$, have

$$f^* : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}(U))$$
$$\varphi \longmapsto \varphi \circ f$$

(ie fns in str sheaf of Y pull back to fns in str sheaf of X.)

Exerc: Convince yourself that this recovers earlier defs of morphism in std examples, eg

smooth mflds, QPV's

Then have notion of isom.

Def. Fix an alg closed field k. A prevariety (over k) is a ringed space $(X, \mathcal{O}_X)$ s.t. X has a finite open cover

$$X = \bigcup_i V_i$$

s.t.

$$(V_i, \mathcal{O}_{V_i} =_{\text{def}} \mathcal{O}_X | V_i)$$

is isom to affine alg set.

Ex. Any QPV.

Main Question (for us): Why bother?

Answer: can glue!

- There are many situations where you want to construct new varieties by gluing processes.

- In practice one doesn't usually leave the world of quasi-projective varieties, but it would be painful (and not terribly instructive) to have to produce quasi-proj embeddings.

- So useful to be able to bypass those questions.

Gluing (see Gathmann, Ch 5, for details)

- Let X_1, X_2 be two prevars, suppose given open sets

$$X_1 \supseteq U_{12} \qquad\qquad U_{2,1} \subseteq X_2$$

and isom

$$U_{12} \xrightarrow[\cong]{f} U_{21}$$

Define gluing of X_1, X_2 along f as follows:

- As sets: $$X = (X_1 \amalg X_2)/(a \sim f(a))$$

so have

$$X_1 \overset{i_1}{\subseteq} X \overset{i_2}{\supseteq} X_2$$

- $U \subseteq X$ open $\Leftrightarrow$ $i_1^{-1}(U) \subseteq X_1$, $i_2^{-1}(U) \subseteq X_2$ open.

- Structure sheaf $\mathcal{O}_X$:

$$\mathcal{O}_X(U) = \{\varphi : U \to k \mid i_1^*\varphi \in \mathcal{O}_{X_1}(i_1^{-1}U),\ i_2^*\varphi \in \mathcal{O}_{X_2}(i_2^{-1}U)\}.$$

Check: The resulting $(X, \mathcal{O}_X)$ is a pre-variety.

Rmk: Similarly can glue 3 or more prevars: need compatibility on triple overlaps. See Gathmann, Construction 5.6.

Ex. $X_1 = X_2 = \mathbb{A}^1$, $U_1 = U_2 = \mathbb{A}^1 - \{0\}$

Glue via $\underset{z}{U_1} \longrightarrow \underset{w}{U_2}$: $w = f(z) = \frac{1}{z}$:

Resulting var is $X = \mathbb{P}^1$

Ex. $X_1 = X_2 = \mathbb{A}^1, \quad U_1 = U_2 = \mathbb{A}^1 - \{0\}$

· But now glue by identity:

$$\mathbb{A}^1 \supseteq U_1 \xrightarrow{f} U_2 \subseteq \mathbb{A}^1, \quad f(t) = t.$$

· Resulting var is "$\mathbb{A}^1$ w origin doubled!"

($/\mathbb{C}$: get a non-Hausdorff space locally isom to $\mathbb{C}$.)

Will want to exclude this sort of thing: will involve cond on the product of X with itself.

Thm: Given two prevars X, Y can construct product

$$P = X \times Y,$$

together w projections

$$\pi_X : P \to X, \quad \pi_Y : P \to Y$$

P is characterized by the property that given any prevar T, and morphs

$$f_X : T \to X, \quad f_Y : T \to Y$$

$$\begin{tikzcd} T \arrow[dr, dashed, "f"] \arrow[drr, bend left, "f_Y"] \arrow[ddr, bend right, "f_X"'] & & \\ & P \arrow[r, "\pi_Y"'] \arrow[d, "\pi_X"] & Y \\ & X & \end{tikzcd}$$

$\exists !$

$$f : T \to P$$

as shown.

(So taking $T = \{pt\}$, see $P = X \times Y$ as sets.)

"Pf." Take affine coverings

$$\{U_i\} \text{ of } X, \quad \text{and } \{V_j\} \text{ of } Y,$$

and glue the products $U_i \times V_j$. (See Gath, Prop 5.15).

Separatedness—

Exerc: A top space X is Hausdorff iff diagonal

$$\Delta \subseteq X \times Y$$

is a closed subset (in the product topology).

Defs. A prevariety is separated if the diagonal

$$\Delta_X \subseteq X \times X$$

is a closed set. (NB: as usual, $X \times X$ doesn't have prod. topology)
A variety is a separated prevariety

Exs. (1). Any QPV is separated.

(2) $\mathbb{A}^1$ w origin doubled is not separated.

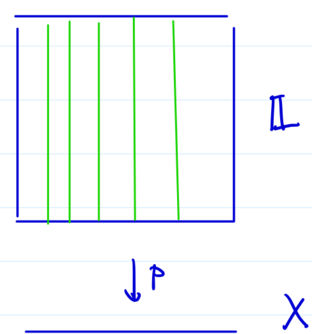
Exerc. Let X be a prevar. If X is separated the intersection of two affine open subsets of X is affine. Statement can fail if X not separated.

Line Bundles and Divisors

- Sections of line bundles play same role in alg geom that smooth fns do in real mfld theory.

- We'll start w. a geometric defn. We'll arrive at a more useful reformulation later.

Let X be a variety (or prevar) $/k=\bar{k}$



Intuitively: lb

$$p: \mathbb{L} \to X$$

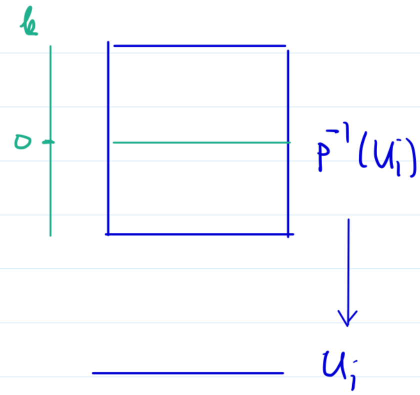
is "locally trivial fam of 1-dim vector spaces $/k$" parametrized by X.

More formally:

Def: A line bundle on X is a var (or prevar) $\mathbb{L}$, together w a morphism

$$p: \mathbb{L} \longrightarrow X$$

st the following hold:



(1). $\exists$ open covering $\{U_i\}$ of X for which one has isoms

$$\begin{tikzcd} p^{-1}(U_i) \arrow[rr, "\varphi_i", "\cong"'] \arrow[dr, "p"'] & & U_i \times \mathbb{A}^1 = U_i \times k \arrow[dl, "pr_1"] \\ & U_i & \end{tikzcd}$$

commuting w projections to U_i.

(So $\mathbb{L}$ is locally a product of X w $\mathbb{A}^1$, but we haven't yet built in condition to guarantee that the fibres have str of vector spaces.)

Set

$$U_{ij} = U_i \cap U_j$$

(<u>Plan</u>: have two identifications of $p^{-1}(U_i \cap U_j)$ w $U_{ij} \times k$, one coming from φ_i and one coming from φ_j. This gives rise to comparison maps which we require to be linear on each fibre.)

Define

$$\varphi_{ij} : U_{ij} \times k \xrightarrow{\cong} U_{ij} \times k$$

to be

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} \quad \text{(as indicated)}$$

$$\begin{tikzcd}
U_j \times k & \supseteq & U_{ij} \times k \arrow[rr, "\varphi_{ij}", "\cong"'] & & U_{ij} \times k & \subseteq & U_i \times k \\
& p^{-1}(U_j) \arrow[ul, "\varphi_j"', "\cong"] & \supseteq & p^{-1}(U_{ij}) \arrow[ul, "\varphi_j"', "\cong"] \arrow[ur, "\varphi_i", "\cong"'] \arrow[d] & \subseteq & p^{-1}(U_i) \arrow[ur, "\varphi_i", "\cong"'] \arrow[d] & \\
& U_j \arrow[from=u] & \supseteq & U_{ij} & \subseteq & U_i &
\end{tikzcd}$$

Require that

$$\varphi_{ij}(x, v) = (x, g_{ij}(x) \cdot v)$$

where

$$g_{ij} : (U_i \cap U_j) \longrightarrow k^* = GL(1, k)$$

is nowhere vanishing reg fn on $U_i \cap U_j$. (The g_{ij} are <u>transition functions</u>.)

Ie. on each fibre φ_{ij} is given by isom of 1 dim vector spaces varying regularly w x.

<u>Rmks</u>. (1). My indexing convention is that g_{ij} = "to i from j"

(2). Analogous defn in other geom settings: define real or cx lbs on top space, sm mfld, etc.

<u>Ex</u>.
$$\begin{tikzcd} \mathbb{L} = X \times \mathbb{A}^1 \arrow[dr, "p"'] & & \arrow[dl, "pr_1"] \\ & X & \end{tikzcd}$$

: This is trivial lb: written

$$\mathbb{1}_X \quad \text{or} \quad \mathcal{O}_X$$

<u>Ex</u> Take

$$X = \mathbb{P}^1 = \left\{ \begin{matrix} \text{1-dim vector subspaces} \\ \Lambda \subseteq k^2 \end{matrix} \right\}$$

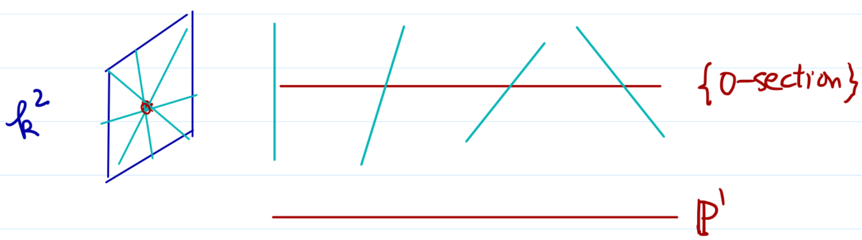
We consider (once again) the natural incidence corresp

$$\begin{tikzcd} \mathbb{P}^1 \times k^2 \supseteq \mathbb{L} \arrow[d, swap, "\mu"] \arrow[dr, "p"] & = \{ ([\Lambda], v) \mid v \in \Lambda \} \\ k^2 & \mathbb{P}^1 \end{tikzcd}$$

(So μ = blowing up of origin in $\mathbb{A}^2 = k^2$). So

$$p^{-1}([\Lambda]) = \Lambda \subseteq k^2,$$

ie every fibre of p is copy of $k = \mathbb{A}^1 \subseteq \mathbb{A}^2 = k^2$.



Let's show $\mathbb{L}$ a lb, and find its trans. fns. Work w std open cover of $\mathbb{P}^1$:

$$U_0 = \{ [1, \tfrac{X_1}{X_0}] \} \qquad U_1 = \{ [\tfrac{X_0}{X_1}, 1] \}$$

$$p^{-1}(U_0) = \{ ([1, \tfrac{X_1}{X_0}], t \cdot (1, \tfrac{X_1}{X_0})) \} \qquad p^{-1}(U_1) = \{ [\tfrac{X_0}{X_1}, 1], s(\tfrac{X_0}{X_1}, 1) \}$$

$$\cong \downarrow \varphi_0 \qquad\qquad \cong \downarrow \varphi_1$$

$$U_0 \times \mathbb{A}^1 \qquad\qquad U_1 \times \mathbb{A}^1$$

$$\varphi_0([X_0, X_1], (t, t(\tfrac{X_1}{X_0}))) = ([X_0, X_1], t) \qquad \varphi_1([X_0, X_1], (s\tfrac{X_0}{X_1}, s)) = ([X_0, X_1], s)$$

$\varphi_1 \circ \varphi_0^{-1}$

$$([X_0, X_1], t) \to ([X_0, X_1], (t, t(\tfrac{X_1}{X_0}))) = ((t \cdot \tfrac{X_1}{X_0}) \cdot \tfrac{X_0}{X_1}, (t \tfrac{X_1}{X_0})) \longrightarrow ([X_0, X_1], t \cdot (\tfrac{X_1}{X_0}))$$

I.e. $\boxed{g_{10}([X_0, X_1]) = (\tfrac{X_1}{X_0}).}$

Similarly:

$$\begin{tikzcd} \mathbb{P}^n \times k^{n+1} \arrow[dr] & \supseteq & \mathbb{L} = \{([\Lambda], v) \mid v \in \Lambda\} \arrow[dl, "p"] \\ & \mathbb{P}^n & \end{tikzcd}$$

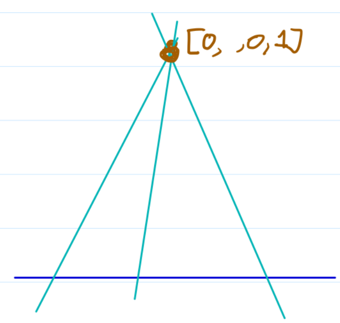
Transition fns (wrt usual covering)

$$g_{ij} = \left(\frac{x_i}{x_j}\right)$$

Vocab: This is called Hopf lb or $\mathcal{O}_{\mathbb{P}^n}(-1)$.

Ex. Take

$$\mathbb{L} = \mathbb{P}^{n+1} - \{[0,\dots,0,1]\},$$



and define $p : \mathbb{L} \longrightarrow \mathbb{P}^n$ to be projection from $[0, ,0,1]$:

$$p([x_0,\dots,x_{n+1}]) = [x_0, , x_n]$$

Then

$$p^{-1}(pt) = \mathbb{P}^1 - [0, ,0,1]$$
$$= \mathbb{A}^1.$$

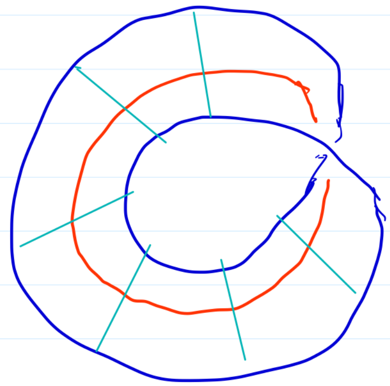
Check: this is lb, w trans fns $\quad g_{ij} = \dfrac{x_j}{x_i}$

Vocab: This is hyperplane bundle, or $\mathcal{O}_{\mathbb{P}^n}(1)$.

Ex. Let

$$\mathbb{L} = \text{Mobius band}$$
$$p : \mathbb{L} \longrightarrow S^1 \quad \text{proj onto central circle}$$

You can think of fibres as copies of $\mathbb{R}$, and can realize $\mathbb{L}$ as total space of lb on S^1.



Trans fns:

$$U_0 = S^1 - \{\text{nbd of } \infty\}$$
$$U_1 = S^1 - \{\text{nbd of } 0\}$$

$U_0 \cap U_1$ has 2 conn comps: L & R. Then can take

$$g_{RL} = \begin{cases} 1 & \text{on } L \\ -1 & \text{on } R \end{cases}$$

<u>Variant</u>: Vect Bundle of rk r on X.

Given by
$$p : E \longrightarrow X,$$
w.
$$p^{-1}(U_i) \underset{\varphi_i}{\overset{\simeq}{\longrightarrow}} U_i \times k^r.$$

Now transitions given by
$$g_{ij} : U_i \cap U_j \longrightarrow GL_r(k),$$
ie.
$$\varphi_i \circ \varphi_j^{-1}(x, v) = (x, g_{ij}(x) \cdot v),$$

(Exerc: work this out!)

<u>Ex.</u> X sm mfld of dim n. Tang bundle TX is vect. bundle of rk n. Exerc: what are transition matrices?

