Math 583

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Intro. to Riemann Sfs

Fall 2021
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I. Elliptic Functions

Refs: Ahlfors, Ch. 7
(Hartshorne, Ch IV, §3)

By way of Intro, will briefly recall classical theory of elliptic fns. This will allow us to understand Riemann sfs of genus $g = 1$. Much of rest of course will be devoted to extending this picture to sfs of genus $g > 1$.

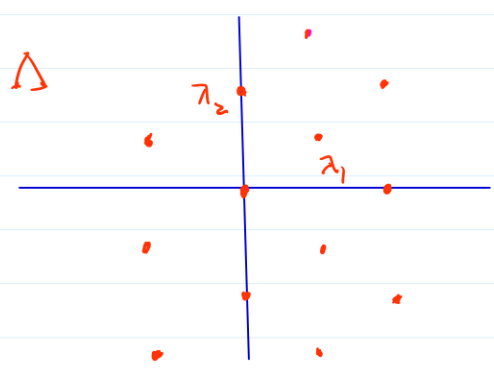
- Fix

$$\lambda_1, \lambda_2 \in \mathbb{C}, \qquad \text{lin ind } / \mathbb{R}.$$

- Let

$$\Lambda = \Lambda(\lambda_1, \lambda_2) = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$$

be lattice that they span.



$\Lambda \subseteq \mathbb{C}$ is discrete sg, abstractly free abelian grp of rk $= 2$.

Def. An elliptic fn (wrt Λ) is mero fn $f(z)$ on $\mathbb{C}$ that is periodic wrt Λ:

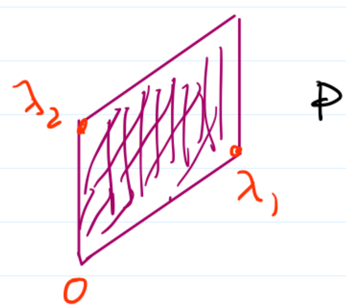
$$f(z+\lambda) = f(z) \qquad \forall\ z \in \mathbb{C},\ \lambda \in \Lambda.$$

Goal: Understand existence and properties of such fns.

(Rmk: $\{\text{Ell fns}\} = \left\{ \begin{array}{l} \text{mero fns on} \\ X = \mathbb{C}/\Lambda : \end{array} \right.$  $\}$)

Prop. The only analytic ell fns are constants.

Notation: We denote by $P \subseteq \mathbb{C}$ the closed region bounded by the parallelogram spanned by λ_1, λ_2 ("period parallelogram")



$P_a = P + a$: transl by $a \in \mathbb{C}$.

Proof of Prop: Assume $f(z)$ entire and Λ-periodic. Then $f(z)$ bounded on P, hence (by periodicity) bdd on $\mathbb{C}$. So $f(z)$ const by Liouville's Thm.

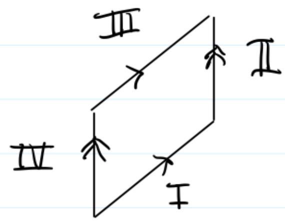
Now assume $f(z)$ mero and Λ-periodic. Choose $a \in \mathbb{C}$ s.t. $f(z)$ has no zeroes or poles on ∂P_a.

Thm ("Residue Thm") Sum of residues of $f(z)$ in P_a is zero.

Pf. By classical residue thm,

$$\sum \begin{pmatrix} \text{residues of} \\ f(z) \text{ in } P_a \end{pmatrix} = \frac{1}{2\pi i} \int_{\partial P_a} f(z)\, dz$$

Decompose ∂P_a into four segments as shown:



$$\partial P_a = \mathrm{I} + \mathrm{II} - \mathrm{III} - \mathrm{IV},$$

and by periodicity:

$$\int_{\mathrm{I}} = \int_{\mathrm{III}}, \quad \int_{\mathrm{II}} = \int_{\mathrm{IV}}. \qquad \text{QED}$$

Cor. There is no elliptic fn having a single simple pole in P_a

Prop. Any non-constant ell fn has same number of zeroes as poles (counting multiplicities) in P_a.

Pf. Consider the ell fn $f'(z)/f(z)$. Have

$$\mathrm{res}_b\left(\frac{f'}{f}\right) = \mathrm{ord}_b(f(z)) \qquad \begin{pmatrix} \text{pos if } b \text{ a zero} \\ \text{neg if } b \text{ a pole} \end{pmatrix}$$

But

$$\sum \mathrm{res}(f'/f) = 0$$

We can also say something about the <u>location</u> of the zeroes and poles of $f(z)$.

<u>Thm</u> ("First half of Abel's Thm") Let $f(z)$ be a non-const ell fn, and let

$$\begin{matrix} p_1, \dots, p_d \\ q_1, \dots, q_d \end{matrix} \in P_\alpha$$

be the zeroes and poles of $f(z)$ inside P_α (repeated according to their multiplicities). Then

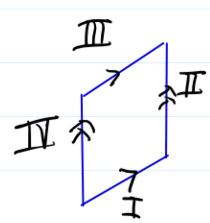
$$\Sigma p_i \equiv \Sigma q_i \pmod{\Lambda}$$

<u>Pf.</u> By classical residue thm,

$$\Sigma p_i - \Sigma q_i = \frac{1}{2\pi i} \int_{\partial P} z \frac{f'(z)}{f(z)}\, dz.$$

As before, write

$$\partial P_\alpha = \mathrm{I} + \mathrm{II} - \mathrm{III} - \mathrm{IV}$$



Let's compare

$$\int_{\mathrm{I}} z \cdot \frac{f'(z)}{f(z)} \quad \& \quad \int_{\mathrm{III}} z \cdot \frac{f'(z)}{f(z)}$$

Note:

$$z \in \text{Side}(\mathrm{I}) \iff z + \lambda_2 \in \text{side}(\mathrm{III}).$$

So

So.

$$\int_{III} z \cdot \frac{f'(z)}{f(z)} = \int_{I} (z+\lambda_2) \frac{f'(z+\lambda_2)}{f(z+\lambda_2)}$$

$$= \int_{I} (z+\lambda_2) \frac{f'(z)}{f(z)}.$$

$$= \int_{I} z \frac{f'(z)}{f(z)} + \lambda_2 \int_{I} \frac{f'(z)}{f(z)}$$

So

$$\frac{1}{2\pi i}\left(\int_{III} - \int_{I} \right) = \lambda_2 \cdot \frac{1}{2\pi i} \int_{I} d\log f(z).$$

But

$$\frac{1}{2\pi i} \int_{I} d\log f(z) = \text{winding no about } 0 \text{ of } f|I$$

$$\in \mathbb{Z}.$$

So $\frac{1}{2\pi i}\left(\int_{III} - \int_{I} \right) \in \mathbb{Z} \cdot \lambda_2$. Sim. $\frac{1}{2\pi i}\left(\int_{IV} - \int_{II} \right) \in \mathbb{Z} \cdot \lambda_1$

So

$$\frac{1}{2\pi i} \int_{\partial P_\alpha} \in \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2. \quad \text{QED}$$

Construction of Elliptic Fns

Note that so far we haven't established the existence of non-trivial ell fns.

Will do so via Weierstrass fn $\wp(z)$.

This term thrown in to ensure conv. Eg $\sum \frac{1}{\lambda^2}$ not convgt.

Define:

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right) \qquad (*)$$

Thm. RHS of $(*)$ converges unif on compact sets disj from Λ to mero fn $\wp(z)$. This is elliptic fn having poles of order 2, w. zero residue, at the pts of Λ, and no other sings. Moreover, $\wp(z)$ is an even fn.

Sketch of pf: For convergence see Ahlfors, Ch. 7, §3.1.

- Statement about sings is clear, as is fact that $\wp(z)$ is even
- Need to prove $\wp(z)$ actually periodic wrt Λ. For this, consider

$$\wp'(z) = -\frac{2}{z^3} - \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \frac{2}{(z-\lambda)^3}$$

$$= -2 \sum \frac{1}{(z-\lambda)^3}.$$

This is clearly periodic wrt Λ. Now fix $\lambda \in \Lambda$.

Then

$$\left(\wp(z+\lambda) - \wp(z)\right)' \equiv 0$$

So for each $\lambda \in \Lambda$,

$$\wp(z+\lambda) - \wp(z) = \text{const } c_\lambda.$$

Take $\lambda = \lambda_1$, plug in $z = -\frac{\lambda_1}{2}$:

$$\wp\left(\frac{\lambda_1}{2}\right) - \wp\left(-\frac{\lambda_1}{2}\right) = c_{\lambda_1}:$$

$\wp$ even $\Rightarrow c_{\lambda_1} = 0$. Sim, $c_{\lambda_2} = 0$. So $\wp$ periodic.

Formulary:

$$\boxed{\begin{aligned} \wp(z) &= \frac{1}{z^2} + \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right) \\ \wp'(z) &= -2 \cdot \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3} \end{aligned}}$$

<u>Prop.</u> Laurent series for $\wp(z)$ is:

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1)\, G_{2k+2}\, z^{2k},$$

where

$$G_{2m} = \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \frac{1}{\lambda^{2m}} \qquad (m > 1)$$

Pf. Say $|z| < |\lambda|$. Then

$$\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}\cdot\left(\frac{1}{(\frac{z}{\lambda}-1)^2} - 1\right) \qquad (*)$$

Now $\frac{1}{(r-1)^2} = 1 + 2r + 3r^2 + \ldots$ for $|r|<1$, so

$$\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}\left(2\cdot\frac{z}{\lambda} + 3\left(\frac{z}{\lambda}\right)^2 + \ldots\right)$$

Now plug this into series defining $\wp(z)$. Terms involving odd powers of z cancel (since $\wp(z)$ even), so see:

$$\wp(z) = \frac{1}{z^2} + \left(\sum_{\lambda\neq 0}\frac{1}{\lambda^4}\right)3z^2 + \left(\sum_{\lambda\neq 0}\frac{1}{\lambda^6}\right)5z^4 + \ldots$$

QED

Note:

$\left\{\begin{array}{l}\text{all ell. fns}\\ \text{wrt fixed } \Lambda\end{array}\right\}$ form a field.

Exercise: This field is generated by $\wp$ & $\wp'$, ie.

$$\left\{\begin{array}{c}\text{Ell. fns wrt}\\ \Lambda\end{array}\right\} = \mathbb{C}(\wp(z), \wp'(z))$$

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We next want to show that $\wp$ & $\wp'$ satisfy a poly relation, ie. that $\wp(z)$ satisfies a differential eqn of a certain shape

We'll give two proofs: first by counting dims, and then by explicit computation.

Def: Given $k \geq 0$, define

$$V_k = \left\{ \begin{array}{l} \text{all ell} \\ \text{fns } f \end{array} \middle| \begin{array}{l} f \text{ has poles of order} \leq k \text{ on } \Lambda \\ \text{analy off } \Delta \end{array} \right\}$$

This is a $\mathbb{C}$ v.s, and

$$\begin{array}{ccccccc} V_0 & \subseteq & V_1 & \subseteq & V_2 & \subseteq & \cdots \\ \| & & \| & & & & \\ \mathbb{C} & & \mathbb{C} & & & & \end{array}$$

Thm ("Riemann-Roch").

If $k \geq 1$,

$$\dim_{\mathbb{C}} V_k = k.$$

Pf. Ok for $k=1$. So suffices to prove

$$(*) \qquad \dim V_{i+1} = \dim V_i + 1 \qquad \text{for } i \geq 1.$$

(1°). Show $\dim V_{i+1} > \dim V_i$:

- Can find $a, b \geq 0$ s.t. $i+1 = 2a+3b$. Then

$$\wp(z)^a \cdot \wp'(z)^b \in V_{i+1}, \notin V_i$$

(2°). Show $\dim V_{i+1} \leq \dim V_i + 1$:

- Fix $f, g \in V_{i+1}$. Will show f, g lin dep in V_{i+1}/V_i.
- Write

$$f = \frac{a}{z^{i+1}} + HOT, \quad g = \frac{b}{z^{i+1}} + HOT$$

Then $bf - ag \in V_i$, as claimed. □

Now let's start writing down elements:

$$\begin{array}{ccccccccccc} V_0 = V_1 & \subset & V_2 & \subset & V_3 & \subset & V_4 & \subset & V_5 & \subset & V_6 \\ \cup & & \cup & & \cup & & \cup & & \cup & & \cup \\ 1 & & \wp & & \wp' & & \wp^2 & & \wp\wp' & & \wp^3, (\wp')^2 \end{array}$$

<u>Cor</u> (of RR). The indicated fns are lin dep/$\mathbb{C}$. I.e. $\exists$ a poly relation

$$(\wp')^2 = a\wp^3 + b\wp\wp' + c\wp^2 + d\wp' + e\wp + f$$

for some $a, b, c, d, e, f \in \mathbb{C}$.