Linear Algebra of lbs and vbs.

Say $p: \mathbb{L} \longrightarrow X$ a lb, or $q: \mathbb{E} \to X$ a v.b. of rk r

$p^{-1}(x)$ carries canon (up to isom) str of 1-dim vs
$q^{-1}(x)$ " " " " " r-dim "

Ie. use local trivializn

$$p^{-1}(U_i) \longrightarrow U_i \times k, \quad q^{-1}(U_i) \longrightarrow U_i \times k^r$$

to realize fibres as v.s.; and note that we've required two such to differ by linear automorphism. Esp:

Metathm: Any naturally defined (indep of basis) operation on vector spaces carries over to vector bundles & lbs.

Ex. Let L, M be 1-dim v.s. / k. Then

$$\mathrm{Hom}(L, M) = 1\text{-dim v.s.} / k,$$

So if $\mathbb{L} \longrightarrow X$, $\mathbb{M} \longrightarrow X$ are lbs, expect

$$\mathbb{H}\mathrm{om}(\mathbb{L}, \mathbb{M}) \longrightarrow X$$

to be another lb. Note that if

$$L \xrightarrow{\cdot g} L, \quad M \xrightarrow{\cdot h} M.$$

are isoms given by mult by $g, h \in k^*$, then we get induced isom

$$\mathrm{Hom}(L, M) \xrightarrow{\cdot gh^{-1}} \mathrm{Hom}(L, M)$$

given by mult by gh^{-1}. "So" if $\mathbb{L}$, $\mathbb{M}$ are described by trans fns

g_{ij}, h_{ij} wrt given cover $\{U_i\}$, then expect $\mathbb{H}om(\mathbb{L},\mathbb{M})$ given by $g_{ij}^{-}\cdot h_{ij}^{-1}$. (Check!)

<u>Ex.</u> If L is 1-dim v.s, then

$$Hom(L,L) = k \ : \ 1\text{-dim v.s. w. canon basis}$$

"So"

$$\mathbb{H}om(\mathbb{L},\mathbb{L}) = \mathbb{1}_X.$$

<u>Ex.</u> If $\mathbb{L} \to X$ is lb, and $d \in \mathbb{Z}$, then

$$\mathbb{L}^{\otimes d} \text{ is also a lb on } X:$$

if $\mathbb{L}$ has trans fns g_{ij}, then $\mathbb{L}^{\otimes d}$ has trans fns g_{ij}^d.

<u>Line & Vector Bundles via Trans Fns:</u>

Let $\mathbb{L} \to X$ be a lb on X, and write

$$g_{ij} : U_i \cap U_j \longrightarrow k^* \qquad (*)$$

be the trans fns wrt trivializing open cover $\{U_i\}$. Then

$$g_{ji} = g_{ij}^{-1}. \qquad (**)$$

<u>Also:</u> On triple open covers $U_{ijk} = U_i \cap U_j \cap U_k$, the g_{ij} satisfy

$$g_{ij} \cdot g_{jk} = g_{ik} \qquad (***).$$

(Check!). Conversely:

<u>Prop:</u> Let $\{U_i\}$ be an open cover of X, and suppose given reg fns

$$g_{ij} : U_i \cap U_j \longrightarrow k^*$$

satisfying (**), (***). Then $\exists$ lb $p: \mathbb{L} \to X$ w. trivializn over the U_i giving rise to trans fns g_{ij}. Sim for vbs.

<u>Sketch:</u> Start w. the varieties $U_i \times \mathbb{A}^1$. Then glue via

$$U_j \times \mathbb{A}^1 \supseteq U_{ij} \times \mathbb{A}^1 \xrightarrow{\cong} U_{ij} \times \mathbb{A}^1 \subseteq U_i \times \mathbb{A}^1$$
$$(x,t) \longmapsto (x, g_{ij}(x)\cdot t)$$

<u>Homoms of vbs:</u>

Consider vbs

$$p: \mathbb{E} \to X \quad , \quad q: \mathbb{F} \to X$$

of rks e, f on X. <u>Homom</u> is reg. map

$$\begin{tikzcd} \mathbb{E} \arrow[rr, "u"] \arrow[dr, "p"'] & & \mathbb{F} \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

that is linear on each fibre.

More concretely: say $\{U_i\}$ open cover on which both $\mathbb{E}$ and $\mathbb{F}$ trivialize, say w trans matrices g_{ij}, h_{ij}. Consider

$$\begin{tikzcd} p^{-1}(U_j) \arrow[r, "a"] \arrow[d, "\varphi_j", "\cong"'] & q^{-1}(U_j) \arrow[d, "\psi_j"', "\cong"] \\ U_j \times k^e \arrow[r, "\psi_j \circ a \circ \varphi_j^{-1}"] & U_j \times k^f \end{tikzcd}$$

Then a given locally on U_j by

$$(x,v) \longmapsto (x, a_j(x)\cdot v)$$

where

$$a_j : U_j \longrightarrow \mathrm{Hom}(k^e, k^f) = M_{f\times e}$$

On $U_{ij} = U_i \cap U_j$ we have compatibility cond

$$\begin{tikzcd} U_{ij} \times k^e \arrow[r, "a_j"] \arrow[d, "\cdot g_{ij} = \varphi_i \circ \varphi_j^{-1}"'] & U_{ij} \times k^f \arrow[d, "\psi_i \circ \psi_j^{-1} = \cdot h_{ij}"] \\ U_{ij} \times k^e \arrow[r, "a_i"] & U_{ij} \times k^f \end{tikzcd}$$

$$\boxed{a_i \cdot g_{ij} = h_{ij} \cdot a_j}$$

ie

$$\boxed{\begin{aligned} a_i &= h_{ij}\, a_j\, (g_{ij})^{-1} \\ &= h_{ij}\, a_j \cdot g_{ji} \end{aligned}}$$

<u>Special case</u>: Suppose $e = f$, &

$(*)$ $\quad a : \mathbb{E} \longrightarrow \mathbb{F}$ is isom

Then a_i invertible and

$$(*) \iff h_{ij} = a_i\, g_{ij}\, a_j^{-1}$$

Especially:

If $\mathbb{L} \to X$ is given by data $\{U_i, g_{ij}\}$, then

$$\mathbb{L} \cong 1_X \iff g_{ij} = a_i^{-1} \cdot a_j \text{ for nowhere van fns } a_i : U_i \to k^*$$

$$\left(\underset{b_i = a_i^{-1}}{\iff} \quad g_{ij} = b_i \cdot b_j^{-1} \text{ for nowhere van. } b_i \right)$$

So, roughly:

$$\left(\begin{array}{c}\text{isom classes of lbs}\\ \text{trivializing on } \mathfrak{U}=\{U_i\}\end{array}\right)$$

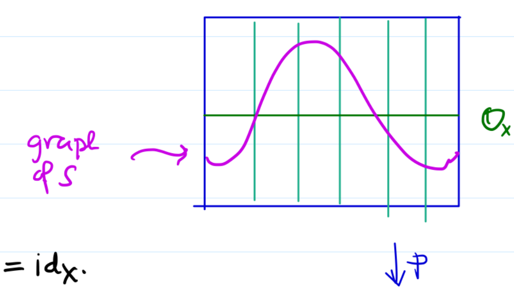
$$\|$$

$$\frac{\left\{ g_{ij}\in \mathcal{O}_X^*(U_i\cap U_j) \;\middle|\; g_{ij}\cdot g_{jk}=g_{ij} \right\}}{\left\{ g_{ij} \;\middle|\; g_{ij}=b_i\cdot b_j^*,\ b_i\in \mathcal{O}_X^*(U_i)\right\}}$$

We will later see this as $H^1(\mathfrak{U},\mathcal{O}_X^*)$. Since not all lbs trivialize on one open cover, to describe all lbs would need to discuss passing to refinements.

Global sections

Consider vb $p: \mathbb{E}\longrightarrow X$.



<u>Def.</u> Global section of $\mathbb{E}$ is

$$s: X\longrightarrow \mathbb{E} \quad \text{s.t.} \quad p\circ s=id_X.$$

Ie., for each $x\in X$, $s(x)\in \mathbb{E}(x)=p^{-1}(x)$.

<u>Local description:</u> Say $\mathbb{E}\leftrightarrow\{U_i, g_{ij}\}$

See that

$$(\varphi_i\circ s)(x)=(x, s_i(x)),$$

$$\begin{array}{ccc} & & U_i\times k^e \\ & & \cong\uparrow \varphi_i \\ U_i & \xrightarrow[s]{} & p^{-1}(U_i)\end{array}$$

where

$$s_i: U_i\longrightarrow k^e \qquad \text{ie. } s_i \text{ a vector of reg fns on } U_i.$$

<u>Check:</u> compatibility cond is

$$\boxed{s_i = g_{ij} \cdot s_j}$$

i.e.

$$\Gamma(X, \mathbb{E}) \underset{\text{def}}{=} \{\text{glob sections of } \mathbb{E}\}$$
$$\cong \{ (U_i, s_i) \mid g_{ij} s_j = s_i \}$$

<u>Ex.</u> $X = \mathbb{P}^1$, $\mathbb{L} = (\text{hyperplane lb}) = \mathcal{O}_{\mathbb{P}^1}(1)$.

Transfn: $g_{10} = \dfrac{X_0}{X_1}$.

$$\Gamma(U_0, \mathbb{L}) \ni s_0 = a_0 + a_1\left(\frac{X_1}{X_0}\right) + a_2\left(\frac{X_1}{X_0}\right)^2 + \cdots$$

$$\Gamma(U_1, \mathbb{L}) \ni s_1 = b_0 + b_1\left(\frac{X_0}{X_1}\right) + b_2\left(\frac{X_0}{X_1}\right)^2 + \cdots$$

Cond:

$$s_1 = \left(\frac{X_0}{X_1}\right) s_0$$

$$b_0 + b_1\left(\frac{X_0}{X_1}\right) + b_2\left(\frac{X_0}{X_1}\right)^2 + \cdots = \left(\frac{X_0}{X_1}\right)\left(a_0 + a_1\left(\frac{X_1}{X_0}\right) + a_2\left(\frac{X_1}{X_0}\right)^2 + \cdots\right)$$

$$b_1 = a_0, \quad b_0 = a_1 \qquad b_2 = \cdots = 0, \quad a_2 = \cdots = 0$$

So

$$\Gamma(\mathbb{P}^1, \mathbb{L}) \cong \{ c_0 X_0 + c_1 X_1 \} = \{\text{linear forms}\}!$$

<u>Similarly:</u> If $\mathbb{L}$ is hyperplane lb on $\mathbb{P}^n$ $\left(g_{ij} = \dfrac{X_j}{X_i}\right)$, then

$$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \cong \{\text{linear forms}\}$$

<u>Exerc:</u> $\mathbb{L} = \mathcal{O}_{\mathbb{P}^n}(d) = (\text{hyperplane lb})^{\otimes d}$: $g_{ij} = \left(\frac{x_j}{x_i}\right)^d$

Then $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \cong \left\{ \begin{matrix}\text{homog polys deg } d \text{ in} \\ x_0, \ldots, x_n.\end{matrix} \right\}$

On the other hand:

<u>Ex</u>: $\mathbb{L} =$ Hopf lb on $\mathbb{P}^1 = \mathcal{O}_{\mathbb{P}^1}(-1)$: $g_{10} = \frac{X_1}{X_0}$

$$s_0 = a_0 + a_1 \frac{X_1}{X_0} + \cdots$$

$$s_1 = b_0 + b_1 \frac{X_0}{X_1} + \cdots$$

$s_1 = \frac{X_1}{X_0} s_0$: only soln $= s_0 = s_1 = 0$

i.e. $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$.

Sim: $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d)) = 0 \quad (d > 0)$
$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-d)) = 0 \quad$ "

<u>Rmk</u>: If $s \in \Gamma(X, \mathbb{L})$, then Zeros$(s) \subseteq X$ well-defined.

<u>Variant</u>: when X irred, can define rational section of lb.

<u>Pull-backs of lbs</u>. Let $\mathbb{L} \xrightarrow{p} X$ be lb, and let

$$f : Y \longrightarrow X$$

be morphism. Then get pull-back lb

$$f^* \mathbb{L} \longrightarrow Y :$$

If $\mathbb{L}$ trivializes over $\{U_i\}$ (cover of X), then $f^*\mathbb{L}$ trivializes over $V_i = f^{-1}(U_i)$ via pulling back

$$\varphi_i : p^{-1}(U_i) \longrightarrow U_i \times k :$$

If $\{g_{ij}\}$ are trans fns for $\mathbb{L}$, then $f^* g_{ij} = g_{ij} \circ f$ are trans fns for $f^*\mathbb{L}$.

Linear Series and Maps to Projective Space:

Let $\mathbb{L}$ be a lb on var X, and consider $\Gamma(X,\mathbb{L}) :=$ all sections of $\mathbb{L}$ over X. This is a v.s. / k. For orientation, we'll later prove:

> **Thm:** If X is projective, then
> $$\dim_k \Gamma(X,\mathbb{L}) < \infty.$$

However for now we'll use this only in passing.

Let
$$V \subseteq \Gamma(X,\mathbb{L})$$
be a finite dimensional subspace, say $\dim V = r+1$. Such a V is called a <u>linear series</u>.

<u>Def.</u> V is <u>basepoint-free</u> ("bpf") if for each $x \in X$, $\exists\ s \in V$ s.t
$$s(x) \neq 0.$$

(Ie. sections in V don't simultaneously vanish at any pt.)

<u>Ex</u> $X = \mathbb{P}^1$, $\mathbb{L} = \mathcal{O}_{\mathbb{P}^1}(3) = (\text{hyperplane lb})^{\otimes 3}$. Recall
$$\Gamma(\mathbb{P}^1,\mathbb{L}) = \text{all cubic forms.}$$

- $V = \langle x^3, y^3 \rangle$: base-pt free
- $V = \langle x^3, xy^2 \rangle$: not bpf both vanish at $[0,1]$

Rmk: Can happen that $\Gamma(X,\mathbb{L}) \neq 0$ but $V = \Gamma(X,\mathbb{L})$ not bpf. But this doesn't occur when $X = \mathbb{P}^n$.

<u>Prop / Def</u>: Let $V \subseteq \Gamma(X, \mathbb{L})$ be a bpf linear series, with $\dim V = r+1$. Choose a basis

$$s_0, \ldots, s_r \in V.$$

Then V gives rise to a morphism:

$$\varphi = \varphi_V : X \longrightarrow \mathbb{P}^r$$

by the rule

$$\varphi(x) = [s_0(x), \ldots, s_r(x)] \in \mathbb{P}^r \qquad (*)$$

<u>Explanation</u>: Sections of lbs don't have well-defined values, so $s_\alpha(x)$ not an elt of k. Meaning of $(*)$ is as follows.

- Choose open cover $\mathcal{U} = \{U_i\}$ on which $\mathbb{L}$ trivializes.
- So on U_j, s_α locally given by reg fn $f_{j,\alpha} \in \mathcal{O}_X(U_j)$.
- Then for $x \in U_j$, $\varphi(x) = [f_{j,0}(x), \ \ , f_{j,r}(x)]$.
- On $U_i \cap U_j$,

  $$f_{i,\alpha} = g_{ij} \cdot f_{j\alpha},$$

  indep of α

  so

  $$[f_{i,0}(x), \ldots, f_{i,r}(x)] = [f_{j,0}(x), \ldots, f_{j,r}(x)]$$

  as pts of $\mathbb{P}^r$.
- See that φ is morphism since locally given by vector of regular functions.

Addendum: In the situation of the Prop/Def

$$\mathbb{L} \cong \varphi^* \mathcal{O}_{\mathbb{P}^r}(1).$$

Pf: Exerc.

Divisors:

- Let $\mathbb{L}$ be a lb on X (irred n-dim var), $0 \neq s \in \Gamma(X, \mathbb{L})$ a non-zero section.
- Then locally defined by one equation, so

$$Z = (s = 0) \subseteq X$$

has pure codim $= 1$.

- But s may vanish to order > 1 along some subvar. Eg

  If $X = \mathbb{A}^2$, then

  $s = y \in \Gamma(\mathbb{A}^2, \mathcal{O})$ vanishes simply along x-axis,

  but

  $s' = y^2$ vanishes to order 2.

- Want some device to keep track of such multiplicities.

Def. Let X = irred var of dimension n. A <u>(Weil) divisor</u> on X is a finite formal $\mathbb{Z}$-linear combination of irred codim 1 subsets. So

$$\mathrm{Div}(X) \ni D = \sum n_i E_i, \quad E_i \subsetneq X \text{ irred codim } 1$$

Equivalently, can write $D = \sum n_E \cdot E$, sum over all codim 1 subvars, all but finitely many n_E are 0. D is <u>effective</u> iff all $n_E \geq 0$

These form a group (the free abel grp on the irred codim 1 subvars).

Given $s \in \Gamma(X, \mathbb{L})$, want to define

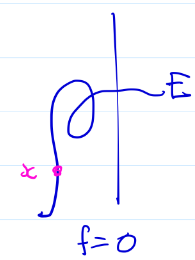
$$\mathrm{div}(s) = \left(\begin{matrix} \text{divisor of zeros} \\ \text{of } s \end{matrix} \right) \in \mathrm{Div}(X).$$

- Issue is to attach multiplicities to irred comps of $s=0$,
- If X has singularities in codim 1, this is rather subtle.
- The natural setting in which to do this is when X is <u>normal</u> (so codim 1 local rings are UFD's).
- However I will assume for simplicity that X is <u>non-singular</u>, where we can simultaneously make all the constructions we'll discuss.

So: assume V <u>non-singular</u>, let

$$0 \neq f \text{ be a regular fn on } V.$$

Say $E \subsetneq V$ be an irred comp of $(f=0)$.



Choose a point $x \in E$ that doesn't lie on any irred comp of $(f=0)$, and consider

$$\mathcal{O}_x X = \text{local ring},$$
$$\Psi$$
$$f : (\text{image of) } f \text{ in local ring}$$

$\mathcal{O}_x X$ is UFD, and $(f=0)$ is irred in nbd of x, so

$$f = (\text{unit}) \cdot \pi^e \quad \text{some irred } \pi \in \mathcal{O}_x X.$$

Define

$$e = \mathrm{ord}_E(f) > 0.$$

<u>Check</u>: indep of choice of x.

Now consider

$$0 \neq s \in \Gamma(X, \mathbb{L}),$$

and let $E \subseteq X$ be irred comp of $(s=0)$. Then s locally given by a regular fn, and we can use previous discussion to define

$$e = \mathrm{ord}_E(f).$$

<u>Check</u>: indep of local trivialization. (Reason: $g_{ij} \in \mathcal{O}_X^* X$)

<u>Upshot</u>: When X non-sing (or normal), and $0 \neq s \in \Gamma(X, \mathbb{L})$, we have defined

$$\mathrm{div}(s) = \sum n_E E, \quad n_E \geq 0. \qquad (*)$$

<u>Ex</u>: $\mathbb{L} = \mathcal{O}_{\mathbb{P}^n}(d)$, so s is homog poly of deg d. Then $\mathrm{div}(s)$ just reflects prime factorizn of s:

$$s = \prod f_i^{e_i}, \quad f_i \text{ irred homog},$$

$$\mathrm{div}(s) = \sum e_i \cdot (f_i = 0).$$

$\circledast$

It's a remarkable fact that <u>when X is non-singular</u> (but not in general when X is only normal) can also go other way around.

<u>Thm</u>: Let X be a non-sing irred var of dim n, and let D be an effective divisor on X. Then:

$$\exists \text{ lb } \mathbb{L} \text{ on } X, \text{ and } 0 \neq s \in \Gamma(X, \mathbb{L})$$

s.t.

$$\mathrm{div}(s) = D.$$

$\circledast$ <u>Variant</u>: If s a rational section of a lb, can define $\mathrm{div}(s)$: but now not effective.

Notation: $\mathbb{L} = \mathcal{O}_X(D)$.

We will prove by introducing:

Cartier Divisors - Let

$$X = \text{irred variety.}$$

Def. A Cartier divisor on X consists of:

(i) Open cover $\{U_i\}$ of X

(ii) Rational fns f_i defined on U_i, $f_i \not\equiv 0$, s.t.

$$f_i / f_j \in \mathcal{O}_X^*(U_i \cap U_j)$$

Two such $\{U_i, f_i\}$, $\{U_i, h_i\}$ determine same C-divisor if

$$f_i / h_i \in \mathcal{O}_X^*(U_i) \text{ all } i. \qquad (*)$$

Ex. $X = \mathbb{P}^1$, $U_{01} = \mathbb{P}^1 - \{0,1\}$, $U_{1\infty} = \mathbb{P}^1 - \{1,\infty\}$, $U_{0\infty} = \mathbb{P}^1 - \{0,\infty\}$

Consider

$$\left(U_{01}, \tfrac{z-1}{z}\right), \left(U_{1\infty}, z-1\right), \left(U_{0\infty}, 1\right)$$

. This is equiv to trivial C-div.

A Cartier div. $\{U_i, f_i\}$ determines in evident way a C-div on a refinement $\{V_\alpha\}$ of $\{U_i\}$. We consider $\{U_i, f_i\}$ and $\{V_\alpha, h_\alpha\}$ equivalent if they satisfy $(**)$ on a common refinement.

Def. Say $\{U_i, f_i\}$ effective if f_i regular on U_i

Def. Set of all such a group CDIV under mult of the f_i.

Prop / Def. Assume X non-sing (or normal). Then there is a natural homom

$$\begin{array}{ccc} \mathrm{div}: C\text{-}\mathrm{Div}(X) & \longrightarrow & \mathrm{Div}(X) \\ \psi & & \psi \\ \{U_i, f_i\} & \longmapsto & D, \end{array}$$

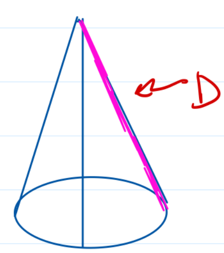
where D is the unique divisor on X s.t.

$$D|U_i = \mathrm{div}(f_i)$$

((Cond (ii) in the def guarantees that $\mathrm{div}(f_i)|U_{ij} = \mathrm{div}(f_j)|U_{ij}$). This homom takes eff divisors to eff divisors

Rmk: On normal but sing vars, map $\underbrace{}_{(\mathrm{div.})}$ might not be surj; eg.

Ruling of cone over conic is not Cartier.



Thm. Assume X non-sing. Then map Φ an isom.

Pf. By linearity, can assume $D = E$ is irred & effective. Now fix any $x \in X$. Since $\mathcal{O}_x X$ a UFD, $\exists$

$$f_x \in \mathcal{O}_x X \quad \text{ideal of } E \text{ in } \mathcal{O}_x X$$

Then $\exists$ $U_x \ni x$ nbd st. f_x regular in U_x, and

$$(f_x) = \text{ideal of } (E \cap U_x).$$

X covered by finitely many such nbds, so get (U_i, f_i) s.t.

$$\mathrm{div}(f_i) = E \cap U_i.$$

Same argument proves result stated earlier:

<u>Thm:</u> D an eff divisor on non-sing X. Then $\exists$ lb $\mathbb{L}$ on X, and

$$s \in \Gamma(X, \mathbb{L}),$$

s.t.

$$\operatorname{div}(s) = D.$$

<u>Pf.</u> By previous Thm, can find C-div. (U_i, f_i) whose divisor is D. Take

$$\mathbb{L} = \text{lb defined by } g_{ij} = \frac{f_i}{f_j} \in \mathcal{O}_X^*(U_i \cap U_j)$$

Then

$$f_i = g_{ij} f_j,$$

so the f_i patch together to define a section of $\mathbb{L}$. $\square$

<u>Next Question:</u> When is $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$?

<u>Lemma:</u> Consider two sections

$$s_1, s_2 \in \Gamma(X, \mathbb{L})$$

of fixed lb on X, w. $s_2 \not\equiv 0$. Then $\exists$ rat fn $\phi \in k(X)$ s.t.

$$s_1 = \phi \cdot s_2,$$

ie. can view quotient

$$\frac{s_1}{s_2} \text{ as a rat } \underline{\underline{\text{fn.}}}$$

<u>Pf</u> Say $\mathbb{L} \leftrightarrow \{U_i, g_{ij}\}$. Then

$$s_\alpha \leftrightarrow \{U_i, f_{\alpha,i}\}$$

where $f_{\alpha i}$ a reg fn on U_i s.t.

$$f_{\alpha i} = g_{ij} f_{\alpha j} \quad \text{on } U_i \cap U_j$$

Follows that

$$\frac{f_{1i}}{f_{2i}} = \frac{f_{1j}}{f_{2j}} \quad \text{on } U_i \cap U_j,$$

so

$$\frac{s_1}{s_2} \underset{loc}{=} \frac{f_{1i}}{f_{2i}} \quad \text{a well-defined rat fn!} \quad \square$$

Now let

$$D_1 = \operatorname{div}(s_1), \quad D_2 = \operatorname{div}(s_2).$$

Then

$$D_2 = D_1 + \operatorname{div}(s_2/s_1) = D_2 + \operatorname{div}(\text{rat fn.})$$

<u>Def.</u> Say two divisors D_1, D_2 are *linearly equivalent*,

$$D_1 \equiv D_2,$$

if $\quad D_2 = D_1 + \operatorname{div}(f), \; f \in k(X)^*$. (Divisors of rat fns are called <u>principal divisors</u>)

<u>Defs / Exercises:</u> Assume X non-sing proj

(1). If D_1, D_2 are effective, then

$$\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2) \text{ iff } D_1 \equiv D_2.$$

(2). Define

$$\operatorname{Pic}(X) = \text{Isom classes of lbs on } X.$$

Then

$$\operatorname{Pic}(X) \cong \operatorname{Div}(X) / \operatorname{Prin}(X)$$

(3). $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}\cdot[\text{hyperplane lb}]$, ie. any lb on $\mathbb{P}^n$ is isom to $\mathcal{O}_{\mathbb{P}^n}(d)$ some $d \in \mathbb{Z}$

(Hint: we know any eff divisor on $\mathbb{P}^n$ is Zeroes(F_d), F_d homog)

(4). Given D, define

$$\mathcal{L}(D) = \{ f \in k(X)^* \mid D + \mathrm{div}(f) \geq 0 \} \cup \{0\}$$

Then

$$\mathcal{L}(D) \cong \Gamma(X, \mathcal{O}_X(D)). \quad \boxtimes$$

Sheaves & Presheaves, contd.

Two motivational Questions -

(1) Let X be a variety (or mfld, or...), and let

$$p: \mathbb{E} \longrightarrow X$$

be a v.b. on X. Then we can define sheaf $\mathcal{O}_X(\mathbb{E})$ via

$$\mathcal{O}_X(\mathbb{E})(U) = \Gamma(U, \mathbb{E}) : \text{ sections of } \mathbb{E} \text{ over } U.$$

<u>Question</u>: How do we recognize sheaves that are of this form?

(2). Let $h: \mathbb{E} \longrightarrow \mathbb{F}$ be a homom of vbs on X.

<u>Question</u>: Is there some way to make sense of $\ker(h)$, $\mathrm{coker}(h)$?

Can't expect these to be vbs. Eg on $X = \mathbb{A}^1$ w. coord t, consider

$$h: \mathbb{1}_X \xrightarrow{\cdot t} \mathbb{1}_X \quad \text{mult by } t.$$

i.e. h is

$$\begin{aligned} X\times k &\longrightarrow X\times k \\ (t,v) &\longrightarrow (t, tv) \end{aligned}.$$

On $\mathbb{A}^1 - \{0\}$, h is an isom, but $h(0)$ is zero. So "coker(h)" should be supported at $0 \in \mathbb{A}^1$.

We'll start w some definitions:

<u>Def.</u> $(X, \mathcal{O}_X)$ a ringed space. A (pre)-sheaf of $\mathcal{O}_X$-modules is a (pre)-sheaf $\mathcal{F}$ s.t.

$\mathcal{F}(U)$ a module over $\mathcal{O}_X(U)$ for all $U \subseteq X$,

and

$\rho^U_V : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is module homom, in the sense that

$\forall\ m \in \mathcal{F}(U)$, $\phi \in \mathcal{O}_X(U)$,

$$\rho(\phi \cdot m) = "(\phi|V)." \rho^U_V(m).$$

<u>Ex.</u> $p: \mathbb{E} \to X$ a v.b., then $\mathcal{O}_X(\mathbb{E})$ an $\mathcal{O}_X$-module.

(Can multiply sections of $\mathbb{E}|U$ by fns on U)

<u>Def.</u> $\mathcal{F}_1, \mathcal{F}_2$ (pre)-sheaves of $\mathcal{O}_X$-modules. A homom

$h : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ is given by

$\mathcal{O}_X(U)$-homoms $h(U) : \mathcal{F}_1(U) \longrightarrow \mathcal{F}_2(U)$ that commute w restrns

<u>Ex.</u> $\mathcal{O}_X$-module $\mathcal{F}$ is $\mathcal{O}_X(\mathbb{E}) \Longleftrightarrow$ locally free of finite rank.

Ex. $h: E \to F$ homom of vbs $\rightsquigarrow h: \mathcal{O}_X(E) \to \mathcal{O}_X(F)$.

Def. $\mathcal{F}$ a sheaf. A subsheaf $\mathcal{S} \subseteq \mathcal{F}$ is a sheaf $\mathcal{S}$ w.

$$\mathcal{S}(U) \subseteq \mathcal{F}(U) \quad \forall\, U$$

(compatibly w. restrictions)

Ex. $h: \mathcal{F}_1 \to \mathcal{F}_2$: then define $\ker(h) \subseteq \mathcal{F}_1$ by

$$\ker(h)(U) = \ker(\mathcal{F}_1(U) \longrightarrow \mathcal{F}_2(U)).$$

Ex. X = var, $Y \subseteq X$ closed subvar. Ideal sheaf of Y in X is

$$\mathcal{I}_{Y/X} \subseteq \mathcal{O}_X \text{ given by } \mathcal{I}_{Y/X}(U) = \{f \in \mathcal{O}_X(U) \mid f|(Y \cap U) \equiv 0\}$$

Ask: what about quotients?

- Consider sheaf $\mathcal{F}$, and subsheaf $\mathcal{S} \subseteq \mathcal{F}$.
- Define

$$\mathcal{Q}(U) = \mathcal{F}(U)/\mathcal{S}(U).$$

- This is presheaf but not in general a sheaf!