<u>Thm</u>. In fact,

$$(\wp')^2 = 4\wp^3 + 60 \cdot G_4 \wp + 140 G_6.$$

<u>Notation</u>: set

$$g_2 = 60 G_4 = 60 \sum_{\lambda \neq 0} 1/\lambda^4$$

$$g_3 = 140\, G_6 = 140 \sum_{\lambda \neq 0} 1/\lambda^6 .$$

Then

$$\boxed{(\wp')^2 = 4\wp^3 - g_2 \wp - g_3}$$

<u>Rmk:</u> Write formally $w = \wp(z)$. Then the diff eqn says

$$\frac{dw}{dz} = \sqrt{4w^3 - g_2 w - g_3} ,$$

so

$$z = \int \frac{dw}{\sqrt{4w^2 - g_2 w - g_3}}$$

("elliptic integral": they arise when you try to compute arclength of ellipse)

i.e.

$$z - z_0 = \int_{\wp(z_0)}^{\wp(z)} \frac{dw}{\sqrt{4w^2 - g_2 w - g_3}}$$

i.e. "$\wp(z)$ arises by inverting elliptic integral".

Can make this precise: see Ahlfors, p. 268 and Ch 6, §2

Will later see deeper interpretation as integrals of 1-form on elliptic curve

Sketch of Pf of Thm:

Compute:

$$\wp(z) = \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + HOT$$

$$\wp'(z) = -\frac{2}{z^3} + 6G_4 z + 20G_6 z^3 + HOT$$

$$\wp'(z)^2 = \frac{4}{z^6} - \frac{24}{z^2} - 80G_3 + HOT$$

$$4\wp^3 = \frac{4}{z^6} + \frac{36}{z^2} + HOT$$

$$60G_2\wp = \frac{60G_2}{z^2} + (\quad)z^2 + HOT$$

So:

$$(\wp')^2 - 4\wp^3 + 60G_2\wp + 140G_3$$

is analy ell fn which vanishes at 0. Hence $\equiv 0$ $\square$

Next idea: study map

$$\begin{array}{cccc} \varphi: & \mathbb{C}-\Lambda & \longrightarrow & \mathbb{C}^2 \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & z & \longmapsto & (\wp(z), \wp'(z)) \end{array}$$

Diff eqn means image lies on alg curve

$$y^2 = 4x^3 - g_2 x - g_3.$$

It will be convenient to have at hand language of complex manifolds

Complex Manifolds

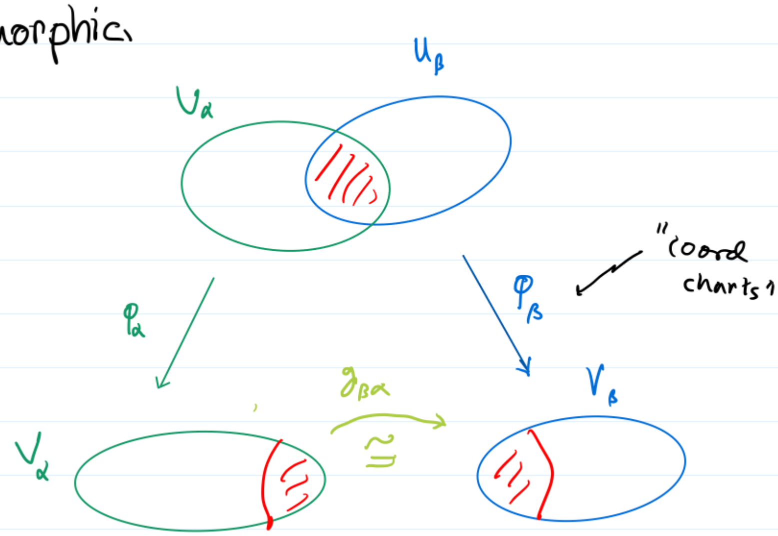
Def. (slightly informal) A complex manifold of (complex) dim n is a Hausdorff space X, with a covering by open sets $U_\alpha \subseteq X$, together w. homeos:

$$\varphi_\alpha : U_\alpha \longrightarrow V_\alpha \overset{\text{open}}{\subseteq} \mathbb{C}^n \qquad (\text{"charts"})$$

s.t. the transition fns

$$\begin{array}{ccc} V_\alpha & & V_\beta \\ \cup\!\mid & & \cup\!\mid \\ \varphi_\alpha(U_\alpha \cap U_\beta) & \xrightarrow[g_{\beta\alpha}]{\qquad\qquad} & \varphi_\beta(U_\alpha \cap U_\beta) \end{array}$$

are biholomorphic.



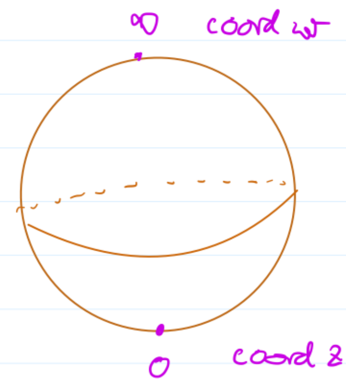
Yoga: any concept that is invariant under biholo mappings makes sense on complex mfld.

Fine Print: strictly speaking, should deal w equivalence classes of such data, so we can tell when they define "same" complex mfld.

Def. Riemann surface is complex mfld of dim = 1.

Exs. (1) Riemann sphere

$$\mathbb{P}^1 = S = \mathbb{C} \cup \{\infty\},$$



Finite coord z ($z=0$ is origin)

Coord near ∞: w ($w=0$ is ∞)

$\boxed{w = \frac{1}{z}}$ (trans fn)

ie. if you glue $\mathbb{C}-\{0\}$ & $\mathbb{C}-\{0\}$ via $z \leftrightarrow 1/w$

$$\begin{array}{ccccc} \text{coord } w & \mathbb{C}-\{0\} & \xleftrightarrow{\text{glue}} & \mathbb{C}-\{0\} & \text{coord } w \\ & z & \longleftrightarrow & 1/w, & \end{array}$$

get sphere

Ex 2: $\Lambda \subseteq \mathbb{C}$ lattice:

$$X = \mathbb{C}/\Lambda : \text{ topologically } X = \text{torus}$$

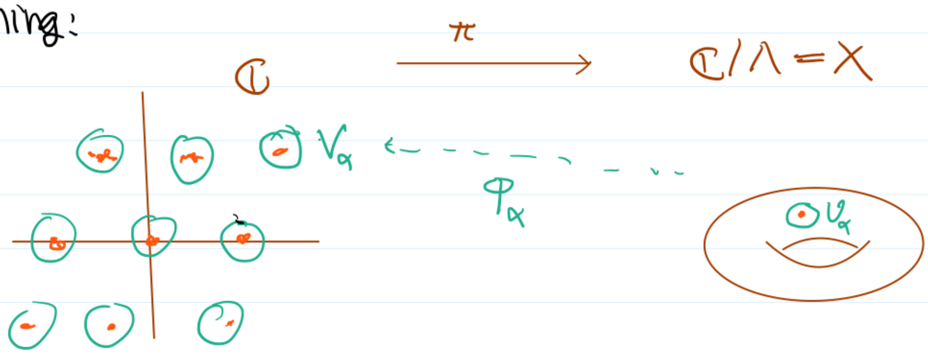
How do we define coord charts?

Consider quotient

$$\pi : \mathbb{C} \longrightarrow X = \mathbb{C}/\Lambda \quad \text{(covering space)}$$

Plan: "Require π to be local analy isom"!

Meaning:



· Choose small open set $U_\alpha \subseteq X$ s.t.

$$\pi^{-1}(U_\alpha) = \coprod V_{\alpha,i}, \qquad V_{\alpha,i} \xrightarrow{\cong} U_\alpha$$

· Pick one $V_{\alpha,i}$ – call it V_α ⟶ so

$$V_\alpha \xrightarrow[\pi]{\cong} U_\alpha \quad \text{(diffeo)}$$

· Use $\pi^{-1}|U_\alpha$: $U_\alpha \xrightarrow[\varphi_\alpha]{\cong} V_\alpha \subseteq \mathbb{C}$ as local coord.

<u>Exerc:</u> trans fns of form $V_\alpha \longrightarrow V_\beta$, $z \longmapsto z + c_{\alpha\beta}$

$c_{\alpha\beta} \in \mathbb{C}$

(3). <u>Projective Space</u>:

<u>Ask</u>: How can we construct complex manifold that compactifies $\mathbb{C}^n$?

I.e. Is there a natural generalization of the construction

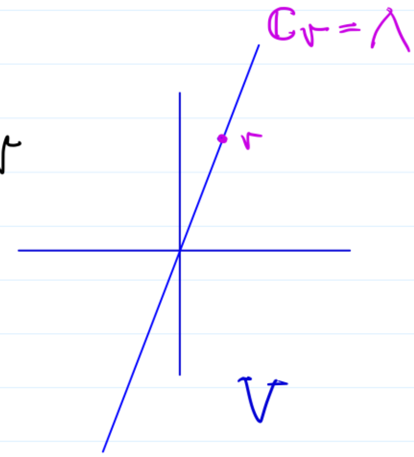
$$\mathbb{C} \subseteq \mathbb{C} \cup \{\infty\} = \text{Riemann sphere?}$$

(Rmk: There are actually many compactifications of $\mathbb{C}^n$, but there is a "simplest" one, which we discuss.)

<u>Def 1</u>. Consider $n+1$ $\mathbb{C}$-vector space $V = \mathbb{C}^{n+1}$.

$$\mathbb{P}^n = \mathbb{CP}^n = \left\{ \begin{matrix} \text{1-dim vector subspaces} \\ \text{of } V \end{matrix} \right\}$$

(So far this only defines $\mathbb{P}^n$ as a set)



Note: to specify 1-dim subspace $\Lambda \subseteq V$
need to choose

$$0 \neq v \in V,$$

and then

$$\Lambda = \mathbb{C} \cdot v ;$$

However, there is ambiguity, because

$$\mathbb{C} v = \mathbb{C} v' \iff v' = \lambda v, \quad \lambda \neq 0.$$

More formally:

$\mathbb{C}^*$ acts on $V = \mathbb{C}^{n+1}$ by scalar mult.

Then

$$\{1\text{-dim subspaces}\} \leftrightarrow (V - \{0\}) / \mathbb{C}^*$$

i.e.

<u>Def. 2:</u> $\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\}) / \mathbb{C}^*$

Then have.

$$h: \mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{P}^n \quad : \text{ quotient by } \mathbb{C}^*\text{-action}$$

$$h(v) = \mathbb{C}\, v,$$

and

$$h^{-1}(\mathbb{C} \cdot v) = \{ \lambda \cdot v \mid \lambda \neq 0 \} = \mathbb{C}^*$$

We topologize $\mathbb{P}^n$ w quotient topology.

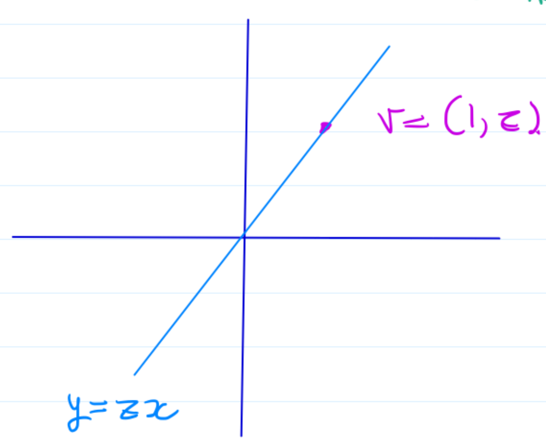
In fact, $\exists!$ str of complex mfld on $\mathbb{P}^n$ that makes h hol map, but we'll take different approach

<u>Ex</u>: $n = 1$

- We can specify any non-vertical line by taking $v = (1, z)$, i.e.

$$\mathbb{C}\, v = \{ y = z \cdot x \}$$

so specify by slope z.



So $\{ \text{all lines} \} - \{ \text{one vertical line} \} \cong \mathbb{C}$. (slope of line)

"So"

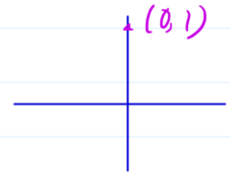
$$\mathbb{P}^1 = \mathbb{C} \cup \{ \text{vertical line} \} = \mathbb{C} \cup \{\infty\}$$

This suggests:

$$\mathbb{P}^1 = \text{Riemann sphere}.$$

To see what's happening near vertical line, consider

$$v' = (w, 1) \quad \text{for } w \in \mathbb{C}$$



(vertical line $\Leftrightarrow$ $w = 0$)

Here, see

$$\{\text{all lines}\} - \{\text{horiz line}\} \longrightarrow \mathbb{C} \qquad (\text{line } x = wy)$$

$$\mathbb{C}(w,1) \longrightarrow w$$

Note that

$$\mathbb{C}(1, z) = \mathbb{C}(w, 1) \Longleftrightarrow w = \frac{1}{z}.$$

So indeed, $\mathbb{P}^1$ obtained by gluing $\mathbb{C} - \{0\}$ to $\mathbb{C} - \{0\}$ via

$$z \longleftrightarrow 1/w$$

<u>Homogeneous coords</u> -

Def 2 gives natural way to describe pts in $\mathbb{P}^n$ via "homog coords"

By Def 2,

$$\mathbb{P}^n = \mathbb{C}^{n+1} - \{0\} / \mathbb{C}^*$$

$$= \{(a_0, \ldots, a_n) \in \mathbb{C}^{n+1}, \text{ not all } a_i = 0\} / \sim$$

where

$$(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n) \text{ if } b_i = \lambda a_i \text{ some } \lambda \neq 0$$

ie.

Point in $\mathbb{P}^n$ described by "homog coords"

$$[a_0, \ , a_n] \quad (\text{not all} = 0)$$

where

$$[a_0, \ , a_n] = [b_0, \ , b_n] \text{ if}$$

$$b_i = \lambda a_i \text{ some } \lambda \neq 0.$$

Let's use this to write down complex coord charts on $\mathbb{P}^n$.