Ex. $X = \mathbb{P}^1$, $Y = \{0, \infty\} \subseteq \mathbb{P}^1$, consider $\mathcal{I}_Y \subseteq \mathcal{O}_{\mathbb{P}^1}$:

$$\mathcal{O}_{\mathbb{P}^1}(U_0)/\mathcal{I}_Y(U_0) = \mathbb{C}\cdot 1_0 \qquad \mathcal{O}_{\mathbb{P}^1}(U_1)/\mathcal{I}_Y(\mathbb{P}^1) = \mathbb{C}\cdot 1_\infty$$

$$\mathcal{O}_{\mathbb{P}^1}(U_0 \cap U_1)/\mathcal{I}_Y(U_0 \cap U_1) = 0$$

So the data

$$1 \cdot 1_0 \,, \qquad 0 \cdot 1_\infty \quad \text{satisfy patching cond.}$$

But this does not lift to a section of

$$\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1)/\mathcal{I}(\mathbb{P}^1) = \text{const.}\,(1_0 + 1_\infty)$$

(Rmk: from a fancier pt of view, we're running into an obstr. in $H^1(\mathbb{P}^1, \mathcal{I}_X)$.)

To define $\mathcal{Q} = \mathcal{F}/\mathcal{I}$, we need to <u>sheafify</u>.

Thurs 4/14

<u>Thm/Def.</u> Let $\mathcal{F}$ be a presheaf on a space X. Then there is a sheaf $\mathcal{F}^+$ on X, together w a homom of presheaves

$$\theta : \mathcal{F} \longrightarrow \mathcal{F}^+$$

characterised by the following property: for any sheaf $\mathcal{G}$, and any homom

$$h : \mathcal{F} \longrightarrow \mathcal{G},$$

$\exists!$

$$h^+ : \mathcal{F}^+ \longrightarrow \mathcal{G}$$

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\theta"] \arrow[dr, "h"'] & \mathcal{F}^+ \arrow[d, dashed, "h^+"] \\ & \mathcal{G} \end{tikzcd}$$

s.t. $h^+ \circ \theta = h$. $\mathcal{F}^+$ is called the <u>sheafification</u> of $\mathcal{F}$, or the <u>sheaf assoc. to the presheaf</u> $\mathcal{F}$.

Constr of $\mathcal{F}^+$ proceeds via a discussion of

<u>Stalks.</u> Consider presheaf $\mathcal{F}$ on space X, and fix $x \in X$. <u>Stalk of $\mathcal{F}$ at x</u> is

$$\mathcal{F}_x = \underset{\text{of } \mathcal{F} \text{ at } x}{\text{germs of sections}} = \varinjlim_{U \ni x} \mathcal{F}(U).$$

Ie elt $\phi \in \mathcal{F}_x$ is represented by pair (U, ϕ_U), $U \ni x$,

$$\phi_U \in \mathcal{F}(U).$$

Define $(U, \phi_U) \sim (V, \phi_V)$ if $\exists$ $x \in W \subseteq U \cap V$ st

$$\rho^U_W(\phi_U) = \rho^V_W(\phi_V).$$

$\mathcal{F}_x$ is equiv classes. Note that if $V \subseteq X$ is any open and $s \in \mathcal{F}(V)$, s gives

$$s_y \in \mathcal{F}_y \text{ for any } y \in V.$$

Ex: X = mfld (w classical topology), $\mathcal{H}^1$ the presheaf

$$\mathcal{H}^1(U) = H^1(U, \mathbb{R})$$

Claim: $(\mathcal{H}^1)_x = 0$ for every x.

(Reason: x has arb small contractible nbds.)

This will imply: $(\mathcal{H}^1)^+ = 0$-sheaf

Idea: sections of sheaf should be determined by stalks

Constr of $\mathcal{F}^+$: For any $U \subseteq X$, define $\mathcal{F}^+(U)$ to be all fns

$$U \longrightarrow \coprod_{p \in U}, \quad p \longmapsto s(p) \in \mathcal{F}_p$$

s.t.

$\forall$ $y \in U$, $\exists$ nbd $p \in V \subseteq U$, and $s \in \mathcal{F}(V)$ s.t.

$$s_y = s(y) \quad \forall\ y \in V.$$

This turns out to be a sheaf

So: for sheaf $\mathcal{F}$, $\mathcal{F}=0 \iff \mathcal{F}_x = 0$ all $x \in X$.

Now return to morphism & inclusion of sheaves

$$h: \mathcal{F}_1 \longrightarrow \mathcal{F}_2, \quad \mathcal{S} \subseteq \mathcal{F}$$

Def. · $\mathrm{Im}(h)$ is the sheaf assoc to $\mathrm{Im}(\mathcal{F}_1(U) \to \mathcal{F}_2(U))$

· h is surj $\iff \mathrm{Im}\, h = \mathcal{F}_2 \iff (\mathcal{F}_1)_x \longrightarrow (\mathcal{F}_2)_x$ surj $\forall x$

· $\mathcal{F}/\mathcal{S}$ is sheaf assoc to presheaf $\mathcal{F}(U)/\mathcal{S}(U)$.

· Seq

$$\mathcal{F}_1 \xrightarrow{a} \mathcal{F}_2 \xrightarrow{b} \mathcal{F}_3 \quad \text{is exact}$$

$$\Updownarrow$$

$$\mathrm{Im}\, a = \ker b$$

$$\Updownarrow$$

$$(\mathcal{F}_1)_x \longrightarrow (\mathcal{F}_2)_x \longrightarrow (\mathcal{F}_3)_x$$

exact $\forall x$.

(*) Consider $Q = \mathcal{F}/\mathcal{S}$.
Global section of Q is represented by data (U_i, f_i) $f_i \in \mathcal{F}(U_i)$ w $f_i - f_j \in \mathcal{S}(U_i \cap U_j)$
$(U_i, f_i) \sim (U_i, f_i')$ if $f_i' - f_i \in \mathcal{S}(U_i)$,
· Then passing to limit over refinement.
(*) $\mathrm{CDiv}(X) = \Gamma(X, \mathcal{K}(X)^*/\mathcal{O}_X^*)$

Ex. $X = C^\infty$ mfld (dim n.) (w classical topology). Define

$$\mathcal{a}^p : \text{sheaf w } \mathcal{a}^p(U) = \{C^\infty \text{ } p\text{-forms on } U\}.$$

Exterior deriv gives complex:

$$\mathcal{a}^\bullet : 0 \to \mathcal{a}^0 \xrightarrow{d_0} \mathcal{a}^1 \xrightarrow{d_1} \mathcal{a}^2 \longrightarrow \cdots \longrightarrow \mathcal{a}^n \to 0$$

Claim: $\ker(d_0) = \mathbb{R}_X$: const sheaf
$\mathcal{a}^\bullet$ exact otherwise. (Poincaré Lemma)

On the other hand, apply $\Gamma(X,-)$: setting

$$A^p(X) = \Gamma(X, \mathcal{a}^p) = \text{space of global } p\text{-forms}$$

$\Gamma(X, \mathcal{a}^\bullet)$ is the global deR cx:

$$0 \to A^0(X) \xrightarrow{d} A^1(X) \xrightarrow{d} A^2(X) \to \cdots \to A^n(X) \to 0$$

This is far from exact. In fact: DeR's thm $\Leftrightarrow$

$$H^p(X, \mathbb{R}) = H^p(\Gamma(X, \mathcal{a}^\bullet))$$

Will see: this is conseq of van. of higher cohom of the $\mathcal{a}^p$.

Sheaves on Alg Vars:

Let V = affine var, $A = k[V]$. Recall: for $0 \neq f \in V$, have

$$V_f =_{\text{def}} \{f \neq 0\} \subseteq V: \text{ these are basis for Z-open sets.}$$

$$k[V_f] = A_f = \{ \text{"}\tfrac{a}{f^e}\text{"} \mid a \in A \}$$

Have $V_f \cap V_g = V_{fg}$, & have nat map:

$$A_f \to A_{fg}, \quad \frac{a}{f^e} \mapsto \frac{ag^e}{f^e g^e}$$

Let M = A-module. Then

$$M_f = \{ \text{"}\tfrac{m}{f^e}\text{"} \} \text{ an } A_f\text{-module.}$$

<u>Prop/Def</u>: Any A-module M defines a unique sheaf $\tilde{M}$ of $\mathcal{O}_V$-modules by the rule

$$\tilde{M}(V_f) = M_f, \quad (*)$$

with the restriction maps

$$\tilde{M}(V_f) \longrightarrow \tilde{M}(V_f \cap V_g) \text{ being } M_f \longrightarrow M_{fg}.$$

The stalk of $\tilde{M}$ at pt $x \in V$ is

$$M_{m_x} = \left\{ \frac{m}{g} \,\middle|\, g(x) \neq 0 \right\}.$$

"<u>Pf</u>": Eqn $(*)$ defines $\tilde{M}$ on basis of open sets, and you can check it satisfies sheaf axiom. There is then a unique way to extend to a sheaf on V. (See Gathmann, Chapt. 14)

<u>Def</u>. Sheaf of form $\tilde{M}$ is called <u>quasi-coherent</u>. It is <u>coherent</u> if M is f.g.

<u>Exs</u>: (1) $M = A = k[V]$: $\tilde{M} = \mathcal{O}_V$

(2). $W \subseteq V$ closed subvar. $I \subseteq k[V]$ ideal of W. Then

$$\tilde{I} = \mathcal{I}_{W/V} : \text{ideal sheaf of } W \text{ in } V$$

Can view $k[W]$ as module over $k[V]$ via $k[V] \twoheadrightarrow k[W]$. This gives surj

$$\mathcal{O}_V \twoheadrightarrow \mathcal{O}_W, \text{ and } \mathcal{O}_W \text{ is coherent } \mathcal{O}_V\text{-module.}$$

- More generally, any q-coh $\mathcal{O}_W$-mod becomes $\mathcal{O}_V$-module.

Non-Example: $V = \mathbb{A}^1$, $A = \mathbb{C}[z]$. Define sheaf in $\mathcal{E}$-top on $\mathbb{A}^1$ by rule

$$\mathcal{H}(U) = \{\text{holo fns on } U \subseteq \mathbb{A}^1\},$$

so eg

$$\mathcal{H}(\mathbb{A}^1) = \text{all entire fns on } \mathbb{A}^1 = \mathbb{C},$$

This is a sheaf of $\mathcal{O}_{\mathbb{A}^1}$-modules, but it is not quasi-coherent, because restr maps are not given by localization. Eg.

$$\mathcal{H}(\mathbb{A}^1 - \{0\}) \supsetneq \mathcal{H}(\mathbb{A}^1)_{\frac{1}{z}} = \{\text{global mero fns w poles only at } 0\}$$

but $\mathcal{H}(\mathbb{A}^1 - \{0\})$ also contains holo fns on $\mathbb{C} - \{0\}$ w. essential sing at 0 eg $e^{1/z}$.

Now consider: $(X, \mathcal{O}_X)$ alg var. Sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is (quasi-) coherent if X has finite affine open cover

$$X = \bigcup_i U_i$$

s.t.

$$\mathcal{F}|U_i \text{ is (quasi-) coherent sheaf on each } U_i.$$

Prop. Cond holds for one (affine) open cover $\Leftrightarrow$ holds for all affine open covers. (See Gathmann)

Ex. Consider $p: \mathbb{E} \to X$ a vb on X. Then $\mathcal{O}_X(\mathbb{E})$ coherent. (In fact, choose affine $U \subseteq X$ s.t.

$$\mathbb{E}|U \cong \mathbb{1}_X^e.$$

Then $\mathcal{O}_X(\mathbb{E})|U \cong \mathcal{O}_U^e$, and this is coherent. (It's sheaf assoc to

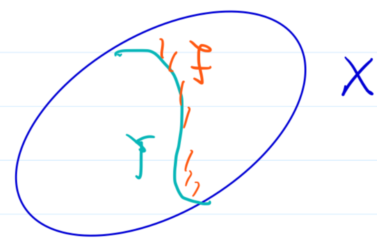
Ex. Support of coh sheaf $\mathcal{F}$ is $\operatorname{supp}(\mathcal{F}) = \{x \mid \mathcal{F}_x \neq 0\}$. This is Z-closed.

free module of rk e.) Special case: $\mathcal{O}_{\mathbb{P}^n}(d)$ coherent.

Ex. Given $Y \subseteq X$ closed, ideal sheaf $\mathcal{I}_{Y/X} \subseteq \mathcal{O}_X$ coherent.

Extension by zero - Let X be a var, and $Y \subseteq X$ a closed subvar, write

$$i : Y \hookrightarrow X \quad \text{for inclusion.}$$



Given a coherent sheaf $\mathcal{F}$ on Y, we define a coherent sheaf $i_*(\mathcal{F})$ on X via

$$i_*(\mathcal{F})(U) = \mathcal{F}(U \cap Y).$$

This is also called "extension of $\mathcal{F}$ by zero": we have

$$\mathcal{F}_x = \begin{cases} 0 & x \notin Y \\ \mathcal{F}_x & x \in Y \end{cases}$$

Algebraically: suppose X, Y affine,

$$A = k[X], \qquad B = k[Y], \text{ so } B = A/I_Y.$$

M is a B-module & $\mathcal{F} = \widetilde{M}$, then $i_*(\mathcal{F})$ is sheaf assoc to A-module

$$M_A =_{\text{def}} M \otimes_B A. \qquad \text{ie Use map } A \to B \text{ to view } M \text{ as } A\text{-module.}$$

Check:

$$\Gamma(Y, \mathcal{F}) = \Gamma(X, i_*(\mathcal{F}))$$

and in general all the cohom properties of $\mathcal{F}$

as a sheaf on Y are unchanged if we extend by zero and view as a sheaf on X.

<u>Notation:</u> often write $\mathcal{F}$ instead of $i_*(\mathcal{F})$.

<u>Example:</u> Given $Y \subseteq X$ as above, view

$$\mathcal{O}_Y = \mathcal{O}_X / \mathcal{I}_Y \quad \text{as a coh sheaf on } X$$

So have short exact seq:

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

of sheaves on X.

<u>Rmk:</u> Let $X \subseteq \mathbb{P}^N$ be a proj var, and $\mathcal{F}$ a coh sheaf on X. As indicated above, "WLOG" we can study $\mathcal{F}$ via extending by zero and viewing it as a coh sheaf on $\mathbb{P}^N$. So,

Study of coh. sheaves on proj vars $\Longleftrightarrow$ study of coherent sheaves on $\mathbb{P}^N$.

Most of main thms will be established in context of coh sheaves on $\mathbb{P}^N$.

<u>Algebraic Constructions</u> —

All the usual operations on modules work in category of coherent sheaves:

<u>Prop:</u> Let $(X, \mathcal{O}_X)$ be a variety, and let $\mathcal{F}, \mathcal{G}$ be coh. sheaves

(i) $\mathcal{F} \oplus \mathcal{G}$ is coh, and more generally any extension of $\mathcal{F}$ by $\mathcal{G}$ is coh

(ii) If $u: \mathcal{F} \to \mathcal{G}$ is a morphism bet coh sheaves

$\ker u$, $\mathrm{im}(u)$, $\mathrm{coker}\, u$ are coherent.