- Take $[a] = [a_0, \ldots, a_n] \in \mathbb{P}^n$.
- Some $a_i \neq 0$; say $a_0 \neq 0$. Then

$$[a_0, \ldots, a_n] = [1, \frac{a_1}{a_0}, \ , \frac{a_n}{a_0}]$$

$\frac{a_i}{a_0} \in \mathbb{C}$ indep of repr of $[a]$ as homog vector

- Formally:

Define
$$\begin{array}{ccc} \mathbb{P}^n \supseteq U_0 = & \{[a_0, \ , a_n] \mid a_0 \neq 0\} \\ \Big\downarrow \varphi_0; & \Big\downarrow \\ \mathbb{C}^n & (\frac{a_1}{a_0}, \ , \frac{a_n}{a_0}) \end{array}$$

Similarly:

$$\begin{array}{ccc} \mathbb{P}^n \supseteq U_i & = \{ [a_0, \ldots, a_n] \mid a_i \neq 0 \} \\ \big\downarrow \cong & \big\downarrow \\ \mathbb{C}^n & \left(\frac{a_0}{a_i}, \ldots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \ldots, \frac{a_n}{a_i} \right) \end{array}$$

<u>Prop:</u> These coord charts give $\mathbb{P}^n$ str. of complex manifold
ie. trans fns are biholomorphic

<u>Ex.</u> $n=1$:

$$\mathbb{P}^1 \ni [a_0, a_1] = p$$

$$U_0 : a_0 \neq 0, \quad \text{coord } z = \frac{a_1}{a_0}, \quad p = [1, z]$$

$$U_1 : a_1 \neq 0 \quad \text{coord } w = \frac{a_0}{a_1}, \quad p = [w, 1]$$

<u>Exerc:</u> Write out all trans fns for $\mathbb{P}^2$

<u>Prop:</u> $\mathbb{P}^n$ is compact.

<u>Pf:</u> Given

$$a = (a_0, \ldots, a_n) \in \mathbb{C}^{n+1} - \{0\},$$

write

$$\|a\| = \left(\sum |a_i|^2 \right)^{1/2}$$

Then

$$[a] \sim \left[\frac{a}{\|a\|} \right], \text{ and } \frac{a}{\|a\|} \in S^{2n+1} \subseteq \mathbb{C}^{2n}$$

Given $u, v \in S^{2n+1}$, have

$$u = \lambda v \iff |\lambda| = 1, \text{ ie. } \lambda \in S^1 \subseteq \mathbb{C}$$

Ie

S^1 acts on S^{2n+1} by scalar mult, and

$$\mathbb{CP}^n = S^{2n+1}/S^1 \ :$$

Vocab: Quotient

$$h : S^{2n+1} \longrightarrow \mathbb{P}^n$$

is called Hopf fibration (eg $S^3 \xrightarrow{S^1} S^2$)

Rmk: $\mathbb{P}^n$ is indeed a compactif. of $\mathbb{C}^n$,

$$\begin{array}{ccc} \mathbb{C}^n & \subseteq & \mathbb{P}^n \\ \upsilon & & \upsilon \\ (z_1, \ , z_n) & \longmapsto & [1, z_1, \ , z_n] \end{array}$$

Provisional Def: Projective alg curve is 1-dim complex submfld of $\mathbb{P}^n$.

Rmk: If

$$F_d = F_d(z_0, \ , z_n) \in \mathbb{C}[z_0, \ , z_n]$$

is homog poly of deg d, and $[a] \in \mathbb{P}^n$ is a pt, the value $F(a) \in \mathbb{C}$ is not-well defined. However the vanishing or not of $F(a)$ is indep of homog repr of $[a]$.

Hence if

$$F_1, \quad , F_p \in \mathbb{C}[z_0, \quad , z_n]$$

are homog (of various degrees), the zero-locus

$$\mathrm{Zeroes}(F_1, \quad , F_p) = \{ [a] \mid F_i(a) = 0 \text{ all } i \} \subseteq \mathbb{P}^n$$

is well-defined. Such sets are called <u>projective algebraic sets</u>. An amazing Thm of Chow says that any compact cx submfld of $\mathbb{P}^n$ is of this form,

<u>Chow's Thm:</u> Let $X \subseteq \mathbb{P}^n$ be a compact complex submanifold. Then X is alg, ie. $\exists$ homog polys

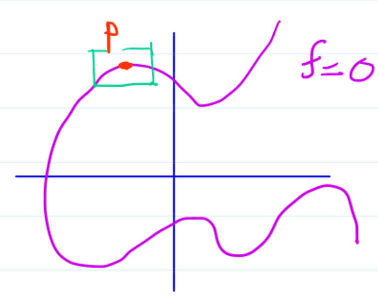
$$F_1, \quad , F_p \in \mathbb{C}[z_0, \quad , z_n] \text{ s.t.}$$

$$X = \mathrm{Zeroes}(F_1, \quad , F_p)$$

In partic, a compact Riemann sf $X \subseteq \mathbb{P}^n$ is alg var of $\dim_{\mathbb{C}} = 1$, i.e., an algebraic curve,

<u>Ex.</u> (Alg curves in $\mathbb{C}^2$). Let $f(x,y) \in \mathbb{C}[x,y]$ be poly. Let

$$X = \{ f(x,y) = 0 \} \subseteq \mathbb{C}^2$$



Assume that for every $p = (a,b) \in X$, either

$$\frac{\partial f}{\partial x}(p), \ \frac{\partial f}{\partial y}(p) \neq 0$$

Then X is 1-dim (non-compact) submfld of $\mathbb{C}^2$.

<u>Sketch</u>. Pt is that analy fn is true for analy fns. So if eg

$$\frac{\partial f}{\partial y}(p) \neq 0,$$

can find nbd $a \in U \subseteq \mathbb{C}$ and analy fn $\phi = \phi(z)$ st

$$X \cap \{\text{nbd of } p\} = \{\text{graph of } \phi \text{ on } U\},$$

which gives local coord. (Details for you!)

<u>Def</u>. X a complex mfld w coord charts $\varphi_i : U_i \longrightarrow V_i \subseteq \mathbb{C}^n$. A fn

$$f : X \longrightarrow \mathbb{C}$$

is <u>analy</u> or <u>holo</u> if all

$$f \circ \varphi_i^{-1} : V_i \longrightarrow \mathbb{C}$$

are holo.

$$\begin{tikzcd} & & X \arrow[r, "f"] & \mathbb{C} \\ & U_i \arrow[ur, phantom, "\subseteq"] \arrow[d, "\varphi_i"'] & & U_j \arrow[ul, phantom, "\supseteq"] \arrow[d, "\varphi_j"] \\ \mathbb{C}^n \supseteq & V_i & & V_j \subseteq \mathbb{C}^n \end{tikzcd}$$

<u>NB</u>: Note on overlaps

$$f \circ \varphi_i^{-1} \text{ holo} \Leftrightarrow f \circ \varphi_j^{-1} \text{ holo}$$

$$\left(f \circ \varphi_i^{-1} = (f \circ \varphi_j^{-1}) \cdot g_{ij} \right).$$

<u>Mero fns</u> defined similarly.

<u>Ex</u>. $\left\{ \begin{matrix} \text{ell} \\ \text{fns} \end{matrix} \right\} = \left\{ \begin{matrix} \text{mero fns} \\ \text{on } \mathbb{C}/\Lambda \end{matrix} \right\}$

<u>Def</u>: A cont mapping

$$f : X \to Y$$

bet cx mflds is <u>analy</u> or <u>holo</u> if it's given in local coords by analy fns.

i.e. After passing to refinements, look for charts

$$\varphi_i : U_i \longrightarrow V_i \subseteq \mathbb{C}^n \quad \text{for } X$$
$$\psi_j : W_j \longrightarrow O_j \subseteq \mathbb{C}^m \quad \text{for } Y$$

s.t.

$$f(U_i) \subseteq W_{j(i)}, \quad \text{and ask that}$$

$$\psi_{j(i)} \circ f \circ \varphi_i^{-1} : V_i \longrightarrow W_{j(i)}$$

be holo.

$$\begin{tikzcd} U_i \arrow[r, "f"] & W_{j(i)} \arrow[d, "\psi_{j(i)}"] \\ \mathbb{C}^n \supseteq V_i \arrow[u, "\varphi_i^{-1}"] \arrow[r, dashed] & W_j \subseteq \mathbb{C}^m \end{tikzcd}$$

<u>Ex.</u> $\pi : \mathbb{C} \longrightarrow \mathbb{C}/\Lambda$ holo.

Tues 8/30

Now return to $\Lambda \subseteq \mathbb{C}$, let $X = \mathbb{C}/\Lambda$. Define

$$\phi : X \longrightarrow \mathbb{P}^2$$

via

$$\text{"} \phi(z) = [\wp(z), \wp'(z), 1] \text{"} \qquad (*).$$

<u>Prop.</u> $(*)$ defines holo map from X to $\mathbb{P}^2$ whose image is contained in

$$E = E_\Lambda$$
$$= \text{closure of } \{ y^2 - 4x^3 + g_2 x + g_3 = 0 \} \subseteq \mathbb{C}^3$$

$$= \{ Y^2 Z - 4X^3 + g_2 X Z^2 + g_3 Z^3 = 0 \}$$

Explanation/Pf: Consider first

$$\tilde{\phi}_0 : \mathbb{C} - \Lambda \longrightarrow \mathbb{P}^2, \quad z \mapsto [\wp(z), \wp'(z), 1],$$

This is certainly analy map, and I claim it extends to holo

$$\tilde{\Phi} : \mathbb{C} \longrightarrow \mathbb{P}^2$$

Consider eg. behavior of $\tilde{\Phi}$ near $0 \in \Lambda$. Have

$$\wp(z) = \frac{1}{z^2} + \text{HOT}, \quad \wp'(z) = -\frac{2}{z^3} + \text{HOT},$$

and for z in punctd nbd of 0:

$$[\wp(z), \wp'(z), 1] = [z^3\wp, z^3\wp', z^3]$$

$z^3\wp' \neq 0$ in nbd of 0

$$= \left[\frac{z^3\wp}{z^3\wp'}, 1, \frac{z^3}{z^3\wp'} \right]$$

these are holo in nbd of 0.

So $\tilde{\phi}_0$ extends, and $\tilde{\phi}(0) = [0,1,0]$.

Next, $\tilde{\Phi}$ is Λ-equiv, so gives

$$\phi : \mathbb{C}/\Lambda \longrightarrow \mathbb{P}^2$$

Diff eqn $\Rightarrow \phi(X) \subseteq E$.

<u>General Principle:</u> X any R.S,

$$f_1, \dots, f_r \in \mathbb{C}(X) =_{\text{def}} \{\text{mero fns on } X\}.$$

Then

$$\begin{array}{ccc} \phi : X & \longrightarrow & \mathbb{P}^r \\ \text{"} \quad \psi & & \cup \quad \text{"} \\ x & \longmapsto & [f_1(x), \dots, f_r(x), 1] \end{array}$$

defines anal map from X to $\mathbb{P}^r$. Moreover, any holo map fr X to $\mathbb{P}^r$ arises in this fashion.

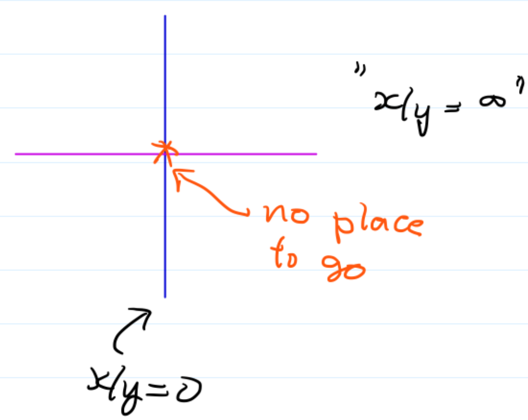
<u>Warning</u>: If $\dim X \geqslant 2$ corresp. statement fails. Eg

$$\frac{x}{y} \text{ a mero fn on } \mathbb{C}^2,$$

but

$$\begin{array}{ccc} \mathbb{C}^2 & \dashrightarrow & \mathbb{P}^r \\ \cup & & \cup \\ (x,y) & \longmapsto & [x/y, 1] = [x, y] \end{array}$$

does not extend to a continuous map.



We aim for

<u>Thm.</u> Let $\Lambda \subseteq \mathbb{C}^2$ be a lattice,

$$X = \mathbb{C}/\Lambda \text{ corresp torus} \qquad g_2 = 60 \sum 1/\lambda^4$$

$$g_1, g_2 \in \mathbb{C} \text{ consts det by } \Lambda \qquad g_3 = 140 \sum 1/\lambda^6$$

(1) Curve

$$E_\Lambda = \text{closure of } \{y^2 - 4x^3 + g_2 x + g_3 = 0\} \subseteq \mathbb{P}^2$$

is non-sing, i.e. a R.S.

(2) Map

$$X \longrightarrow E_\Lambda \subseteq \mathbb{P}^2$$

an isom of cx mflds.

<u>Pf</u>. (1). Check (for you):

$$E_\Lambda \text{ a cx mfld} \iff \text{Poly } 4x^3 - g_2 x - g_3 = 0$$

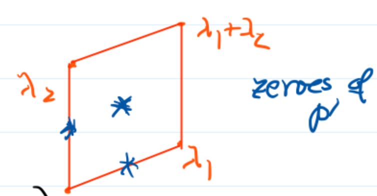
has no repeated root.

(Need to check no sings at ∞)

<u>Roots of $4x^3 - g_2x - g_3 = 0$</u>

Let $\lambda_1, \lambda_2 \in \Lambda$ be basis. Then three zeroes of $\wp'(z)$ (mod Λ) are

$$\frac{\lambda_1}{2}, \quad \frac{\lambda_2}{2}, \quad \frac{\lambda_1 + \lambda_2}{2}$$



<u>Pf</u>: $\wp'(-\frac{\lambda_1}{2}) = \wp'(\frac{\lambda_1}{2})$ by periodicity

but $\wp'(-\lambda_1/2) = -\wp'(\lambda_1/2)$ since $\wp'$ odd.

so $\wp'(\lambda_1/2) = 0$

Follows fr. diff eqn that

$$\wp(\lambda_1/2), \ \wp(\lambda_2/2), \ \wp(\tfrac{\lambda_1+\lambda_2}{2})$$

are roots of $4x^3 - g_2 x - g_3 = 0$.

Want to show these three values are distinct.

· Consider

$$f(z) = \wp(z) - \wp(\lambda_1/2),$$

Vanishes at $z = \lambda_1/2$, and being even has a double zero there. But $f(z)$ has only two zeroes (mod Λ), so

$$f(\lambda_2/2) \neq 0 \quad f((\lambda_1+\lambda_2)/2) \neq 0 \quad \text{etc.} \ \square$$

(2°) Claim: ϕ is surjective and 1-1.

Pf Say $(a,b) \in E$, ie. $b^2 = 4a^3 - g_2 a - g_3$.

$$f(z) = \wp(z) - a$$

Non-trivial ell fn, so has zero, say at z_0, ie. $\wp(z_0) = a$. Then

$$\wp'(z_0)^2 = 4a^3 - g_2 a - g_3,$$

ie

$$\wp'(z_0)^2 = b^2.$$

So $\wp'(z_0) = \pm b$, so $\phi(z_0) = (a,b)$ or $\phi(-z_0) = (a,b)$

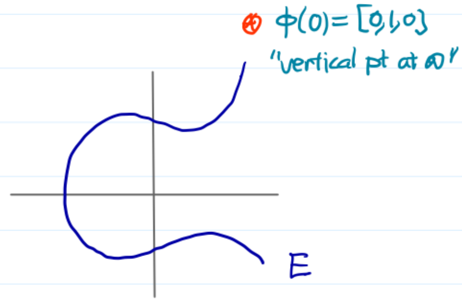
For you: ϕ is 1-1

(3°). Lemma: A 1-1 surj holo map $f: X \to Y$ of Riemann sfs is

Pf. Need to show f^{-1} is holo. This is local question so suffices to note that any holo map locally of form

$$z \mapsto z^e \cdot u(z) \quad u(0) \neq 0$$

If f locally 1–1 then $e=1$, and f is invertible □



<u>Upshot:</u>

$$X_\Lambda = \mathbb{C}/\Lambda \cong E_\Lambda \subseteq \mathbb{P}^2$$

non-singular cubic curve in $\mathbb{P}^2$

(Rmk: will later see that every sm cubic is E_Λ for some Λ)

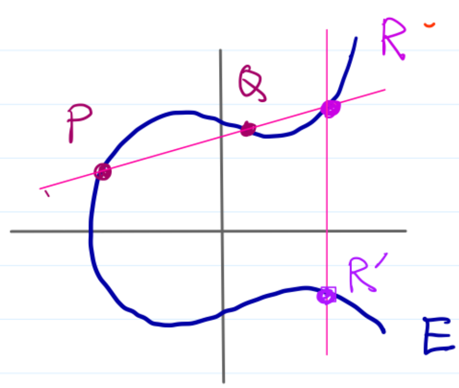
<u>Group Law on E:</u>

- $X = \mathbb{C}/\Lambda$ is a group
- In fact, X is a complex Lie group, ie

$$\text{add}: X \times X \longrightarrow X, \qquad \text{neg}: X \longrightarrow X$$

are holo maps (Exerc!)

- So expect that the points of $E \subseteq \mathbb{P}^2$ form a group.
- This is classical constr that goes as follows:
  - Take $P, Q \in E$.
  - Line joining P, Q meets E at 3rd pt R.
  - Vertical line thru R meets E at R'



Then

$$P + Q = R' \quad \text{in group law}$$

$\Big($ More succinctly: pt at ∞ is $0 \in E$, and

$$P + Q + R = 0 \iff P, Q, R \text{ are collinear} \Big)$$

This is outlined on HW: key pt is

Exer: $\forall\, z_1, z_2 \; (\not\equiv 0,\; z_1 \not\equiv z_2)$

$$\det \begin{vmatrix} \wp(z_1) & \wp'(z_1) & 1 \\ \wp(z_2) & \wp'(z_2) & 1 \\ \wp(-z_1-z_2) & \wp'(-z_1-z_2) & 1 \end{vmatrix} = 0$$

Sept. 1

Mero Fns on $X = \mathbb{C}/\Lambda$

Notation:

$$\mathbb{C}(X) = \text{field of mero fns on } X$$

$$\Big(= \mathbb{C}(\wp, \wp') = \mathbb{C}(E) \Big)$$ ← rational fns on cubic curve E

Recall –

(1). A non-zero $f \in \mathbb{C}(X)$ has same no of zeroes & poles (counting multiplicities)

(2). Given $0 \neq f \in \mathbb{C}(X)$, say f has

poles at $p_1, \dots, p_r \in X$
zeroes at $q_1, \dots, q_r \in X$ (repeat for multiplicities)

Then

$$\sum p_i = \sum q_i \quad (\text{in } X \cong \mathbb{C}/\Lambda)$$

It's remarkable fact that the converse is true:

<u>Thm</u>. (Abel's Thm). Suppose given pts

$$\begin{matrix} p_1, \dots, p_r \\ q_1, \dots, q_r \end{matrix} \in X$$

s.t.

$$\sum p_i = \sum q_i \in X$$

Then $\exists\ 0 \neq f \in \mathbb{C}(X)$ w poles at p_i & zeroes at q_i

To prove will introduce new fn on $\mathbb{C}$.

As before, fix lattice $\Lambda \subseteq \mathbb{C}$. Define

$$\sigma(z) = \sigma(z, \Lambda)$$
$$= z \cdot \prod_{\lambda \neq 0} \left(1 - \frac{z}{\lambda}\right) \cdot e^{(z/\lambda) + \frac{1}{2}(z/\lambda)^2}$$

<u>Prop.</u> (a). The infinite product converges uniformly on compact subsets to define an entire fn. $\sigma(z)$ has simple zeroes on Λ, and no other zeroes

(b).
$$\frac{d^2}{dz^2} \log \sigma(z) = -\wp(z).$$

(c). For any $\lambda \in \Lambda$, $\exists$ $a, b \in \mathbb{C}$ st.
$$\sigma(z+\lambda) = e^{az+b}\,\sigma(z).$$

<u>Pf.</u> (a) See Ahlfors.

$$\frac{-1/\lambda}{1-z/\lambda} = \frac{-1}{\lambda - z}$$

(b). Compute:
$$\log(\sigma) = \log(z) + \sum_{\lambda \neq 0} \left(\log\left(1-\frac{z}{\lambda}\right) + \left(\frac{z}{\lambda} + \frac{1}{2}\left(\frac{z}{\lambda}\right)^2\right)\right)$$

Differentiate twice term by term:
$$\frac{d^2}{dz^2} \log(\sigma) = -\frac{1}{z^2} + \sum \left(-\frac{1}{(z-\lambda)^2} + \frac{1}{\lambda^2} \right)$$
$$= -\wp(z).$$

(c). By periodicity of $\wp$,
$$\frac{d^2}{dz^2}\Big(\log(z+\lambda) - \log(z)\Big) \equiv 0$$

So

$$\log \sigma(z+\lambda) = \log(z) + (az+b)$$

ie.

$$\sigma(z+\lambda) = \sigma(z) \cdot e^{az+b}$$

Pf of Abel's Thm: Fix

$$c_1, \quad , c_s \in \mathbb{C}$$
$$n_1, \quad , n_s \in \mathbb{Z}$$

s.t.

$$\sum n_i = 0, \quad \sum n_i c_i \in \Lambda$$

Need to constr ell fn $f(z)$ w.

$$\mathrm{ord}_{c_i}(f) = n_i \quad , \text{ no other zeroes or poles.}$$

Replacing c_i by translates, can assume:

$$\sum n_i c_i = 0 \quad (\text{Check!})$$

Consider:

$$f(z) = \prod \sigma(z-c_i)^{n_i}$$

This has right zeroes & poles. Need to show f elliptic

Fix $\lambda \in \Lambda$. Then

$$f(z+\lambda) = \prod \sigma(z+\lambda-c_i)^{n_i}$$

$$= \prod \left(\sigma(z-c_i) \cdot e^{a(z-c_i)+b} \right)^{n_i}$$

$$= f(z) \cdot \exp\left(\sum n_i \left(a(z-c_i) + b \right) \right)$$

But

$$\sum n_i = 0 \quad \text{so} \quad \sum n_i a z = 0, \quad \text{and} \quad \sum n_i b = 0$$

Also we assume $\sum n_i c_i = 0$, so

$$\sum n_i c_i a = 0. \quad \text{QED!}$$

Moduli.–

· Want to discuss isom classes of RS's of genus 1.

Consider:

$\Lambda, \Lambda' \subseteq \mathbb{C}$ two lattices

$X = \mathbb{C}/\Lambda$, $X' = \mathbb{C}/\Lambda'$: corresp. RS of genus 1

Ask: When is $X \cong X'$ as cx mflds?

(NB: $X \approx_{\text{diffeo}} X' \approx_{\text{diffeo}} S^1 \times S^1$)

Write:

$0 \in X$, $0' \in X'$ for origin (images of Λ, Λ')

It's enough to study holo mappings

$(*)$ $\quad f : X \to X'$ s.t. $f(0) = 0'$

(Arbitrary map differs from one of these by translation.)

<u>Lemma</u>: Given f as in $(*)$, $\exists\, \alpha \in \mathbb{C}$ s.t. f induced by

$$\mathbb{C} \longrightarrow \mathbb{C}, \quad z \longmapsto \alpha \cdot z$$

s.t.

$$\alpha \Lambda \subseteq \Lambda'.$$

(Pf is on HW) <u>Hint</u>: f covered by map $F : \mathbb{C} \longrightarrow \mathbb{C}$ on univ. covering spaces and F holo. Moreover

$$F(z+\lambda) = F(z) + \lambda' \quad \text{some } \lambda' = \lambda'(z,\lambda).$$

Check this implies F linear.

In particular,

$$X_\Lambda \cong X_{\Lambda'} \iff \begin{array}{c} \Lambda' = \alpha \Lambda \\ \text{for some } \alpha \in \mathbb{C} \end{array} \qquad (*)$$

Ask: How do $\wp, \wp'$ etc behave under transf $z \mapsto \alpha z$?

<u>Ex</u>:
$$\wp(\alpha z, \alpha \Lambda) = \alpha^{-2} \cdot \wp(z, \Lambda)$$
$$\wp'(\alpha z, \alpha \Lambda) = \alpha^{-3} \wp(z, \Lambda)$$

$$g_2(\alpha\Lambda) = \alpha^{-4} g_2(\Lambda), \quad g_3(\alpha\Lambda) = \alpha^{-6} g_3(\Lambda)$$

In particular, this shows how the Weierstrass eqn

$$y^2 = 4x^3 - g_2 x - g_3 \quad \text{transforms}$$

Now let

$$\boxed{\Delta = g_2^3 - 27 g_3^2}$$ (depends on Λ)

This is discr of $4x^3 - g_2 x - g_3$; so $\Delta \neq 0$ since this has distinct roots.

Next, define

$$J = J(\Lambda) = \frac{g_2^3}{\Delta} \in \mathbb{C}$$

Note that

$$J(\Lambda) = J(\alpha \cdot \Lambda),$$

both top and bottom transf by α^{-12}

so J is invariant of isom classes.

Thm. $X_\Lambda \cong X_{\Lambda'} \iff J(\Lambda) = J(\Lambda')$,

and moreover every complex no occurs as J-invariant.

ie.

$$\left\{ \begin{array}{c} \text{isom classes} \\ \text{of } X_\Lambda \end{array} \right\} \underset{J}{\longleftrightarrow} \mathbb{C}$$

One approach: Show that g_2^3/Δ classifies roots of

$$4x^3 - g_2 x - g_3 = 0$$

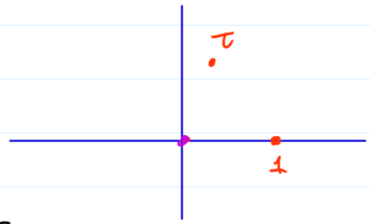
up to changes of coords.