(iii) Define $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ to be sheaf assoc to $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.
If $\mathcal{F}, \mathcal{G}$ are coherent, then so is $\mathcal{F} \otimes \mathcal{G}$.
In fact, if in affine setting

$$\mathcal{F} = \widetilde{M}, \quad \mathcal{G} = \widetilde{N},$$

then

$$\mathcal{F} \otimes \mathcal{G} = \widetilde{(M \otimes_A N)}.$$

(iv) Similarly for $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.

<u>Motivation for Cohom: surjectivity of global sections in exact sequences</u>

<u>Question</u>: Consider a short exact sequence

$$0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0$$

of (say) coherent sheaves on alg var X. Ask: what can we say about surjectivity or not of

$$\Gamma(X, \mathcal{E}) \longrightarrow \Gamma(X, \mathcal{F}) \ ?$$

Before discussing answer, let me explain one setting where question is important.

Consider irred proj var $X \subseteq \mathbb{P}^n$, w ideal sheaf sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_X \longrightarrow 0. \qquad (*)$$

Let's tensor by $\mathcal{O}_{\mathbb{P}^n}(d)$:

<u>Def.</u> If $\mathcal{F}$ is coh sheaf on $\mathbb{P}^n$, write

$$\mathcal{F}(d) = \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(d). \quad (\text{"Twisting by } \mathcal{O}_{\mathbb{P}}(d)\text{"})$$

Since $\mathcal{O}_{\mathbb{P}^n}(d)$ is locally free, $\otimes \mathcal{O}_{\mathbb{P}^n}(d)$ is exact, so we get

$$0 \longrightarrow \mathcal{I}_X(d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow \mathcal{O}_X(d) \longrightarrow 0 \qquad (*)(d)$$

Ask: what are the global sections of these sheaves?

- $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \{\text{homog polys deg } d\}$
- $\mathcal{O}_X(d) = \mathcal{O}_{\mathbb{P}^n}(d)|X$ is lb on X, and

$$\Gamma(\mathbb{P}^n, \mathcal{O}_X(d)) = \Gamma(X, \mathcal{O}_X(d)) = \text{sections of } \mathcal{O}_X(d) \text{ on } X$$

- Have:

$$\begin{array}{rccc} p_d: & \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) & \longrightarrow & \Gamma(X, \mathcal{O}_X(d)) \\ & \psi & & \psi \\ & F & \longmapsto & F|X \end{array} \qquad (\star)_d$$

- So $\boxed{\Gamma(\mathbb{P}^n, \mathcal{I}_X(d)) = \ker(p_d) = (I_X)_d.}$

· It's often important to understand the deg d piece of I_X. In practice we can understand source and target of $(*)_d$, so issue is whether ρ_d is surj.

Ex. Consider

$$u: \mathbb{P}^1 \hookrightarrow \mathbb{P}^3, \quad [s,t] \mapsto [s^6, s^5t, st^5, t^6]$$

let $X = \text{image}$. So

$$X \subseteq \mathbb{P}^3 \text{ curve of deg } 6, \text{ abstr. isom to } \mathbb{P}^1$$

Note:

$$u^* \mathcal{O}_{\mathbb{P}^3}(1) = \mathcal{O}_{\mathbb{P}^1}(5), \text{ i.e. } \mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^1}(6)$$

$$u^* \mathcal{O}_{\mathbb{P}^3}(d) = \mathcal{O}_{\mathbb{P}^1}(5d), \text{ i.e. } \mathcal{O}_X(d) = \mathcal{O}_{\mathbb{P}^1}(6d)$$

Moreover:

$$\rho_1 : \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \longrightarrow \Gamma(X, \mathcal{O}_X(1))$$

this is all quintic polys in s,t

$$z_0, z_1, z_2, z_3 \longmapsto s^6, s^5t, st^5, t^6$$

so $\quad \dim \operatorname{coker}(\rho_1) = 6 - 4 = 2$

all polys deg 12 in s,t

$$\rho_2 : \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \longrightarrow \Gamma(X, \mathcal{O}_X(2))$$

quadr monomials in $z_i z_j$ $\longmapsto$ corresp monomial in s^6, s^5t, st^5, t^6

dim 10 $\qquad\qquad$ dim 13

Check: $\dim \operatorname{im}(\rho_2) = 9$; (im spanned by $s^{12}, s^{11}t, s^{10}t^2, \; s^7t^5, s^6t^6, s^7t^5, \; s^2t^{10}, \ldots$)

Sim:

- ρ_3 not surj
- ρ_d surj for $d \geq 4$.

Note: here you can work everything out by hand. But

Ask: For general $X \subseteq \mathbb{P}^n$, is map

$$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \longrightarrow \Gamma(X, \mathcal{O}_X(d))$$

surjective for $d >> 0$? (Not obvious!) . $\square$

Return to short exact seq

$$0 \to \mathcal{G} \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\rho} \mathcal{F} \to 0, \qquad (*)$$

of coherent sheaves on var X, and fix $s \in \Gamma(X, \mathcal{F})$.

Ask: What is obstruction to lifting s to $\tilde{s} \in \Gamma(X, \mathcal{E})$?

(1°). Since $(*)$ ex seq of coh sheaves, $\exists$ affine cover $\mathcal{U} = \{U_i\}$ of X s.t.

$$0 \to \mathcal{G}(U_i) \xrightarrow{\alpha} \mathcal{E}(U_i) \xrightarrow{\rho} \mathcal{F}(U_i) \to 0 \qquad (*)_i$$

exact $\forall i$, Write $s_i = s|U_i \in \mathcal{F}(U_i)$.

Moreover, can assume $(*)_{ij}$ exact on $U_i \cap U_j$

(2°). We can lift s_i to

$$\tilde{s}_i \in \mathcal{E}(U_i) \quad \text{w.} \quad p(\tilde{s}_i) = s_i.$$

Ask: can we arrange that the $\tilde{s}_i$ patch to give a global section of $\mathcal{E}$?

(3°). Restrict $\tilde{s}_i, \tilde{s}_j$ to $U_{ij} = U_i \cap U_j$. On U_{ij},

$$p(\tilde{s}_i - \tilde{s}_j) = 0 \in \mathcal{F}(U_{ij}),$$

so by exactness of

$$0 \to \mathcal{G}(U_{ij}) \xrightarrow{\alpha} \mathcal{E}(U_{ij}) \xrightarrow{p} \mathcal{F}(U_{ij}) \to 0$$

can write

$$(\tilde{s}_i - \tilde{s}_j)|U_{ij} = \alpha(t_{ij}), \quad \text{some } t_{ij} \in \mathcal{G}(U_{ij})$$

See:

$$t_{ij} = -t_{ji}, \quad t_{ij} + t_{jk} + t_{ki} = 0 \text{ on } U_{ijk} \quad (+)$$

(4°). Check: Suppose we can write

$$t_{ij} = (t_i - t_j)|U_{ij} \quad \text{for } t_\alpha \in \mathcal{G}(U_\alpha).$$

Then, setting $\tilde{s}_i' = \tilde{s}_i - \alpha(t_i)$, the $\tilde{s}_i'$ patch to give a section

$$\tilde{s}' \in \Gamma(X, \mathcal{E})$$

st.

$$p(\tilde{s}') = s.$$

(5°). Now "define"

$$H^1(\mathcal{U}, \mathcal{G}) = \frac{\left\{ \begin{array}{c} \text{collections} \\ t_{ij} \in \mathcal{G}(U_{ij}) \end{array} \middle| \begin{array}{c} t_{ij} = -t_{ji} \\ t_{ij} + t_{jk} + t_{ki} = 0 \end{array} \right\}}{\left\{ t_{ij} \middle| \begin{array}{c} t_{ij} = (t_i - t_j) \mid U_{ij} \\ \text{for } t_\alpha \in \mathcal{G}(U_\alpha) \end{array} \right\}}$$

Then:

> We've constructed a map
>
> $$\Gamma(X, \mathcal{F}) \xrightarrow{\delta} H^1(\mathcal{U}, \mathcal{G}),$$
>
> and $\delta(s) = 0 \iff s$ lifts to $\tilde{s} \in \Gamma(X, \mathcal{E})$.

Plan: For any sheaf X on $\mathcal{F}$, constr cohom groups

$$H^i(X, \mathcal{F}) \qquad (i \geq 0)$$

s.t.

$$H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$$

and short ex seq of sheaves

$$0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0$$

gives long ex seq

$$0 \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{E}) \to H^0(X, \mathcal{F}) \to$$
$$\to H^1(X, \mathcal{G}) \to H^1(X, \mathcal{E}) \to H^1(X, \mathcal{F}) \to$$
$$\to H^2(X, \mathcal{G}) \to \cdots$$

Intro to Sheaf Cohom:

Consider:

$$\begin{aligned}X &= \text{space (esp alg var)}\\ \mathcal{F} &= \text{sheaf on } X \text{ (esp. coh. sheaf)}\\ \mathcal{U} &= \{U_i\} \text{ open cover (esp affine open cover)}\end{aligned}$$

Define:

$$C^p(\mathcal{U},\mathcal{F}) = \prod_{(i_0,\dots,i_p)\in I^{p+1}} \mathcal{F}(U_{i_0}\cap \quad \cap U_{i_p})$$

So $\sigma \in C^p(\mathcal{U},\mathcal{F})$ consists of

$$\sigma_{i_0 \dots i_p} \in \mathcal{F}(U_{i_0}\cap \dots \cap U_{i_p}),$$

for each $(p+1)$-fold intersectn of the U_i. (Don't at moment require the σ's to be alternating).

Define

$$d : C^p(\mathcal{U},\mathcal{F}) \longrightarrow C^{p+1}(\mathcal{U},\mathcal{F})$$

by

$$(d\sigma)_{i_0\dots i_{p+1}} = \sum (-1)^j \left(\sigma_{i_0 \dots \hat{i_j} \dots i_{p+1}} \mid U_{i_0}\cap \quad \cap U_{i_{p+1}}\right)$$

Ex: Say $\sigma \in C^0$, so $\sigma = \{\sigma_i \in \mathcal{F}(U_i)\}$. Then

$$(d\sigma)_{ij} = (\sigma_i - \sigma_j)\mid U_{ij}.$$

Lemma: $d^2 = 0$, so (C^p, d) a cx

Def: $H^p(\mathcal{U},\mathcal{F}) = H^p(C^{\cdot}(\mathcal{U},\mathcal{F}))$

<u>Ex.</u> $H^0(\mathfrak{U}, \mathcal{F}) = \Gamma(X, \mathcal{F})$

<u>Rmk</u> (alt cocycles). Let

$$C^p_{alt}(\mathfrak{U}, \mathcal{F}) = \left\{ \gamma \in C^p \;\middle|\; \begin{matrix} \gamma \text{ is alternating} \\ \text{in the indices} \end{matrix} \right\}$$

Then $C^\cdot_{alt}$ a sub-complex of $C^\cdot$, and

$$H^p(C^\cdot_{alt}) \xrightarrow{\cong} H^p(C^\cdot).$$

([FAC, §22], or [Mumford-Oda, p. 228 ff]) So one can just as well use alt. cycles: we will generally do so.

<u>Prop.</u> Let

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0 \qquad (*)$$

be a short ex seq of sheaves, and let $\mathfrak{U} = \{U_i\}$ be an open covering. Assume that the following property holds:

(**) For any finite collection of indices $i_0, \ldots, i_p \in I$, the seq

$$0 \to \mathcal{G}(U_{i_0 \ldots i_p}) \longrightarrow \mathcal{E}(U_{i_0 \ldots i_p}) \longrightarrow \mathcal{F}(U_{i_0 \ldots i_p}) \to 0$$

is exact.

Then $(*)$ gives rise to a long exact seq

$$0 \to H^0(\mathfrak{U}, \mathcal{G}) \longrightarrow H^0(\mathfrak{U}, \mathcal{E}) \longrightarrow H^0(\mathfrak{U}, \mathcal{F}) \longrightarrow H^1(\mathfrak{U}, \mathcal{G}) \to H^1(\mathfrak{U}, \mathcal{E}) \to \cdots$$

$$\cdots \longrightarrow H^{p-1}(\mathfrak{U}, \mathcal{F}) \longrightarrow H^p(\mathfrak{U}, \mathcal{G}) \longrightarrow H^p(\mathfrak{U}, \mathcal{E}) \longrightarrow H^p(\mathfrak{U}, \mathcal{F}) \to H^{p+1}(\mathfrak{U}, \mathcal{G}) \longrightarrow \cdots$$

Rmk. (∗∗) holds if (∗) is ex seq of coh sheaves on var X, and $\mathcal{U} = \{U_i\}$ is std affine covering

Pf of Prop. Cond (∗∗) guarantees that (∗) gives short ex seq

$$0 \to C^\cdot(\mathcal{U}, \mathcal{G}) \to C^\cdot(\mathcal{U}, \mathcal{E}) \to C^\cdot(\mathcal{U}, \mathcal{F}) \to 0$$

of complexes, which in turn gives long ex seq of coh groups.

Refinements — We've so far defined cohom wrt a specific open cover. Although it turns out it won't be necessary in setting of coh cohom, in general one needs to pass to refinements:

Let

$$\mathcal{U} = \{U_i\}, \quad \mathcal{B} = \{V_j\}$$

be two open covers of X. Suppose that $\mathcal{U}$ is a refinement of $\mathcal{B}$, so for each $i \in I$ one is given $\lambda(i) \in J$ st

$$U_i \subseteq V_{\lambda(i)}.$$

Then

$$U_{i_0 \cdots i_p} \subseteq V_{\lambda(i_0) \cdots \lambda(i_p)},$$

so for any sheaf $\mathcal{F}$, have restr map

$$\mathcal{F}(V_{\lambda(i_0) \cdots \lambda(i_p)}) \longrightarrow \mathcal{F}(U_{i_0 \cdots i_p}).$$

Now define

$$\tau : \tau(\mathcal{B}, \mathcal{U}, \lambda) : C^p(\mathcal{B}, \mathcal{F}) \longrightarrow C^p(\mathcal{U}, \mathcal{F})$$

via

$$(\tau\gamma)_{i_0 \cdots i_p} = \gamma_{\lambda(i_0) \cdots \lambda(i_p)} \mid U_{i_0 \cdots i_p}.$$

This gives morphism $C^\cdot(\mathcal{B}, \mathcal{F}) \to C^\cdot(\mathcal{U}, \mathcal{F})$, hence

$$\tau^* : H^p(\mathcal{B}, \mathcal{F}) \longrightarrow H^p(\mathcal{U}, \mathcal{F})$$

Lemma: Map τ^* on cohom indep of choice of fn $\lambda, I \to J$

Idea: Two different refinement fns $\lambda, \lambda' : I \to J$ give rise to homotopic maps τ, τ'.

Ref: Mumford-Oda, p. 227.

Now we define $\check{C}$ech cohom of $\mathcal{F}$ by passing to limit over all coverings:

Def
$$\check{H}^p(X, \mathcal{F}) = \varinjlim_{\substack{(\text{all covers}) \\ \mathcal{U}}} H^p(\mathcal{U}, \mathcal{F})$$

Concretely, class $a \in \check{H}^p(X, \mathcal{F})$ repr by

$$\alpha_{\mathcal{U}} \in H^p(\mathcal{U}, \mathcal{F}),$$

where

$$a_{\mathcal{U}} \in H^p(\mathcal{U}, \mathcal{F}), \quad \alpha_{\mathcal{U}'} \in H^p(\mathcal{U}', \mathcal{F})$$

represent same class if they both map to same class under a a common refinement.