Modular Group: Any lattice equiv (in above sense) to lattice generated by

$$1, \tau \qquad , \tau \in \mathbb{H} = \text{UHP}$$

Let

$$\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$$



However there is some ambiguity in the choice of τ, because if

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \quad \left(= \begin{matrix}\text{integer matrices} \\ \text{w. det} = 1\end{matrix}\right)$$

Then

$$\mathbb{Z} + \mathbb{Z}\tau = \mathbb{Z}\cdot(a\tau + b) + \mathbb{Z}(c\tau + d)$$

i.e. set

$$\gamma\cdot\tau = \frac{a\tau+b}{c\tau+d} \in \mathbb{H}$$

Then

$$\mathbb{C}/\Lambda_\tau \cong \mathbb{C}/\Lambda_{\gamma\cdot\tau},$$

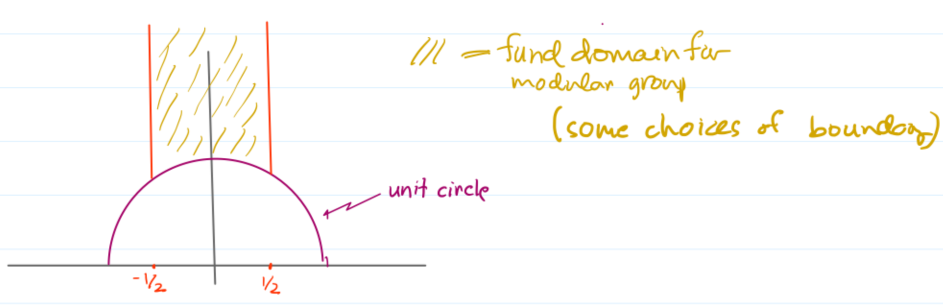
and in fact this is only ambiguity. Noting that $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts trivially, define

$$\Gamma = SL_2(\mathbb{Z})/\pm 1 = PSL_2(\mathbb{Z}) : \text{modular group}$$

This discussion "shows" that

$$\left\{\begin{matrix}\text{isom classes} \\ \text{of ell curves}\end{matrix}\right\} \cong \mathbb{H}/\Gamma$$

To understand this quotient, should look for fund domain for Γ. This is famous picture:



· Can view g_2, g_3 as fns of τ

Ex: $g_2(\gamma\tau) = (c\tau+d)^{-4} g_2(\tau)$

$g_3(\gamma\tau) = (c\tau+d)^{-6} g_3(\tau)$

Then set

$$J(\tau) = \frac{g_2^3}{\Delta}$$

Thm: J defines holo isom of fund domain onto $\mathbb{C}$.

II. Riemann Surfaces — Basics

Consider

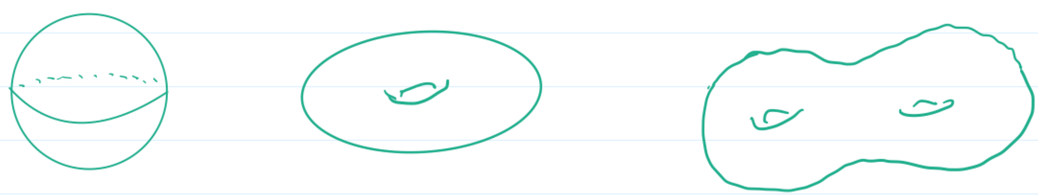
X = compact connected Riemann sf

So, X is closed oriented 2-mfld without boundary.

Because: biholo maps are orientation preserving.

<u>Thm.</u> (Classification of sfs):

X is diffeomorphic to a sf w. $g \geq 0$ handles attached



$g=0$ $\quad$ $g=1$ $\quad$ $g=2$

g is <u>genus</u> of X, and it classifies X up to diffeomorphism.

<u>Topological invariants</u>: if $g(X) = g$, then

$$b_0(X) = 1, \quad b_1(X) = 2g, \quad b_2(X) = 1$$

$$\chi_{top}(X) = 2 - 2g.$$

<u>Rmk</u>: As we've seen in case $g=1$, it's not true that all RS's of given genus are isom as RS's.

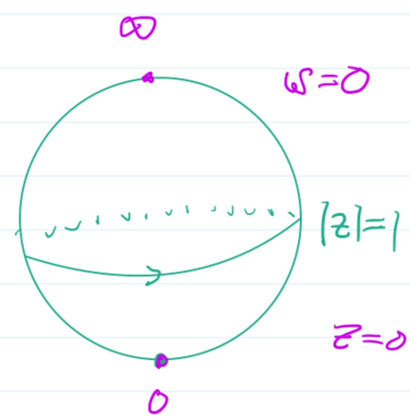
Prop: If X is a compact connected RS, then any holo fn f on X is constant.

Pf. By compactness, $|f|$ attains a maximum at some pt $x \in X$. By maximum modulus principle, f const in nbd of x. By connectedness, f is globally const.

So: as in case of $\mathbb{C}/\Lambda$, we should focus on mero fns.

- What about residues?

Ex. Consider fn z on $\mathbb{C}$, viewed as mero fn on $\mathbb{P}^1$.



$$\mathrm{res}_0\left(\frac{1}{z}\right) = \frac{1}{2\pi i} \int_{|z|=1} \frac{1}{z} = 1$$

If $w = \frac{1}{z}$, then $|z|=1$ is $-(|w|=1)$ (orientation reversed) and

$$\frac{1}{2\pi i} \int_{-(|w|=1)} w = 0 \quad !!$$

What's happening?!

Crucial Fact: Cannot integrate fns on a manifold, and mero fns on a RS don't have residues!

To do integral correctly, need to integrate $\frac{dz}{z}$:

$$\cdot \quad \frac{1}{2\pi i} \int_{|z|=1} \frac{dz}{z} = 1$$

$$z = \frac{1}{w}, \quad \text{so} \quad dz = -\frac{dw}{w^2}, \quad \text{so}$$

$$\frac{dz}{z} = -\frac{dw}{w},$$

and

$$\int_{|z|=1} \frac{dz}{z} = \int_{-|w|=1} -\frac{dw}{w} = 1$$

On a RS: <u>one-forms</u> are the things that can be integrated and have residues.

<u>One-Forms</u>:

<u>Def</u>. A holo 1-form on RS X is C^∞ $\mathbb{C}$-valued 1-form which in local coords can be expressed as

$$\eta \underset{loc.}{=} f(z)\,dz$$

f analy, $dz = dx + i dy$

· Say you have other local coord

$$w = g(z), \qquad z = h(w) \qquad (h = g^{-1})$$

Then

$$dw = g'(z)\,dz, \quad dz = h'(w)\,dw \, .$$

$$\boxed{f(z)\,dz = f(h(w))\cdot h'(w)\,dw}$$

(NB: $f(z)$ analy $\Longleftrightarrow$ $f(h(w))\cdot h'(w)$ analy)

<u>Mero 1-form:</u>

$$\eta = f(z)dz, \quad f(z) \text{ merom.}$$

<u>Ex.</u> $X = \mathbb{P}^1$: $\frac{dz}{z} = -\frac{dw}{w}$

or $dz = -\frac{dw}{w^2}$.

<u>Ex.</u> 1-form dz on $\mathbb{C}$, is invariant under Λ, descends to nowhere van form on $X = \mathbb{C}/\Lambda$ "dz"

On $\mathbb{C}/\Lambda$,

mero fn	$\longleftrightarrow$	mero 1-form
$f(z)$		$f(z)$ "dz"

This is special to genus 1.

<u>Ex.</u> In general, if f a mero fn on X, then define

$$df \underset{loc.}{=} d(f(z)) = f'(z)dz$$

This is well-def. mero 1-form.

<u>Def.</u> If η a mero 1-form on RS X, locally

$$\eta = f(z)dz,$$

then $p \in X$ a zero or pole of η according to whether it's a zero or pole of $f(z)$. Define

$$\mathrm{ord}_p(\eta) = \mathrm{ord}_p(f)$$

Note: poles & zeroes discrete, so if X compact only finitely many

(Exer: Show that this is indep of local expression).

A crucial point is that holo forms are closed.

Prop: Let X be a (possibly) non-compact RS, and let ω be a holo 1-form on X. Then ω is closed, i.e.

$$d\omega = 0$$

this is the "C^∞-mfld d", viewing ω as C^∞ 1-form.

Pf: Write

$$\begin{aligned}
\omega &= f(z)\,dz \\
&= \big(u(x,y) + i\cdot v(x,y)\big)(dx + i\,dy) \\
&= (u\,dx - v\,dy) + i(v\,dx + u\,dy)
\end{aligned}$$

Then

$$\begin{aligned}
d\omega &= d(u\,dx - v\,dy) + i\,d(v\,dx + u\,dy) \\
&= \left(-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) dx \wedge dy + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx \wedge dy \\
&= 0 \quad \text{by CR eqns!}
\end{aligned}$$

<u>Warning:</u> Can define holo forms on higher dim cx mflds. Not true that any holo form is closed, eg

$$\omega = z_1 \, dz_2 \quad \text{on } \mathbb{C}^2.$$

Analogue of Prop is that in dim n, "holo $(n,0)$-form is closed", ie

$$\omega = f(z_1, \ldots, z_n)\, dz_1 \wedge \cdots \wedge dz_n \; : \; \text{closed by CR}$$

<u>However</u>: It's a non-trivial thm that a global holo form on a sm proj var (or compact Kähler mfld) is closed for reasons of Hodge theory.

<u>Rmk</u>: Recall that closed forms on compact mfld determine de Rham coh. classes. It will be very interesting to understand the coh classes of holo forms on compact RS.

<u>Integrals and Residues:</u>

If $\gamma \subseteq X$ is a curve, and η is holo in nbd of γ, then

$$\int_\gamma \eta \in \mathbb{C}$$

is defined.

<u>Def</u>: Let η be a mero 1-form on RS X, and let $p \in X$ be a pole of η. Define

$$\mathrm{res}_p(\eta) = \frac{1}{2\pi i}\int_\gamma \omega,$$

where γ is small pos oriented loop around p.

(By 1CV, this is indep of choice of loop.)

Ex. $X = \mathbb{P}^1$, $\quad \eta = \frac{dz}{z} = -\frac{dw}{w}$. Have

$$\mathrm{res}_0(\eta) = 1, \quad \mathrm{res}_\infty(\eta) = -1$$

Thm. (Residue Thm). Let X be a compact RS, and

$$\eta = \text{mero } 1\text{-form on } X.$$

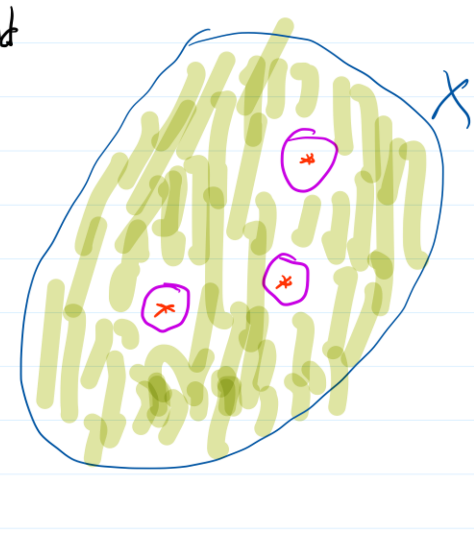
Then

$$\sum \mathrm{res}_p(\eta) = 0$$

(Either sum over all poles, or over all $p \in X$ w. the convention that $\mathrm{res}_p(\eta) = 0$ if η holo at p.)

Proof Choose small (open) disk around each pole, and let

$$M = X - \bigcup(\text{disks})$$



So M is mfld w boundary a union of circles

Then:

$$\sum \mathrm{res}_p(\eta) = \frac{1}{2\pi i}\int_{\partial M} \eta$$

$$\underset{\text{Stokes}}{=} \frac{1}{2\pi i}\int_M d\eta \underset{\substack{\uparrow \\ \text{since } d\eta = 0}}{=} 0$$

Rmk: For those of you who have been reading about sheaves and cohom, here's a cohom interpr of Thm. Let

$$\Omega^1_X = \text{canon bundle, so } \Gamma(X, \Omega^1_X) = \text{holo 1-forms}$$

Pick

$p_1, \ldots, p_d \in X$, and consider

$\Omega^1_X(p_1 + \cdots + p_d)$: lb whose global sections can be identified w. mero 1-forms w poles at p_i.

Then

$$\Omega^1_X \hookrightarrow \Omega^1_X(p_1 + \cdots + p_d),$$

and there is canon identification (via residues)

$$\frac{\Omega^1(p_1 + \cdots + p_d)}{\Omega^1_X} \cong \mathbb{C}_{p_1} \oplus \cdots \oplus \mathbb{C}_{p_d} \quad \text{(sky-scraper sheaf)}$$

Starting w exact seq.

$$0 \to \Omega^1_X \to \Omega^1_X(\Sigma p_i) \to \mathbb{C}_{p_1} \oplus \cdots \oplus \mathbb{C}_{p_d} \to 0,$$

take cohom:

$$\begin{tikzcd} H^0(\Omega^1_X(\sum P_i)) \arrow[r] & \bigoplus_{i=1}^{d} \mathbb{C} \arrow[r, "\partial"] & H^1(X,\Omega^1) \arrow[r] & 0 \end{tikzcd}$$

with $H^1(X,\Omega^1) \cong \mathbb{C}$,

$$\eta \longmapsto \begin{pmatrix}\text{vector of} \\ \text{res}_{P_i}(\eta)\end{pmatrix}$$

$$(a_1, \ldots, a_d) \longrightarrow \sum a_i$$

Exactness shows $\sum \text{res} = 0$. Surjectivity of last map means that if we pick

$$a_1, \ldots, a_d \in \mathbb{C}, \quad \sum a_i = 0,$$

$\exists$ mero η w. $\text{res}_{P_i}(\eta) = a_i$, holo elsewhere. We'll see later how this follows "classically" fr RR.

<u>Cor</u>: Let f be a non-const mero fn on X. Then f has same number of zeroes as poles (counting multiplicities)

<u>Pf</u>. Consider

$$\eta = \frac{df}{f}.$$

Check locally:

$$\text{res}_p(\eta) = \text{ord}_p(f).$$

Sept-12 Apply residue Thm

Correction: why global holo form closed? If ω is holo p-form, then ω is $\bar{\partial}$-harmonic so d-harmonic, so closed.

Thm: Let η, ω be non-zero mero 1-forms on compact RS X. Then

$$\sum \mathrm{ord}_p(\omega) = \sum \mathrm{ord}_p(\eta).$$

i.e.

(# zeroes − # Poles) the same for all mero 1-forms.

Pf. (1°). Main Claim:

$\exists$ mero fn ϕ on X s.t.

$$\eta = \phi \cdot \omega.$$

(i.e. "η / ω" a well-defined mero fn)

Grant claim. Then for each $p \in X$

$$\mathrm{ord}_p(\eta) = \mathrm{ord}_p(\phi) + \mathrm{ord}_p(\omega),$$

so

$$\sum \mathrm{ord}_p(\eta) = \sum \mathrm{ord}_p(\phi) + \sum \mathrm{ord}_p(\omega).$$

But by Thm from last class,

$$\sum \mathrm{ord}_p(\phi) = 0.$$

(2). Pf of Main Claim:

Write locally:

$$\eta = a(z)\,dz, \quad \omega = b(z)\,dz$$

Locally define $\phi = \dfrac{a(z)}{b(z)}$ (mero fn)

Issue: Is ϕ globally well-defined as mero fn?

Consider new coords w:

$$z = h(w) \qquad (h \text{ local analy isom})$$

$$dz = h'(w)\,dw$$

$$\eta = \underbrace{a(h(w))\cdot h'(w)}_{A(w)}\,dw, \quad \omega = \underbrace{b(h(w))\cdot h'(w)}_{B(w)}\,dw$$

$$\eta = A(w)\,dw \qquad\qquad \omega = B(w)\,dw$$

So

$$\phi = \frac{a(z)}{b(z)} = \frac{a(h(w))}{b(h(w))}$$

$$= \frac{A(w)}{B(w)}. \qquad \therefore \text{ so a global mero fn.}$$

Ex. On $\mathbb{P}^1$, $\#\text{zeroes} - \#\text{poles} = -2$

on $\mathbb{C}/\Lambda$: $\#\text{zeroes} - \#\text{poles} = 0$

Branched Coverings & Riemann–Hurwitz Thm

Let X, Y be compact connected RS's, and

$$f : X \longrightarrow Y$$

a non-const holo mapping. Fix pts

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Psi & & \Psi \\ p & & q = f(p) \end{array}$$

(1°). <u>Claim</u>: we can choose local coords

z on X centered at p, w on Y centered at q

st.

$$w = f(z) = z^e \quad \text{for some integer } e \geq 1.$$

<u>Pf.</u> Choose local coords z_1, w centered at p, q, so

$$w = f_1(z_1), \quad f_1(0) = 0.$$

Can write

$$f_1(z_1) = (z_1)^e \cdot u(z_1), \quad u(0) \neq 0$$

Then $\exists$ analy. $v(z_1)$ $v_1(0) \neq 0$, st

$$v_1(z_1)^e = u(z_1).$$

Then set

$$z = z_1 \cdot v_1(z_1) \quad \text{(biholo change of coords)}$$

Then

$$w = z^e$$

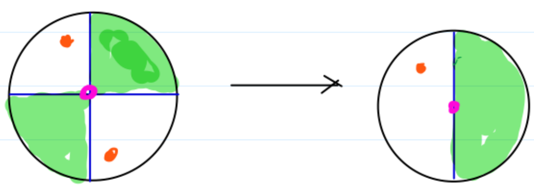
<u>Def</u>: Define

$$e = e_f(p) : \underline{\text{local degree}} \text{ of } f \text{ at } p.$$

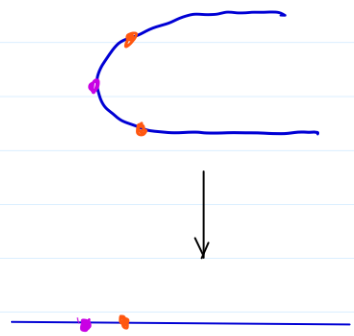
(sometimes called $\mathrm{ord}_p(f)$ or ...)

(2°). Local geometry: Look at local model:

$$\begin{array}{ccc} \Delta & \longrightarrow & \Delta \\ z & \longmapsto & z^e \end{array}$$



$e = 2$



Schematic real pict.

We see:

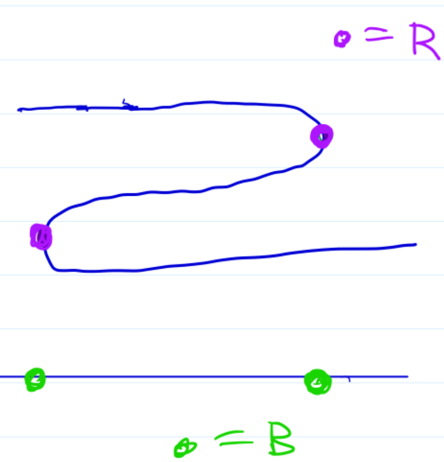
- If $e_f(p) = e$, then f is locally e-to-one near p.
- If $q \neq q' \in Y$ is suff close to q, then
$$\#\left(f^{-1}(q') \cap (\text{small nbd of } p) \right) = e$$
- f locally an isom near $p \iff e_f(p) = 1$.

Def: Say f ramifies at p if $e_f(p) > 1$. Ramification index is
$$r_f(p) = (e_f(p) - 1)$$

(Ram pts are isolated hence finite.)

$$R = \text{ramif. locus} = \{\text{all ram pts}\} \subseteq X$$

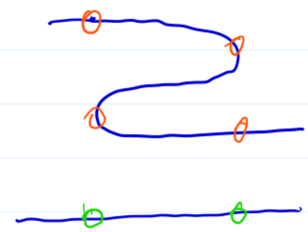
$$B = \text{branch pts} = f(R) \subseteq Y$$



Prop: Given $f : X \to Y$, consider

$$f^\circ : (X - f^{-1}(B)) \longrightarrow (Y - B)$$

Then f° is a connected covering space.



(Remove branch pts, and their full preimages in X)

Pf. For you, or see Miranda
(It's clear that f° a local isom at each pt in source; there is a little argument using compactness to check even covering cond.)

Def: $\deg(f) = $ deg of this covering,

Prop: Consider

$$f : X \longrightarrow Y$$

as before. ("A branched covering.") Fix any $y \in Y$. Then

$$\sum_{x \in f^{-1}(y)} e_f(x) = \deg(f)$$

Sketch: If $y \in Y - B$, this follows since f° a covering space. Now say

$$y \in B : \quad x_1, \ , x_t \in X \text{ the distinct preimages of } y,$$

Choose small nbd $V \ni y$ in such a way that

$$f^{-1}(V) = \coprod_{i=1}^{t} U_i, \qquad U_i = \text{small nbd of } x_i.$$

By our local picture, $U_i \longrightarrow V$ locally like $z \longrightarrow z^{e_i}$,

<u>Thm</u> (Riemann–Hurwitz). Let

$$f : X \longrightarrow Y$$

be non-const anal map bet compact connected RS. Then

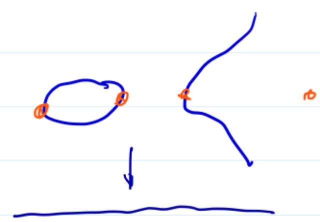
$$(2g(X)-2) - (\deg(f))\cdot(2g(Y)-2) = \mathrm{Ram}(f)$$

where

$$\mathrm{Ram}(f) = \sum_{p \in X} r_p(f) \quad = \text{total ramif.}$$

<u>Ex</u> $X = \mathbb{C}/\Lambda$, $Y = \mathbb{P}^1$. View $\wp(z)$ as mero fn, defining

$$f : X \longrightarrow \mathbb{P}^1$$



Then $\deg(f) = 2$, f ramifies over ∞ and at the three half periods. So

$$\mathrm{Ram}(f) = 4.$$

Check:

$$(2\cdot 0 - 2) - 2\cdot(-2) = 4 \quad \checkmark$$

<u>Ex.</u> $f \in \mathbb{C}[z]$ poly of deg d, defines

$$f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \ \deg d.$$

$f^{-1}(\infty) = \infty$: so total ram over ∞, ie $e_f(\infty) = d-1$. Finite ram:

zeroes of $f'(z)$: total finite ram $= d-1$

So $\mathrm{Ram}(f) = 2d-2$.

Check: $(-2) - d(-2) = 2d-2$

Proof of Riemann-Hurwitz:

- Will prove by triangulating X, Y and computing Euler chars.

- Recall: suppose we triangulate Y in such a way that the triangulation has

$$v_0 \text{ vertices}, \quad v_1 \text{ edges}, \quad v_2 \text{ triangles},$$

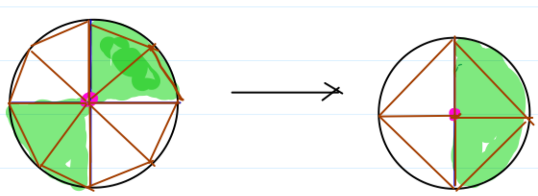
then

$$v_0 - v_1 + v_2 = \chi_{top}(Y) = 2 - 2g(Y)$$

- Given $f : X \longrightarrow Y$, choose a triang of Y that includes all branch points as vertices, and which is suff fine so that it lifts to a triangulation of X, ie.

$$f^{-1}\begin{pmatrix} \text{vertex} \\ \text{edge} \\ \text{tri} \end{pmatrix} \text{ of } Y \Bigg) = \begin{pmatrix} \text{vertex} \\ \text{edge} \\ \text{tri} \end{pmatrix} \text{ of } X \Bigg)$$

Ex: look at picture of simple ram pt.



$e_f(x) = 2$

Sept. 15

Let

v_0, v_1, v_2 : # of simplices in Y

w_0, w_1, w_2 : " X

· Write

$$d = \deg(f)$$

Then

$$\boxed{\begin{aligned} w_1 &= d \cdot v_1 \\ w_2 &= d \cdot v_2 . \end{aligned}}$$

The action occurs over 0-simplices.

· Fix any pt $y \in Y$. Then

$$\sum_{x \in f^{-1}(y)} e_f(x) = d$$

· So

$$\sum_{x \in f^{-1}(y)} (e_f(x) - 1) = d - \# f^{-1}(y)$$

$$\Vert$$

$$\sum_{x \in f^{-1}(y)} r_f(x)$$

ie

$$\# f^{-1}(y) = d - \sum_{x \in f^{-1}(y)} r_f(x)$$

Applying at each of the vertices in Y, see

$$\boxed{w_0 = d \cdot v_0 - \mathrm{ram}(f)}$$

So find

$$d(v_0 - v_1 + v_0) = w_0 - w_1 + w_2 + \mathrm{ram}(f)$$

i.e.,

$$d(2 - 2g(Y)) = (2 - 2g(X)) + \mathrm{ram}(f)$$

i.e.

$$(2g(X) - 2) - d \cdot (2g(Y) - 2) = \mathrm{ram}(f) \quad \text{QED!}$$

∘ ∘

We can also connect / detect ramif via forms.

<u>Prop:</u> Let

$$f : X \longrightarrow Y$$

be a non-const map of R.S, and say η = any mero diff on Y, so

$$f^* \eta = \text{mero diff on } X.$$

Then for any $p \in X$,

$$\mathrm{ord}_p(f^* \eta) = e_f(p) \cdot \mathrm{ord}_{f(p)}(\eta) + r_f(p)$$

<u>Pf.</u> Choose local coords s.t.

$$w = f(z) = z^e.$$

and say

$$\eta = \phi(w)\, dw.$$

Then

$$f^*(\eta) \underset{loc}{=} \phi(z^e).(e.z^{e-1})\,dz$$

Statement follows.

Cor: Let

$$f : X \longrightarrow Y$$

be a br cover of deg d, let η = mero 1-form on Y. Then

$$\begin{pmatrix}\text{total no zeros} \\ -\text{poles of } f^*\eta\end{pmatrix} = d \cdot \begin{pmatrix}\text{total no zeroes} \\ -\text{poles of } \eta\end{pmatrix} + \text{ram}(f).$$

Pf. Ex for you.