** Example: If X is any variety, then

sheaf of abelian group written multiplicatively

$$\mathrm{Pic}(X) =_{def} \left\{ \begin{matrix} \text{isom classes} \\ \text{of lbs} \end{matrix} \right\} \cong \check{H}^1(X, \mathcal{O}_X^*)$$

Idea: Given lb, repr it by data $\{U_i, g_{ij}\}$, $g_{ij} \in \mathcal{O}_X^*(U_{ij})$. Then

$$\{g_{ij}\} \in Z^1(\mathcal{U}, \mathcal{O}_X^*)$$

<u>Check</u>: various choices involved exactly determine a class in $\check{H}^1(X, \mathcal{O}_X^*)$.

Now return to short exact seq

$$0 \to \mathcal{G} \to \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0 \qquad (*)$$

of sheaves on X.

<u>Prop.</u> Assume that either:

(i) $(*)$ is an ex seq of quasi-coh sheaves on alg var X; or

(ii). X is paracompact and Hausdorff, eg a mfld

Then $(*)$ gives long ex seq on $\check{H}^i(X, \cdot)$. In this case we define

$$H^i(X, \mathcal{F}) = \check{H}^i(X, \mathcal{F}).$$

<u>Pf.</u> (i). Affine coverings satisfying condition $(**)$ in previous Prop are cofinal among all coverings

(ii). See ~~Mfd~~ Oda, p. 231.

<u>Rmk.</u> For arb $\mathcal{F}$ on arb X, $\check{H}$ is not the "right" cohom theory. I'll come back to this a little later.

<u>De Rham's Thm</u>: It's instructive to see how sheaf cohom gives rise to quick pf of (weak form of) DeR's Thm.

Key point is:

Prop. Let X be a C^∞ mfld, and let $\mathcal{E}$ be the sheaf of sectns of a C^∞ vb on X. Then

$$H^p(X, \mathcal{E}) = 0 \quad \text{for } p > 0.$$

Pf. Enough to show that if $\mathfrak{U} = \{U_i\}$ is a locally finite cover, then

$$H^p(\mathfrak{U}, \mathcal{E}) = 0 \quad \text{for } p > 0.$$

Crucial pt is $\exists$nce of partition of unity $\{\rho_i\}$ subordinate to $\mathcal{U}$, ie. fns

$$\rho_i : X \longrightarrow \mathbb{R}, \qquad \operatorname{supp}(\rho_i) \subseteq U_i,$$

w.

$$\sum_{i \in I} \rho_i \equiv 1 \quad \text{on } X.$$

We'll show $H^1(\mathfrak{U}, \mathcal{E}) = 0$, general case being conceptually similar but notationally heavier. So say

$$\tau = (\tau_{ij}) \in Z^1_{\text{alt}}(\mathfrak{U}, \mathcal{E}),$$

so

$$\tau_{ij} \in \mathcal{E}(U_{ij}), \quad \tau_{ij} = -\tau_{ji}$$

&

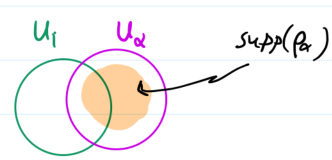
$$\tau_{ij} + \tau_{jk} + \tau_{ki} \equiv 0 \quad \text{on } U_{ijk}.$$

Want to find:

$$\sigma_i \in \mathcal{E}(U_i) \quad \text{s.t.} \quad \tau_{ij} = (\sigma_i - \sigma_j) \mid U_{ij}.$$

Define:

$$\sigma_i = \sum_{\alpha \in I} \rho_\alpha \cdot \tau_{i\alpha} :$$



(Meaning $\operatorname{supp}(\rho_\alpha \tau_{i\alpha}) \subseteq U_\alpha \cap U_i$, extend by 0 to view as elt of $\mathcal{E}(U_i)$)

Then

$$\begin{aligned}\sigma_i - \sigma_j &= \sum_{\alpha\in I} \rho_\alpha \tau_{i\alpha} - \sum_{\alpha\in I} \rho_\alpha \tau_{j\alpha} \\ &= \sum_{\alpha\in I} \rho_\alpha (\tau_{i\alpha} - \tau_{j\alpha}) \\ &= \sum_{\alpha} \rho_\alpha (\tau_{i\alpha} + \tau_{\alpha j}) \\ &= \sum_{\alpha} \rho_\alpha (\tau_{ij}) \\ &= \tau_{ij}\end{aligned}$$

(For general case, see Griffiths–Harris, p.42)

Now consider De-R cx $\mathcal{A}^\bullet$ of sheaves

$$0 \to \mathcal{A}^0 \xrightarrow{d_0} \mathcal{A}^1 \xrightarrow{d_1} \mathcal{A}^2 \to \cdots \to \mathcal{A}^n \to 0$$

Have

$\ker d_0 = \mathbb{R}_X$, otherwise exact.

$H^q(X, \mathcal{A}^p) = 0 \quad q > 0.$

Write:

$A^\bullet = (\Gamma(X, \mathcal{A}^0) \to \Gamma(X, \mathcal{A}^1) \to \cdots)$

= complex of global sections, so $H^k(A^\bullet) = H_{DR}(X)$

Want to conclude:

$$H^k(X, \mathbb{R}) \cong H^k(A^\bullet)$$

Follows from purely formal:

<u>Lemma</u> ("Abstract DeR Thm"). Consider complex of sheaves:

$$\mathcal{A}^\bullet : \qquad 0 \longrightarrow \mathcal{A}^0 \xrightarrow{d^0} \mathcal{A}^1 \xrightarrow{d^1} \mathcal{A}^2 \xrightarrow{d^2} \cdots$$

on space X. Assume

$$\ker(d^0) = \mathcal{F},$$

while $\mathcal{A}^\bullet$ otherwise exact, and assume also that

$$H^q(X, \mathcal{A}^p) = 0 \quad q > 0, \text{ all } p.$$

Finally, suppose we know that short exact seqs of sheaves give long exact sequences in cohom. Then

$$H^k(X, \mathcal{F}) = H^k(A^\bullet),$$

where

$$A^\bullet = \Gamma(X, \mathcal{A}^\bullet) \text{ is complex of global sections determined by } \mathcal{A}^\bullet$$

"Pf. Chop $\mathcal{A}^\bullet$ into short exact sequences and chase resulting diagram."

I'll prove conclusion for $k = 1$. Form diagram:

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \arrow[dr] & & 0 & & \\
& \mathcal{F} \arrow[dr] & & & & \mathcal{E}_2 \arrow[dr] \arrow[ur] & & & \\
0 \arrow[rr] & & \mathcal{A}^0 \arrow[rr, "d^0"] \arrow[dr] & & \mathcal{A}^1 \arrow[rr, "d^1"] \arrow[ur] & & \mathcal{A}^2 \arrow[rr, "d^2"] \arrow[dr] & & \mathcal{A}^3 \\
& & & \mathcal{E}_1 \arrow[ur] \arrow[dr] & & & & \mathcal{E}_3 \arrow[ur] \arrow[dr] & \\
& & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

Diagonal sequences short exact

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \arrow[dr] & & 0 & & \\
& \mathcal{F} \arrow[dr] & & & & \mathcal{E}_2 \arrow[ur] \arrow[dr] & & & \\
0 \arrow[rr] & & a^0 \arrow[rr, "d^0"] \arrow[dr] & & a^1 \arrow[rr, "d^1"] \arrow[ur] & & a^2 \arrow[rr, "d^2"] \arrow[dr] & & a^3 \\
& & & \mathcal{E}_1 \arrow[ur] \arrow[dr] & & & & \mathcal{E}_3 \arrow[ur] \arrow[dr] & \\
& & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

Now look at long exact seqs arising from diag seqs:

<u>Claim</u>: $\ker\left(H^0(a^1) \longrightarrow H^0(a^2) \right) \cong H^0(\mathcal{E}_1)$

<u>Pf</u>: Use:

$$\begin{tikzcd}
& & & 0 \arrow[d] \\
0 \arrow[r] & H^0(\mathcal{E}_1) \arrow[r] & H^0(a^1) \arrow[r] & H^0(\mathcal{E}_2) \arrow[d] \\
& & & H^0(a^2) \arrow[d] \\
& & & {}
\end{tikzcd}$$

<u>So</u>:

$$H^1(A') = \operatorname{coker}\left(H^0(a^1) \longrightarrow H^0(\mathcal{E}_1) \right).$$

But: $H^1(a^0) = 0$, so from ex seq on left get

$$H^0(a^0) \longrightarrow H^0(\mathcal{E}_1) \longrightarrow H^1(\mathcal{F}) \longrightarrow \boxed{0} = H^1(a^0).$$

QED!

This leads to "official" defn of $H^*(X, \mathcal{F})$ as a derived functor.

Idea: start w a "resoln" of $\mathcal{F}$ via a complex of sheaves

$$\mathcal{a}^\bullet : \quad 0 \longrightarrow \mathcal{a}^0 \xrightarrow{d^0} \mathcal{a}^1 \xrightarrow{d^1} \mathcal{a}^2 \xrightarrow{d^2} \mathcal{a}^3 \xrightarrow{d^3} \cdots,$$

ie.

$$\mathcal{F} = \ker(d^0), \qquad \mathcal{a}^\bullet \text{ otherwise exact.}$$

Choose the $\mathcal{a}^i$ in such a way that they should have vanishing higher cohom.

Then it has to be true that $H^k(X, \mathcal{F}) = H^k(A^\bullet)$, where $A^\bullet = \Gamma(X, \mathcal{a}^\bullet)$.

Grothendieck: takes $\mathcal{a}^i$ to be injective sheaves. There are also other choices in the literature

Remains to show this gives "good" theory.

Tues 4/26

Vanishing for Affine Varieties.

Thm. Let V be an affine var, and let $\mathcal{F}$ be a (quasi-) coherent sheaf on V. Then

$$H^p(V, \mathcal{F}) = 0 \quad \text{for } p > 0.$$

Sketch of Pf. Write $A = k[V]$, and say $\mathcal{F} = \tilde{M}$ for an A-module M. Choose

$$f_1, \ldots, f_r \in A \text{ generating the unit ideal,}$$

and let

$$\mathcal{U} = \{V_{f_i}\} \text{ be the corresp open cover}$$

It suffices to show that

$$H^p(\mathcal{U}, \tilde{M}) = 0 \quad \text{for } p > 0,$$

since such open covers are cofinal among all open covers.

Cech complex $C^\bullet_{alt}(\mathcal{U}, \tilde{M})$ has form:

$$\begin{array}{ccccccc} \bigoplus_i M_{f_i} & \longrightarrow & \bigoplus_{i<j} M_{f_i f_j} & \longrightarrow & \bigoplus_{i<j<k} M_{f_i f_j f_k} & \longrightarrow & \cdots \\ \| & & \| & & \| & & \\ C^0_{alt} & & C^1_{alt} & & C^2_{alt} & & \end{array}$$

So we need to show this is exact starting at C^1. Argument is parallel to pf of acyclicity of C^∞ sheaves. As we did there, I'll focus on the case $p = 1$.

So fix alternating cocycle

$$\tau = (\tau_{ij}) \in C^1_{alt}(\mathcal{U}, \tilde{M}):$$

ie.

$$\tau_{ij} = \frac{m_{ij}}{f_i^{e_i} f_j^{e_j}} \in M_{f_i f_j}$$

w.

$$\tau_{ij} + \tau_{jk} + \tau_{ki} = 0 \in M_{f_i f_j f_k}.$$

Want to write

$$\tau_{ij} = \tau_i - \tau_j.$$

Since the f_α generate unit ideal, so do the $f_\alpha^{e_\alpha}$. So $\exists$ $g_\alpha \in A$ s.t

$$\sum g_\alpha f_\alpha^{e_\alpha} = 1$$

Write

$$\tau_j = \sum_\alpha g_\alpha f^{e_\alpha} \tau_{j\alpha}$$

$$= \sum g_\alpha f_\alpha^{e_\alpha} \frac{m_{j\alpha}}{f_j^{e_j} f_\alpha^{e_\alpha}} \in M_{f_j}.$$

Moreover,

$$\tau_i - \tau_j = \sum_\alpha g_\alpha f_\alpha^{e_\alpha} (\tau_{i\alpha} - \tau_{j\alpha})$$

$$= \sum_\alpha g_\alpha f_\alpha^{e_\alpha} (\tau_{i\alpha} + \tau_{\alpha j})$$

$$= \sum_\alpha g_\alpha f_\alpha^{e_\alpha} (\tau_{ij})$$

$$= \tau_{ij}. \qquad \text{QED.}$$

This has various consequences. To begin with

<u>Thm</u> (Thm of Leray). Let $\mathcal{F}$ be a quasi-coherent sheaf on a variety X, and let $\mathfrak{U} = \{U_i\}$ be any affine covering of X. Then

$$H^i(X, \mathcal{F}) = H^i(\mathfrak{U}, \mathcal{F}).$$

Ie cohom of $\mathcal{F}$ can be computed from any one affine covering.

<u>Idea</u>: Let $\mathfrak{B} = \{V_j\}$ be another affine covering. Suffices to show that

$$H^i(\mathfrak{U}, \mathcal{F}) \cong H^i(\mathfrak{B}, \mathcal{F})$$

by an isom compatible w restriction maps when $\mathfrak{U}$ is a refinement of $\mathfrak{B}$.

Can form big double complex $C^{\cdot,\cdot}(\mathfrak{U},\mathfrak{B};\mathcal{F})$ whose terms are

$$\mathcal{C}^{p,q} = \mathcal{F}(U_{i_0}\cap \cap U_{i_p}\cap V_{j_0}\cap \cap V_{j_q})$$

Then

$$H^i(\mathfrak{U},\mathcal{F}) = H^i(\ker(C^{\cdot,0}\to C^{\cdot,1}))$$

$$H^i(\mathfrak{B},\mathcal{F}) = H^i(\ker(C^{0,\cdot}\to C^{1,\cdot}))$$

But it follows formally from vanishing thm that both of these are isom to comology of total complex assoc to $C^{\cdot,\cdot}$. See Mfd-Oda, p. 234 ff.

As another consequence, we find

<u>Thm.</u> Let X be a variety of dim n, and $\mathcal{F}$ a quasi-coherent sheaf on X. Then

$$H^i(X,\mathcal{F}) \simeq 0 \quad \text{for } i > n = \dim X.$$

<u>Pf.</u> We'll prove under the additional assumption that X is quasi-projective.

It suffices to prove

<u>Claim:</u> X has an affine coverings $\mathfrak{U} = \{U_i\}$ by $n+1$ or few open subsets.

Then $C^p_{alt}(\mathfrak{U},\mathcal{F}) = 0$ for $p > n$, so our assertion follows from previous Thm.

For claim, say

$$X \subseteq \mathbb{P}^N \text{ loc closed,}$$

$$\overline{X} \subseteq \mathbb{P}^N \text{ proj closure of } X$$

$$X = \overline{X} - F, \quad F \subseteq \overline{X} \text{ closed}$$

<u>Lemma:</u> Let $\overline{X} \subseteq \mathbb{P}^N$ be proj var of dim n, $F \subseteq \overline{X}$ closed set. Then if $d \gg 0$, $\exists$ $n+1$ homog. polys

$$P_0, \ , P_n \text{ of deg } d$$

not vanishing identically on $\overline{X}$, s.t.

$$F = \overline{X} \cap \{P_0 = \ = P_n = 0\}$$

<u>Pf:</u> Fun exercise!

Now let

$$U_i = \overline{X} - \{P_i = 0\}.$$

Then $U_i \subseteq X$ is affine, and

$$X = \cup U_i. \quad \text{QED!}$$

Serre's Thm on Global Generation

Let $\mathbb{P} = \mathbb{P}^n$ be proj space (won't really care about dim), let

$$\mathcal{F} = \text{coh sheaf on } \mathbb{P}.$$

Recall that for $d \in \mathbb{Z}$,

$$\mathcal{F}(d) =_{\text{def}} \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(d).$$

- If $d << 0$, then typically $\Gamma(\mathbb{P}, \mathcal{F}(d)) = 0$.

  Ex. Say $X \subseteq \mathbb{P}$ a closed subvar, $\mathcal{F} = \mathcal{I}_{X/\mathbb{P}}$. Then we saw that

  $$\Gamma(\mathbb{P}, \mathcal{I}_{X/\mathbb{P}}(d)) = \begin{array}{l}\text{homog polys of deg } d \\ \text{vanishing on } X,\end{array}$$

  and of course this $= 0$ if $d < 0$ (or d small).

- Serre proved that if $d >> 0$, then $\mathcal{F}(d)$ has many global sections. To make this precise we start with:

<u>Prop/Def.</u> Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}$. TFAE:

(a). $\exists$ finitely many global sections

$$s_0, \ldots, s_r \in \Gamma(X, \mathcal{F})$$

s.t. the "values"

$$s_i(x) \in \mathcal{F}/m_x\mathcal{F}$$

generate the vector space $\mathcal{F}/m_x\mathcal{F}$ $\forall x \in \mathbb{P}$

(b) $\exists$ finitely many global sections

$$s_0, \ldots, s_r \in \Gamma(X, \mathcal{F})$$

s.t. the germs

$(s_i)_x \in \mathcal{F}_x$ span the $\mathcal{O}_x\mathbb{P}$-module $\mathcal{F}_x$ $\forall x$

(c). $\mathcal{F}$ is a quotient of a trivial bundle of rk $r+1$; ie $\exists$ surj map

$$\mathcal{O}_{\mathbb{P}}^{r+1} \longrightarrow \mathcal{F} \longrightarrow 0$$

of sheaves on $\mathbb{P}^n$.

If these hold, one says that $\mathcal{F}$ is globally generated, or generated by its global sections.

Sketch of Pf of Equivalence:

(a) $\Rightarrow$ (b): Nakayama's Lemma.

(b) $\Rightarrow$ (c). Note that if $s \in \Gamma(X, \mathcal{F})$, then s defines

$$\mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{F} \quad \text{with} \quad \begin{array}{ccc} \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) & \longrightarrow & \Gamma(\mathbb{P}, \mathcal{F}) \\ \cup & & \cup \\ 1 & \longmapsto & s \end{array}.$$

(Exerc!). So the s_i give

$$\mathcal{O}_{\mathbb{P}}^{r+1} \longrightarrow \mathcal{F},$$

surj on stalks (hence surj as a map of sheaves).

(c) $\Rightarrow$ (a). Take values at x.

Ex. $\mathcal{I}_{X/\mathbb{P}}(d)$ glob gen $\Rightarrow X$ cut out by hypsfs of deg d.

Thm (Serre). Let $\mathcal{F}$ be a coh sheaf on $\mathbb{P}$. Then $\exists$ an integer

$$d_0 = d_0(\mathcal{F})$$

s.t. $\mathcal{F}(d)$ is glob gen $\forall\, d \geqslant d_0$.

<u>Lemma.</u> Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}$, and let

$$U_\alpha = \{ x_\alpha \neq 0 \} \subseteq \mathbb{P}$$

be one of the standard open sets.

(a) Suppose $s \in \Gamma(\mathbb{P}, \mathcal{F})$ is a section whose restriction to U_α is zero:

$$s|U_\alpha = 0 \in \Gamma(U_\alpha, \mathcal{F}|U_\alpha).$$

Then

$$x_\alpha^m s = 0 \in \Gamma(\mathbb{P}, \mathcal{F}(m)) \text{ for some } m>0.$$

(b). Let $t \in \Gamma(U_\alpha, \mathcal{F}|U_\alpha)$ be a section. Then $\exists\ m>0$ s.t.

$$x_\alpha^m \cdot t \in \Gamma(U_\alpha, \mathcal{F}(m)|U_\alpha)$$

extends to a global section in $\Gamma(\mathbb{P}, \mathcal{F}(m))$.

<u>Sketch</u> (M-O, Lemma 4.1, p.97). I'll prove (b)

- On U_i, $\exists\ k[U_i]$-module M_i s.t. $\mathcal{F}|U_i = \tilde{M}_i$. So

$$\mathcal{F}(U_i \cap U_\alpha) = (M_i)_{(x_\alpha/x_i)}.$$

So $\exists\ m = m_i$

$$s_i = x_\alpha^m\, t \in \mathcal{F}(m_\alpha)|U_i \qquad (*).$$

We can choose one value of m s.t. (*) holds for every i.

- Now on $U_i \cap U_j$ s_i & s_j are defined, and

$$(s_i - s_j) | U_\alpha \cap U_i \cap U_j = 0 \in (M_i)_{(\frac{x_\alpha}{x_i})} \cong (M_j)_{(\frac{x_\alpha}{x_j})}$$

Then $\exists$ $m' = m'_{i,j}$ s.t.

$$x_\alpha^{m'} \cdot (s_i - s_j) = 0 \text{ on } U_i \cap U_j,$$

and again can take m' uniform in i, j. So

$$x_\alpha^{m+m'} t \text{ extends to glob section } \in \Gamma(\mathcal{F}(m+m')).$$

Pf of Thm: Given $\mathcal{F}$ coh on $\mathbb{P}$, consider restr

$$\mathcal{F} | U_\alpha = \tilde{M}_\alpha.$$

Can find finitely many sections

$$s_{\alpha 0}, \quad , s_{\alpha r} \in \Gamma(U_\alpha, \mathcal{F} | U_\alpha)$$

that generate $\mathcal{F}$ over U_α. By lemma, $\exists$ m s.t. $x_\alpha^m s_{\alpha i}$ extend to glob sections

$$t_{\alpha i} \in \Gamma(\mathbb{P}, \mathcal{F}(m))$$

that generate $\mathcal{F}(m)$ on U_α. Putting together the $t_{\alpha i}$ over all α proves the statement

Cohom of Line Bundles on Proj Space

We work on $\mathbb{P}^n$ w homog coords $T_0, \dots, T_n$ and consider std affine cover

$$U_i = \{T_i \neq 0\}$$

so

$$k[U_i] = k\left[\frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right]$$

Recall

$$\mathcal{O}_{\mathbb{P}^n}(d) = \mathcal{O}_{\mathbb{P}}(d)$$

is sheaf of sections of lb w trans fns

$$g_{ij} = \left(\frac{T_j}{T_i}\right)^d.$$

<u>THM.</u> For any $d \in \mathbb{Z}$, cohom of $\mathcal{O}_{\mathbb{P}}(d)$ is as follows

(i) $$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(d)) = \begin{cases} \text{homog polys of deg } d & \text{when } d \geq 0 \\ 0 & \text{〃} \quad d < 0 \end{cases}$$

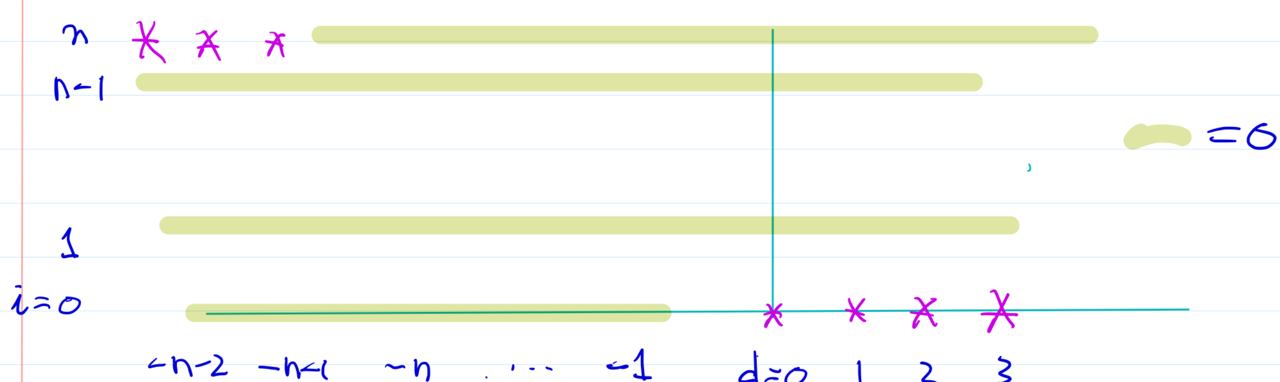
(ii) For $1 \leq i \leq n-1$,

$$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = 0 \quad \text{for all } d$$

(iii) $$H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1-d))^*$$

(so $\dim H^n(\mathcal{O}(-n-1)) = 1$, $\dim H^n(\mathcal{O}(-n-2)) = n+1$, etc)

Schematic diagram



Will prove in several steps, and will establish for all d at once

Step 1: Interpretation of Cech complex

- Recall (Thm of Leray) that we can compute cohom from the covering $\mathfrak{U} = \{U_i\}_i$
- Let

$$S = k[T_0, \dots, T_n].$$

viewed as graded ring. Write S_d for deg d comp.

- Consider localizn

$$S_{T_i} = S\left[\tfrac{1}{T_i}\right] = k\left[T_0, \dots, T_n, \tfrac{1}{T_i}\right]$$

Then S_{T_i} is also graded, w.

$$(S_{T_i})_d = \left\{ \frac{F_{d+e}}{T_i^e} \;\middle|\; F \in S_{d+e} \right\} \quad \left(\begin{array}{l}\text{i.e. count } 1/T_i \text{ as} \\ \deg = -1\end{array}\right)$$

Sim $S_{T_{i_0} \cdots T_{i_p}}$ is graded

- Now fix $d \in \mathbb{Z}$, let $L = \mathcal{O}_{\mathbb{P}^n}(d)$. So

$$L(U_i) \cong k\left[\tfrac{T_0}{T_i}, \dots, \tfrac{T_n}{T_i}\right],$$

but when we work on double or higher overlaps there are different choices of isoms

Observe: There is a canonical identification

$$L(U_i) = (S_{T_i})_d\,,$$

$$L(U_{i_0 \dots i_p}) = (S_{T_{i_0} \cdots T_{i_p}})_d$$

in such a way that the restrictions

$$L(U_i) \longrightarrow L(U_i \cap U_j)$$

are the canonical maps

$$(S_{T_i})_d \longrightarrow (S_{T_i T_j})_d$$

<u>Pf.</u> Make the usual identification

$$L(U_i) \xrightarrow[\cong]{\varphi_i} k\left[\frac{T_0}{T_i}, \dots, \frac{T_n}{T_i}\right]$$

wrt which trans fns of $\mathcal{O}_{\mathbb{P}^n}(d)$ are $g_{ij} = \left(\frac{T_j}{T_i}\right)^d$.

The identification

$$L(U_i) = k\left[\frac{T_0}{T_i}, \dots, \frac{T_n}{T_i}\right] \xrightarrow[\Phi_i]{\cdot T_i^d} (S_{T_i})_d$$

is given by mult by T_i^d. Then on double overlap have

$$\begin{tikzcd}
k\left[\frac{T_0}{T_i}, \dots, \frac{T_n}{T_i}\right]_{T_j} \arrow[r, "T_i^d"] & (S_{T_i})_d \arrow[dr] & \\
& & (S_{T_i T_j})_d \\
k\left[\frac{T_0}{T_j}, \dots, \frac{T_n}{T_j}\right]_{T_i} \arrow[uu, "\cdot \left(\frac{T_j}{T_i}\right)^d"] \arrow[r, "T_i^d"'] & (S_{T_j})_d \arrow[ur] &
\end{tikzcd}$$

as required. □

So: Can identify

$$\check{C}^{\bullet}_{\text{all}}(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n}(d))$$

with the degree d piece of the Cech-type complex of graded modules $C^{\bullet}$

$$\bigoplus_{i_0} S_{T_{i_0}} \xrightarrow{d} \bigoplus_{i_0 < i_1} S_{T_{i_0} T_{i_1}} \xrightarrow{d} \bigoplus_{i_0 < i_1 < i_2} S_{T_{i_0} T_{i_1} T_{i_2}} \xrightarrow{d} \cdots$$

$$\cdots \xrightarrow{d} \bigoplus_{i_0 < \dots < i_{n-1}} S_{T_{i_0} \cdots T_{i_{n-1}}} \xrightarrow{d} S_{T_{i_0} \cdots T_{i_n}} \longrightarrow 0$$

Step 2: Computation of H^0, H^n.

- We already know H^0. (Or do as exercise from previous step).
- H^n is coker of

$$\bigoplus_{i_0 < \dots < i_{n-1}} S_{T_{i_0} \cdots T_{i_{n-1}}} \xrightarrow{d} S_{T_0 \cdots T_n}$$

- $S_{T_0 \cdots T_n}$ is spanned by all monomials

$$\frac{1}{T_0 \cdots T_n} \cdot T_0^{a_0} \cdots T_n^{a_n}, \quad a_i \in \mathbb{Z}.$$

- What is image of d? Consider eg the piece

$$S_{T_0 \cdots T_{n-1}} \longrightarrow S_{T_0 \cdots T_n}$$

Image is spanned by monomials

$$\frac{1}{T_0 \cdots T_n} T_0^{a_0} \cdots T_{n-1}^{a_{n-1}} T_n^{a_n}, \quad a_n \geq 1$$

In general, see

$\mathrm{Im}(d)$ spanned by monomials

$$\frac{1}{T_0 \cdots T_n} \cdot T_0^{a_0} \cdots T_n^{a_n}$$

with $a_i \geq 1$ for some i.

So:

$H^n =$ coker d spanned by

$$(*) \qquad \frac{1}{T_0 \cdots T_n} \cdot T_0^{a_0} \cdots T_n^{a_n} \quad \text{w all } a_i \leq 0$$

So

$H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1-k))$ spanned by monomials $(*)$ w $\sum -(a_i) = k$.

So H^n as claimed.

Remains to show that $H^i(C_\cdot) = 0$ for $1 \leq i \leq n-1$

For this we pause proof and make some remarks on Koszul complexes.

Koszul complexes.

Ex. Consider $A = k[x,y]$ and $f,g \in A$ w no common factor,
eg

$$f = x^N, \quad g = y^N.$$

Claim: Have exact sequence

$$0 \longrightarrow A \xrightarrow{\binom{g}{-f}} A^2 \xrightarrow{(f,g)} A \longrightarrow A/(f,g) \to 0 \qquad (*)$$

$$\binom{a}{b} \longmapsto af+bg$$

$$1 \longrightarrow (g,-f)$$

(Use unique factorization to show that

$$af+bg=0 \iff a=gh, \ b=-fh \text{ some } h.)$$

Note that up to sign, $(*)$ (w last term omitted) is

$$(A \xrightarrow{-f} A) \otimes (A \xrightarrow{g} A).$$

Def Let $A =$ Noetherian ring,

$$f_0, \ , f_n \in A.$$

Koszul complex assoc to the f_i is

$$K = K(f): \quad (A \xrightarrow{f_0} A) \otimes (A \xrightarrow{f_1} A) \otimes \quad \otimes (A \xrightarrow{f_n} A).$$

Explicitly, K is

$$K. \quad A \longrightarrow A^{\oplus n+1} \longrightarrow \Lambda^2(A^{\oplus n+1})$$

$$1 \longrightarrow \Sigma f_i e_i$$

$$e_\alpha \longmapsto \Sigma f_i e_i \wedge e_\alpha$$

(so the maps are $\wedge(\Sigma f_i e_i)$.)