Cor. Let X be compact $\underbrace{\text{RS}}_{\text{(of genus } g)}$ that carries a non-trivial mero fn. Then

$$\begin{matrix}\#(\text{zeroes} - \text{poles}) \text{ of any} \\ \text{mero 1 form}\end{matrix} = 2g-2.$$

Pf: View mero fn as defining

$$f : X \longrightarrow \mathbb{P}^1 \quad \text{of deg } d$$

(So d = no of zeroes = # poles). By RH

$$(2g(X)-2) + 2d = \text{total ram.}$$

Apply previous Cor to $f^*(\frac{dz}{z})$:

Find

$$\underset{\text{of } f^*\left(\frac{dz}{z}\right)}{\#(\text{zeroes} - \text{poles})} - 2\left(\underbrace{\underset{\text{of } \frac{dz}{z}}{\#\text{ zeroes} - \text{poles}}}_{=-2}\right) = \text{total ramf}$$

Comparing these formulae, find

$$\underset{\text{of } f^*\left(\frac{dz}{z}\right)}{\#(\text{zeroes} - \text{poles})} = 2g(X) - 2.$$

But we've seen expression on LHS same for all mero 1-forms. QED.

<u>Remarks on $\exists$nce of mero fns</u>. Let

$$X = \text{any compact RS},$$

It's a (non-trivial) Thm that X always carries non-const mero fns. There are several approaches to proving this:

(1°). Study analytic properties of X. (See books of Donaldson or Varolin.)

(2°). Invoke Math 545 – let ω be a pos $(1,1)$-form on X. Then

$$d\omega = 0 \quad (\text{trivially}),$$

ie X is Kahler. Scale ω s.t. $\int_X \omega \in \mathbb{Q}$. Then Kodaira embedding thm implies $\exists$ holo embedding

$$X \hookrightarrow \mathbb{P}^N$$

But for proj mflds, $\exists$nce of mero fns is trivial: use ratios z_i/z_j of homog coords.

(3°). Deal w. algebraic curves: i.e. limit attention to RS's X that are assumed to admit embedding

$$X \subseteq \mathbb{P}^N$$

(which, as we've just said, is actually all RS's.)

We'll generally follow (3°): but just keep in mind that in fact everything we say holds for all compact RS's.

III. Examples & Constructions

Hyperelliptic Curves

Fix poly. $f_d(z) \in \mathbb{C}[z]$ of deg d w distinct roots:

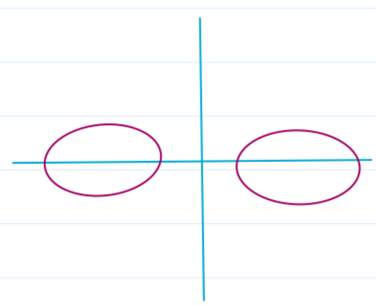
$$f_d = (z-a_1)\cdot \ldots \cdot (z-a_d)$$

· Now consider curve $X_0 \subseteq \mathbb{C}^2$ defined by

$$X_0 = \{ y^2 - f(z) = 0 \}$$

· Implicit fn thm $\Rightarrow$ X_0 a (non-compact) RS. Have deg $=2$ mapping

$$\pi_0 : X_0 \longrightarrow \mathbb{C}$$
$$(z,y) \longmapsto z$$



π has simple ramif pts at $(a_i, 0)$.

So when $d=3$, we're essentially back to our elliptic curves.

Goal: Compactify picture:

$$\begin{tikzcd} X_0 \arrow[d, "\pi_0"'] & \subseteq & X \arrow[d, "\pi"] \\ \mathbb{C} & \subseteq & \mathbb{P}^1 \end{tikzcd}$$

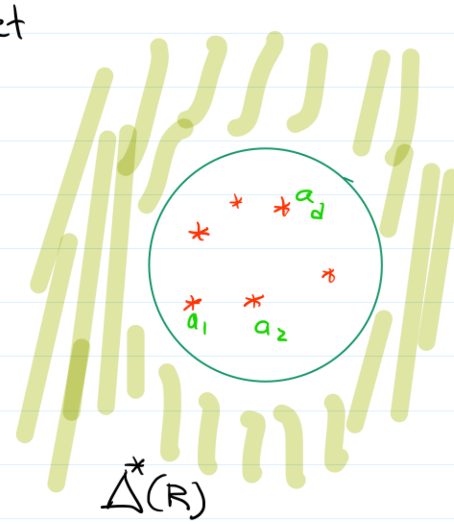
by adding one or two pts over ∞.

Have to convince oneself one can do this, and decide whether to add one pt or two.

Fix $R \gg 0$ s.t. $|a_i| < R$ all i, set

$$\Delta^*(R) = \{ |z| > R \}$$

$$X^*(R) = \{ \pi^{-1}(\Delta_R) \},$$



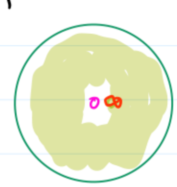
so have:

$$\begin{tikzcd} X^*(R) \arrow[d, "\pi"'] & \subseteq & X_0 \arrow[d] \\ \Delta^*(R) & \subseteq & \mathbb{C} \end{tikzcd}$$

and $X^*(R) \longrightarrow \Delta^*(R)$ is an unbranched covering space of degree 2.

Now look at what's happening over ∞. Set $w = 1/z$. Then

$$\Delta^* = \Delta^*(R) = \{ 0 < |w| < 1/R \}$$ is a punctured nbd of ∞:



and by topology, Δ^* has only two degree two covering spaces:

(i) Trivial covering

$$Y_{triv} = \Delta^* \amalg \Delta^* \longrightarrow \Delta^*$$

(ii) Connected covering

$$Y_{conn} = \{ 0 < |u| < \tfrac{1}{\sqrt{R}} \} \longrightarrow \Delta^*$$
$$u \longmapsto u^2$$

So

$$X^*(R) \longrightarrow \Delta^*(R)$$

is isom as a covering to (i) or (ii), and then we can compactify to a deg 2 (possibly) branched covering

$$X(R) \longrightarrow \{ |w| < \tfrac{1}{R} \}$$

by adding two points over ∞ (call them ∞_1 & ∞_2) in case (i), and 1 pt over ∞ in case (ii). So we get

$$\begin{array}{ccc} X_0 & \subseteq & X \\ \downarrow & & \downarrow \text{deg } 2 \\ \mathbb{C} & \subseteq & \mathbb{P}^1 \end{array}$$

It remains to distinguish between Cases (i) & (ii) in our setting.

<u>Claim</u>: Case (i) occurs $\Longleftrightarrow$ d is even
Case (ii) occurs $\Longleftrightarrow$ d is odd

Classical explanation: Fix $z_0 \in \Delta^*(R)$, and analytically continue branch of

$$y = \sqrt{f_d(z)} \qquad (= \text{pts of } \Delta^*(R))$$

around circle of radius R. After one revolution, either:

(a). You either return to same branch

(b). You have changed sign.

Then

$$(a) \iff \text{case (i)} \iff d \text{ even}$$
$$(b) \iff \text{case (ii)} \iff d \text{ odd}$$

Sketch of more formal explanation:

- Let $S \subseteq \mathbb{P}^1$ be a finite set,

$$\pi : Y \longrightarrow \mathbb{P}^1 - S$$

a two sheeted covering. (For us: $S = \{a_1, \ , a_d, \infty\}$).

- Fix $w_0 \in \mathbb{P}^1 - S$, let

$$\{y_1, y_2\} = \pi^{-1}(w_0).$$

- Have canonical map:

$$\rho : \pi_1(\mathbb{P}^1 - S, w_0) \longrightarrow \mathrm{Perm}\{y_1, y_2\} = \{\pm 1\} \qquad \text{(Monodromy repr)}$$

defined as follows: given loop $\gamma \in \pi_1$, lift γ to $\tilde{\gamma}_1, \tilde{\gamma}_2$ based at y_1, y_2. Then

$$\rho(\gamma) = \mathrm{perm}(\tilde{\gamma}_1(1), \tilde{\gamma}_2(1))$$

· Now say $a \in S$, take γ_a a small loop around a.

When we restrict Y to small punctured disk Δ^* around a, have:

$$\pi^{-1}(\Delta^*) \longrightarrow \Delta^* \text{ is}$$

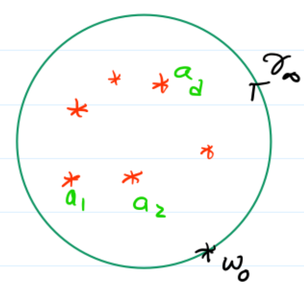
· trivial, ie $\Delta^* \sqcup \Delta^* \longrightarrow \Delta^* \iff \rho(\gamma_a) = id$

· non-trivial, ie $\Delta^* \to \Delta^*$, $\iff \rho(\gamma_a) =$ non-triv perm.

Now go back to hyperell curve $X_0 \longrightarrow \mathbb{C}$, apply w.

$$S = \{a_1, \quad , a_d, \infty\} \subseteq \mathbb{P}^1$$

Take $\gamma_\infty =$ large circle. WTS:



$$\rho(\gamma_\infty) = \begin{cases} \text{trivial if } d \text{ even} \\ \text{non-triv if } d \text{ odd} \end{cases}$$

· But monodromy around each br pt is non trivial and if we order the a_i's suitably, then

$$\gamma_\infty = \gamma_{a_1} \quad \cdot \gamma_{a_d} \text{ in } \pi_1(\mathbb{P}^1 \setminus S, w_0).$$

So
$$\rho(\gamma_\infty) = \rho(\gamma_{a_1}) \cdot \quad \cdot \rho(a_d)$$
$$= (-1)^d \quad \text{QED}$$

HW: If $d = 2g+1,\ 2g+2$, then

$$\text{genus}(X_d) = g. \qquad (X_d = \text{compactif of } X_0)$$

Differentials

Go back to

$$X_0 = \{\, y^2 - f_d(x) = 0 \,\} \subseteq \mathbb{C}^2$$

- Consider:

$$\omega_0 = \frac{dx}{y}\Big|\, X_0$$

ie. ω_0 is mero 1-form on $\mathbb{C}^2$ restr to X_0

Claim: ω_0 holo on X_0.

Pf. ω_0 clearly holo away from $(y=0) \cap X_0$. Now let

$$\phi(x,y) = y^2 - f_d(x).$$

be the defining eqn of X_0. Then $d\phi \,|\, X_0 = 0$,

$$2y\,dy = f'(x)\,dx \quad \text{on } X_0.$$

So

$$\frac{dx}{y}\Big|\, X_0 = \frac{2dy}{f'(x)}\Big|\, X_0.$$

But $f'(x) \neq 0$ when $y = 0$.

Ex: Go back to case $d=3$, so $X_0 =$ ell curve. Recall

$$\mathbb{C}-\Lambda \xrightarrow{p} X_0, \quad p(z) = (\wp, \wp').$$

So

$$p^*(\omega_0) = \frac{d\wp}{\wp'} = \frac{\wp'(z)\,dz}{\wp'(z)} = dz,$$

i.e. ω_0 is plane incarnation of our old friend dz.

Consider next:

$$x^k \frac{dx}{y}.$$

This extends to mero diff on X_d. When is it holo?

HW: $\quad x^k\,dx/y$ holo on $X_{2g+1}, X_{2g+2} \iff 0 \leq k \leq g-1$

Hint: Say $d = 2g+2$, let ∞_1, ∞_2 be the two pts over ∞. Then

$$\mathrm{ord}_{\infty_1}(x) = \mathrm{ord}_{\infty_2}(x) = -1$$

$$\mathrm{ord}_{\infty_1}(y) = \mathrm{ord}_{\infty_2}(y) = -(g+1),$$

$$\mathrm{ord}_{(\infty \text{ on } \mathbb{P}^1)}(dx) = -2$$

(For second formula: mero fn y has $2g+2$ zeroes on X_0, so must have $2g+2$ total poles at ∞. But $(x,y) \to (x,-y)$ extends to invol'n $X_0 \to X_0$ that takes ∞_1 to ∞_2...)

Plane Curves:

Consider alg. curve

$$X = \{H(Z_0, Z_1, Z_2) = 0\} \subseteq \mathbb{P}^2, \quad \deg H = d.$$

(H homog of deg d). We assume:

The three partials $\frac{\partial H}{\partial z_i}$ have no common zeroes

(Ex. Then implicit fn thm $\Rightarrow$ X is R.S.)

We want to write down holo & mero 1-forms on X. Will analyze what happens in local coords.

Consider $U_2 = \mathbb{C}^2 = \{z_2 \neq 0\}$: local coords

$$x = \frac{z_0}{z_2}, \quad y = \frac{z_1}{z_2},$$

$$X_2 = (X \cap U_2) \text{ defined by}$$

$$f(x,y) = H(x,y,1).$$

Consider

$$\frac{dx}{\partial f/\partial y}\Big|_{X_2} = \frac{-dy}{\partial f/\partial x}\Big|_{X_2}:$$

As above, this is holo 1-form on X_2. More generally, for any poly

$$p = p(x,y) \in \mathbb{C}[x,y],$$

$$\eta = \eta_p \underset{\text{def}}{=} p(x,y)\,\frac{dx}{\partial f/\partial y}$$

is holo form on X_2.

Ask: When does η_P extend to holo form on all of X?

· Let's see what happens on $\mathbb{C}^2 = U_0 = \{Z_0 \neq 0\}$

Local coords

$$U_2 \qquad\qquad\qquad U_0$$

$$x = \frac{Z_0}{Z_2} \qquad\qquad s = \frac{Z_1}{Z_0}$$

$$y = \frac{Z_1}{Z_2} \qquad\qquad t = \frac{Z_2}{Z_0}$$

Transitions:

$$s = y/x \qquad\qquad x = 1/t$$
$$t = 1/x \qquad\qquad y = s/t$$

Say $g(s,t) = H(1,s,t)$ is local eqn of $X_0 = (X \cap U_0)$.