Not in general exact. However, recall that $f.$ is a <u>regular sequence</u> if all the maps

$$A \xrightarrow{\cdot f_0} A, \qquad A/f_0 \xrightarrow{\cdot f_1} A/f_0, \quad \dots$$

are injective.

<u>Prop.</u> Assume A regular (or C-M), and the f_i are a regular sequence. Then

$$K. = K(f_0, \ , f_n) \text{ acyclic} \qquad (\text{Ref.}$$

<u>Ex.</u> This applies when

$$A = S. = k[T_0, \ , T_n], \quad f_i = T_i^M \qquad \square$$

Now let's return to Cech complex $C^\bullet$,

$$\begin{array}{ccccccc} S & \longrightarrow & \bigoplus_i S_{T_i} & \longrightarrow & \bigoplus S_{T_{i_0} T_{i_1}} & \longrightarrow & \\ \| & & \| & & \| & & \\ C^{-1} & & C^0 & & C^1 & & \end{array} \quad ; \text{ call this } \tilde{C}^\bullet$$

Adding term $C^{-1} = S$ on left, we want to show to show this is acyclic.

Now choose $M > 0$, and consider "truncation" of $\tilde{C}^\bullet$, defined by

$$\tilde{C}^p_M = \prod_{i_0 < \cdots < i_p} S \cdot \frac{1}{(T_{i_0} \cdots T_{i_p})^M}$$

(ie we bound powers of vars in denoms.)

These fit together in complex $\tilde{C}_M$:

$$\begin{array}{ccccccc}
C^{-1}_M & & C^0_M & & C^1_M & & \\
\| & & \| & & \| & & \\
S & \longrightarrow & \bigoplus S \cdot \frac{1}{T_i^M} & \longrightarrow & \Lambda^2\left(\bigoplus S \frac{1}{T_i^M}\right) & \longrightarrow & \\
1 & \longmapsto & \sum T_i^M \frac{e_i}{T_i^M} & & & &
\end{array}$$

We see:

Up to grading, $\tilde{C}^\bullet_M$ can be identified with the Koszul cx on $T_0^M, \ldots, T_n^M$.

So by Prop, $H^i(\tilde{C}^\bullet_M) = 0 \quad i < n$.

But $\tilde{C}^\bullet = \varinjlim_M \tilde{C}^\bullet_M$, follows that $\tilde{C}^\bullet$ acyclic.

QED

Serre's Thms.

Thm. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. Then

(a). The cohom groups $H^i(\mathbb{P}, \mathcal{F})$ are finite dim v.s. $/k$ $\forall i$

(b) $\exists\, m_0 = m_0(\mathcal{F})$ s.t.

$$H^i(\mathbb{P}, \mathcal{F}(m)) = 0 \quad \text{for } m \geq m_0,\ i > 0.$$

Pfs will be by descending induction on i. If $i > n$ nothing to prove.

(1°). Prove for $i = n$.

- We know that $\exists\, m_1 \gg 0$ st. $\mathcal{F}(m_1)$ glob gen.

- So have:

$$\mathcal{O}_{\mathbb{P}^n}^N \longrightarrow \mathcal{F}(m_1) \to 0$$

Equiv:

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^N(-m_1) \longrightarrow \mathcal{F} \to 0 \qquad (*)$$

Now observe that $H^n(\mathbb{P}^n, \bullet)$ is right exact on seqs of coh sheaves, since $H^{n+1}(\mathbb{P}^n, \text{any coh}) = 0$. So from $(*)$ see

$$H^n(\mathcal{O}_{\mathbb{P}^n}(-m_1))^{\oplus N} \twoheadrightarrow H^n(\mathcal{F})$$

But group on left is f. dim by our computation of $H^n(\text{line})$, so

So $H^n(\mathbb{P}, \mathcal{F})$ f. dim'l.

Also, know $H^n(\mathcal{O}_{\mathbb{P}^n}(k)) = 0$ for $k > -n$. So see

$$H^n(\mathbb{P}, \mathcal{F}(m)) = 0, \quad m \geq m_1 - n.$$

(2°). We assume Thm known $\forall$ coh sheaves and $i = n$. Now we prove for $i = n-1$.

Get from (*) an ex seq:

$$H^{n-1}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(-m_1)^N) \longrightarrow H^{n-1}(\mathbb{P}^n, \mathcal{F}) \longrightarrow H^n(\mathcal{F}_1)$$

By comp of cohom of lbs and ind hypoth applied to $H^n(\mathcal{F}_1)$, we see that outer terms are f. dim so

$$\dim H^{n-1}(\mathcal{F}) < \infty$$

Continuing, get finite dimensionality of $H^i(\mathbb{P}, \mathcal{F})$ $\forall i$.

(3°). Again applying the case $i = n$ to $\mathcal{F}_1$, we can find m_2 s.t.

$$H^n(\mathcal{F}_1(m)) = 0 \quad \text{for } m \geq m_2$$

and if $n-1 \geq 1$, then also

$$H^{n-1}(\mathcal{O}_{\mathbb{P}^n}(m - m_1)) = 0 \quad \text{for } m \geq m_2$$

So get $H^{n-1}(\mathbb{P}^n, \mathcal{F}(m)) = 0$ $m \geq m_2$. Again conclude by descending ind on i.

Ex. Consider sm rat curve

$$\mathbb{P}^1 \cong C \hookrightarrow \mathbb{P}^3$$

of deg d, embedded by

$$f_0, f_1, f_2, f_3 \in k[s,t]_d.$$

We discussed before the map

$$\begin{array}{cccccc} \rho_k : & H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k)) & \xrightarrow{\rho_k} & H^0(C, \mathcal{O}_C(k)) & = & H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(dk)) \\ & \Psi & & \Psi & & \\ & P_k(z_0, z_1, z_2, z_3) & \longrightarrow & P_k(f_0, \ldots, f_3) & & \end{array}$$

and we asked whether ρ_k surj for $k \gg 0$. But ρ_k is piece of coh. on seq assoc to

$$0 \to \mathcal{I}_C(k) \to \mathcal{O}_{\mathbb{P}^3}(k) \to \mathcal{O}_C(k) \to 0$$

and so we see

$$\operatorname{coker} \rho_k \underset{k \geq 0}{=} H^1(\mathbb{P}^3, \mathcal{I}_{\mathbb{P}^3}(k)).$$

So indeed ρ_k surj for $k \gg 0$

Sim: for any $X \subseteq \mathbb{P}^r$,

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}}(k)) \twoheadrightarrow H^0(X, \mathcal{O}_X(k)) \quad \text{for } k \gg 0.$$

Serre Duality:

Recall: we saw:

$$H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-d-n-1))^*,$$

(although we were a little vague on the duality). This is a special case of the

Serre Duality Thm: Let $X \subseteq \mathbb{P}^r$ be a smooth proj var of dim n. Define the canonical lb to be

$$\omega_X = \det(\Lambda^n T_X^*) \qquad \left(\begin{array}{l}\text{top exterior power}\\ \text{of cotang bundle.}\end{array}\right)$$

Then $H^n(X, \omega_X) = k$, and for any lb L on X, have

$$H^i(X, L) \cong H^{n-i}(X, \omega_X \otimes L^*)$$

Ex. $\omega_{\mathbb{P}^n} = \mathcal{O}(-n-1)$, so recover earlier duality

(duality fr perfect pairing $H^i(X, L) \otimes H^{n-i}(X, \omega_X \otimes L^*) \longrightarrow H^n(X, \omega_X) = k$)

There are many ways of proving this, but one is to reduce (non-trivially) to a computation w lbs on $\mathbb{P}^r$.

Ex. Consider sm hypersf $X_d \subseteq \mathbb{P}^{n+1}$, defined by $(F=0)$. View $\mathcal{O}_X$ as sheaf on $\mathbb{P}^{n+1}$. Then have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(-d) \xrightarrow{.F} \mathcal{O}_{\mathbb{P}^{n+1}} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

(ie. $d_{X/\mathbb{P}^{n+1}} = \mathcal{O}_{\mathbb{P}}(-d)$). Can show:

$$\omega_X = \mathcal{O}_X(d-n-1).$$

Fix $k \in \mathbb{Z}$ and $\otimes$ by $\mathcal{O}_{\mathbb{P}^n}(d-n-1-k)$. Get

$$0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-n-1-k) \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(d-n-1-k) \longrightarrow \mathcal{O}_X(d-n-1-k) \longrightarrow 0$$

Using our computation of coh for lbs find

$$\begin{array}{ccccc} 0 \to H^n(\mathcal{O}_X(d-n-1-k)) & \longrightarrow & H^{n+1}(\mathcal{O}(-n-1-k)) & \longrightarrow & H^{n+1}(\mathcal{O}_{\mathbb{P}^{n+1}}(d-n-1-k)) \\ & & \| & & \\ & & H^0(\mathbb{P}^{n+1}, \mathcal{O}(k))^* & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^{n+1}}(-d+k))^* \end{array}$$

ie. find

$$H^n(\mathcal{O}_X(d-n-1-k))^*$$
$$\|$$
$$\text{coker}\left(H^0(\mathcal{O}_{\mathbb{P}}(k-d)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}}(k))\right) = H^0(X, \mathcal{O}_X(d)) \;. \; \square$$

Riemann-Roch:

<u>Thm</u> (Classical RR): Let L = lb deg d on curve C of genus g. Then

$$h^0(L) - h^0(\omega_C \otimes L^*) = d + 1 - g$$
$$\| \text{ serre D}$$
$$h^0(L) - h^1(L).$$

There was a similar classical result for sfs, but until the 1950's it wasn't even clear what the right question was in higher dims

<u>Euler char</u>: Let $\mathcal{F}$ be a coh sheaf on proj var X. Define

$$\chi(X, \mathcal{F}) = \sum_{i \geq 0} (-1)^i \dim H^i(X, \mathcal{F}) \quad \in \mathbb{Z}$$

Long ex seq of sheaf cohom $\Rightarrow$ χ additive across exact seqs:

Given: $\quad 0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$

have

$$\chi(X, \mathcal{F}) = \chi(X, \mathcal{F}_1) + \chi(X, \mathcal{F}_2).$$

So χ well behaved. Eg:

Classical RR on curves: $\chi(C, L) = \underbrace{\deg(L)}_{\text{top info about } L} + \underbrace{1 - g}_{\text{top info about } C}$

Serre – Kodaira: Suggested that for sm proj var X, RR problem was to compute $\chi(X, L)$ in terms of char classes of L, X.

Hirzebruch-R-R: For sm proj X of dim L,

$$\chi(X, L) = \int_X ch(L) \cdot Td(X)$$

cohom classes depending on L (eg $c_1(L), c_1(L)^2, \dots$)

expression in $c_i(X) = c_i(TX)$.

GAGA. We work $/\mathbb{C}$.

Write $\mathbb{P}^n$ for $\mathbb{P}^n/\mathbb{C}$ w Zar top, and $\mathbb{P}^n_{anal}$ for $\mathbb{P}^n$ w classical top. Sim, if

$$X \subseteq \mathbb{P}^n \text{ is alg subvar} \Rightarrow X_{an} \subseteq \mathbb{P}^n_{an} \text{ analy subv}$$
$$\mathcal{F} \text{ on } \mathbb{P}^n \text{ coh alg sheaf} \Rightarrow \mathcal{F}_{an} \text{ on } \mathbb{P}^n_{an}, \text{ coh. anal sheaf}$$

Have natural map

$$(*) \qquad H^i(\mathbb{P}^n, \mathcal{F}) \longrightarrow H^i(\mathbb{P}^n_{an}, \mathcal{F}_{an})$$

(alg cocycle determines anal one)

Thm (Serre) Any coherent analy sheaf on $\mathbb{P}^n_{an}$ is of form

$$\mathcal{F}_{an} \text{ for some coh alg sheaf } \mathcal{F} \text{ on } \mathbb{P}^n$$

Moreover, maps $(*)$ are isoms

<u>Cor.</u> (Chow's Thm). Any analy subvar $V \subseteq \mathbb{P}^n_{an}$ is algebraic subset

(<u>Pf</u>: $\mathcal{O}_V$ is (coh) analy sheaf hence the analytification of coh alg sheaf. So support is alg subvar.)

<u>Idea</u>: (1). Prove thm for $\mathcal{O}_{\mathbb{P}^n}(k)_{analy}$: ie. same cohom as $\mathcal{O}_{\mathbb{P}^n}(k)$

(2). Prove analogue of Serre's Thms for coh analy sheaves. Esp:

If $\mathcal{F}'$ is any coh analy sheaf on $\mathbb{P}^n$, then

$$\mathcal{F}' \otimes \mathcal{O}_{\mathbb{P}^n}(m)_{an} \text{ glob gen for } m \gg 0.$$

(These are much deeper in analy categ)

Granting this, one argues as follows:

- Let $\mathcal{F}'$ = coh analy sheaf. Then by (2), can write $\mathcal{F}'$ as a global coker of form:

$$\mathcal{O}_{\mathbb{P}^n}(-m_1)^{N_1}_{an} \xrightarrow{u'} \mathcal{O}_{\mathbb{P}^n}(-m_0)^{N_0}_{an} \longrightarrow \mathcal{F}' \longrightarrow 0$$

<u>Claim:</u> u' is given by a matrix of homog polys

<u>Pf</u>: u' is given by mx of global sections of $\mathcal{O}_{\mathbb{P}^n}(m_1-m_0)_{an}$, and by (1), these are homog polys of deg m_1-m_0.

- Follows that u' is analy of alg map

$$\mathcal{O}_{\mathbb{P}^n}(-m_1)^{N_1} \xrightarrow{u} \mathcal{O}_{\mathbb{P}^n}(-m_0)^{N_0} \longrightarrow \mathcal{F} \longrightarrow 0,$$

so

so

$$\mathcal{F}' = \mathcal{F}_{an}.$$

- Remains to show that get isom on cohom. Again argue by descending induction on i:

- For $i = n$, have

$$\begin{tikzcd} H^n(\mathcal{O}_{\mathbb{P}^n_{an}}(-m_1)_{an})^{N_1} \arrow[r] & H^n(\mathcal{O}_{\mathbb{P}^n_{an}}(-m_0)_{an})^{N_0} \arrow[r] & H^n(\mathcal{F}_{an}) \arrow[r] & 0 \\ H^n(\mathcal{O}_{\mathbb{P}^n}(-m_1))^{N_1} \arrow[r] \arrow[u, "\cong"] & H^n(\mathcal{O}_{\mathbb{P}^n}(-m_0)^{N_0}) \arrow[r] \arrow[u, "\cong"] & H^n(\mathcal{F}) \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

- Follows that RH vert map also an isom. Now proceed by inductn