Then

$$f(x,y) = H\left(\tfrac{1}{t}, \tfrac{s}{t}, 1\right)$$
$$= \frac{1}{t^d} H(1,s,t) = \frac{1}{t^d} g(s,t)$$

I.e.

$$f(x,y) = \frac{1}{t^d}\, g(s,t) = x^d g\left(\tfrac{y}{x}, \tfrac{1}{x}\right)$$

$$g(s,t) = t^d \cdot f\left(\tfrac{1}{t}, \tfrac{s}{t}\right).$$

So

$$\partial g/\partial s = t^d \frac{\partial f}{\partial y}\left(\frac{1}{t}, \frac{s}{t}\right) \cdot \frac{1}{t}$$

$$= t^{d-1} \frac{\partial f}{\partial y}\left(\frac{1}{t}, \frac{s}{t}\right),$$

so

$$\frac{\partial f}{\partial y}\left(\frac{1}{t}, \frac{s}{t}\right) = \frac{1}{t^{d-1}} \frac{\partial g}{\partial s}(s,t)$$

Now let's go back to

$$\eta = \eta_p = p(x,y) \frac{dx}{\partial f/\partial y}$$

and rewrite in s & t coords:

$$\eta = \frac{p\left(\frac{1}{t}, \frac{s}{t}\right) \cdot \left(-\frac{1}{t^2}\, dt\right)}{1/t^{d-1} \; \frac{\partial g}{\partial s}(s,t)}$$

$$= -t^{d-3}\, p\left(\frac{1}{t}, \frac{s}{t}\right) \frac{dt}{\partial g/\partial s}$$

i.e. η extends to holo 1-form on X_0 provided that

$$t^{d-3} p\left(\tfrac{1}{t}, \tfrac{s}{t}\right) \text{ holo.}$$

Now for poly $p(x,y)$,

$$t^{d-3} p\left(\tfrac{1}{t}, \tfrac{s}{t}\right) \text{ is poly in } s,t \text{ "} \Leftrightarrow \text{" } \deg p \leq d-3$$

A similar computation holds for $U_1 = \{z_1 \neq 0\}$ and one finds:

<u>Thm.</u> Let

$$X \subseteq \mathbb{P}^2$$

be smooth curve of degree d. Then have

$$\left\{ \begin{array}{l} \text{polys of deg} \\ \leq d-3 \text{ in} \\ 2 \text{ vars.} \end{array} \right\} \hookrightarrow \left\{ \begin{array}{l} \text{Holo 1-forms} \\ \text{on } X \end{array} \right\}.$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ p(x,y) & \longmapsto & \text{"} p(x,y) \dfrac{dx}{\partial f/\partial y} \text{"} \end{array}$$

Can view space on LHS as homog polys of deg = d-3

Sept. 22

<u>Rmks.</u> (1) Will see later this is isom.

(2). This is special case of "adjunction formula"

What about singular curves?

- Most Riemann sfs do not arise as sm plane curves.

- However will "see" that all curves can be "realized" as plane curves w nodes. So we want to study 1-forms on these.

<u>Set-Up:</u> Consider

$$X' = \text{compact RS}$$

$$\mu : X' \longrightarrow \mathbb{P}^2 \quad \text{holo map, gen 1-1 over image}$$

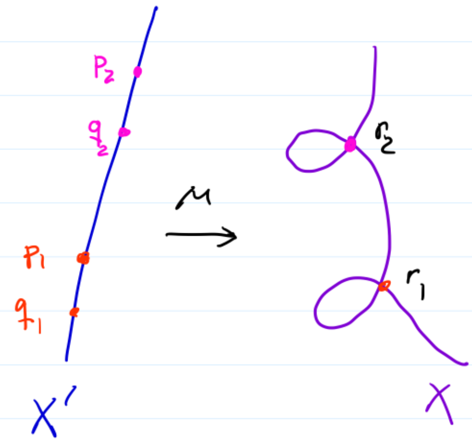
s.t.

$X =_{\text{def}} \mu(X')$ is curve of deg d w only simple nodes as sings

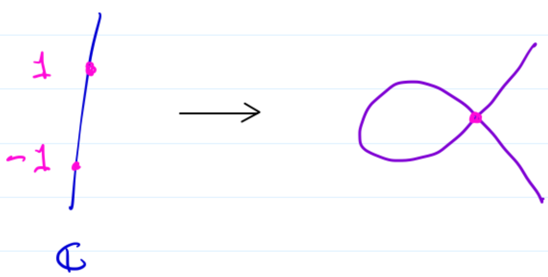
Ie. μ is isom away from finitely many pairs of pts

$p_1, q_1, \ldots, p_s, q_s \in X'$

which are mapped by μ to sing pts of X w. local analy equ $zw = 0$.



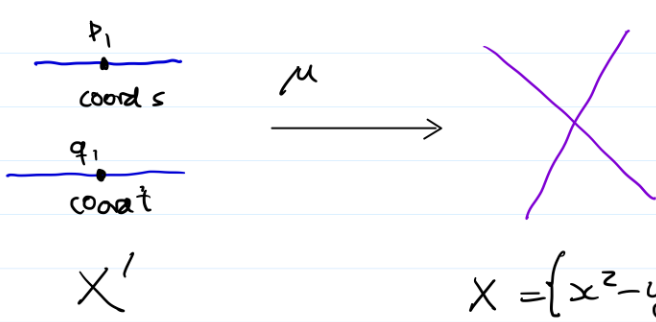
<u>Ex.</u> $\mu : \mathbb{C} \longrightarrow \mathbb{C}^2 \quad \mu(t) = (t^2 - 1, t^3 - t)$



$y^2 = x^2(x+1)$

· Want to understand what forms on $\mathbb{P}^2$ pull back to holo forms on X.

· Question local, so can take nbds of p_1, q_1 as follows:



$\mu(s) = (s, s)$
$\mu(t) = (t, -t)$

$X = \{x^2 - y^2 = 0\}$: $f(x,y) = x^2 - y^2$

Consider

$$\eta = \eta_p = p(x,y)\frac{dx}{\partial f/\partial y} = -2p(x,y)\frac{dx}{y}$$

$$\mu^*(\eta) = \begin{cases} -2p(s,s)\cdot\frac{ds}{s} & \text{near } p_1 \\ 2p(t,-t)\frac{dt}{t} & \text{"} \quad q_1 \end{cases}$$

See:

$$\boxed{\begin{array}{c} \mu^*(\eta) \text{ holo in nbd of } p_1, q_1 \\ \Updownarrow \\ p(0,0) = 0. \end{array}}$$

This "shows":

<u>Thm'</u>: Let

$$\mu : X' \to X \subseteq \mathbb{P}^2$$

be as in the set-up, and let

$$r_1, \ldots, r_\delta \in X$$

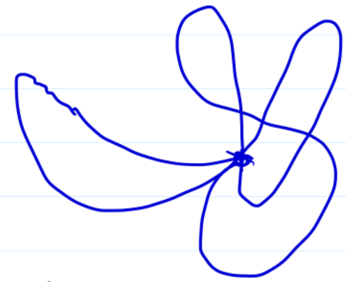
be the double pts of X. Then

$$\left\{\begin{array}{l} \text{homog polys of} \\ \text{deg } d-3 \text{ van at} \\ r_1, \ldots, r_\delta \end{array}\right\} \hookrightarrow \left\{\begin{array}{c} \text{holo 1-forms} \\ \text{on } X \end{array}\right\}$$

<u>Remarks on Sing Curves</u>:

Let $F = F(Z_0, Z_1, Z_2)$ be an arbitrary irred homog poly of deg d, let

$$X = \{F = 0\} \subseteq \mathbb{P}^2$$



This is plane curve of deg d that may have complicated sings.

<u>Fact</u>: $\exists!$ Riemann sf X', and

$$\mu : X' \longrightarrow X \subsetneq \mathbb{P}^2$$

(μ' is called "resoln of sings")

s.t. μ is isom away from sing pts of X.

<u>Constructions</u>:

(1). Griffiths, Chapter II

(2). Riemann surface of $f(x,y) = 0$ (Ahlfors, Chapt. 8)

(3). Algebraically: normalization

(4). "Embedded resoln" of curves on sf.

· Study of singular pts very interesting, eg:

Given two (germs of) sings

$$(X, 0) \quad , \quad (X', 0') \in \mathbb{C}^2$$

can you find local biholo isom

$$(\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$$

that takes X to X'? (No!)

· To prove Riemann-Roch, we will need to produce mero fns. The most down-to-earth way of doing this is to realize a Riemann sf as a plane curve w nodes

Proving everything in detail is painful and not terribly enlightening. So I'll occasionally limit myself to indicating main ideas.

Realizing Riemann sfs as plane curve w nodes.

Set-up:

$$X = \text{Compact RS}$$

Assume: X admits embedding

$$X \subseteq \mathbb{P}^N$$

as smooth alg curve, i.e. submfld cut out by homog polys.

("Recall" every X has such an embedding).

Thm: There is holo mapping

$$\mu : X \longrightarrow \mathbb{P}^2$$

that is everywhere an immersion (ie. $d\mu_x \neq 0 \ \forall x$), and that is an embedding away from finitely many pairs of pts

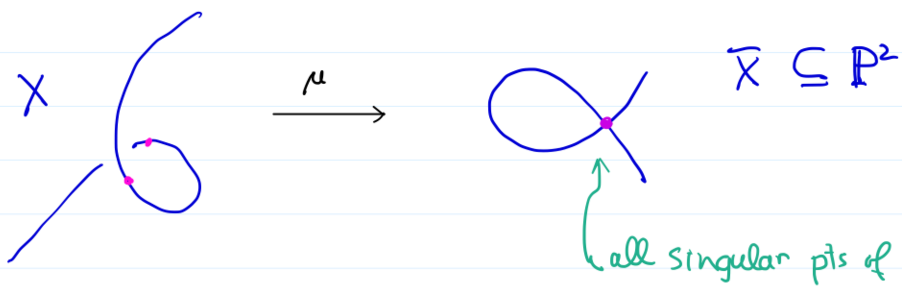
$$p_1, q_1 ; \ \dots , p_\delta, q_\delta \in X$$

which are mapped to ordinary double pts

$$r_1, \dots, r_\delta \in \overline{X} =_{\text{def}} \mu(X).$$

The image $\bar{X} \subseteq \mathbb{P}^2$ is an alg curve of some degree d, with δ ord double pts but no other sings.

<u>Picture</u> (as before)



all singular pts of $\bar{X}$ are locally (in suitable analy coords) $zw = 0$

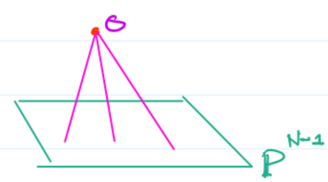
<u>Intuition</u>: We think of $\bar{X}$ as "realizing" X as plane curve w nodes

<u>Note</u>: $\bar{X} \not\cong X$, but $\bar{X} \sim_{\text{bir}} X$, and sings of $\bar{X}$ are suff simple that we can use $\bar{X}$ to compute anything we want to know about X. (Eg we've already seen how to constr mero 1-forms on X in terms of $\bar{X}$)

<u>Idea of Pf</u>: Consider $X \subseteq \mathbb{P}^N$, and suppose first $N \geq 4$. We'll show first that we can find new embedding

$$X \hookrightarrow \mathbb{P}^3.$$

· For this, pick $o \in \mathbb{P}^N$, $p \notin X$. Then proj from o defines a map



$$\pi = \pi_o : X \longrightarrow \mathbb{P}^{N-1}$$

(eg if $o = [0, .., 0, 1]$, $\pi_o([z_0, , z_N]) = [z_0, .., z_{N-1}]$).

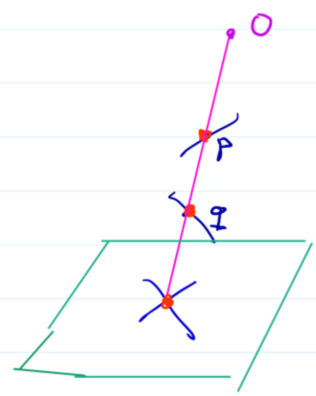
<u>Claim:</u> If $N \geq 4$ and O is general, then π_O is an embedding

$$\pi_O : X \hookrightarrow \mathbb{P}^{N-1}.$$

<u>Sketch of Pf of Claim:</u> π_O <u>fails</u> to be an embedding if and only if:

(a) $\pi_O(p) = \pi_O(q)$ for $p \neq q \in X$

(b) $d\pi_O(p) = 0$.



Now

(a) $\iff$ $O \in \overline{pq}$ (secant line joining p & q)

(b) $\iff$ $O \in \mathbb{T}_p X$ (embedded tang line to X at p.) ← a limiting case of (a) when $p = q$.

So we want to show that if $N \geq 4$, we can choose O to avoid both of these possibilities.

For this consider:

$$\mathbb{P}^N \supseteq \mathrm{Sec}(X) \underset{\text{def}}{=} \begin{matrix}\text{(Zariski)} \\ \text{Closure} \\ \text{of}\end{matrix} \left(\bigcup_{\substack{p,q \in X \\ p \neq q}} \overline{pq} \right)$$

See: (a) or (b) holds for $O \iff O \in \mathrm{Sec}(X)$. (Need to think about.)

So to prove Claim, it's suff to show:

Subclaim: $Sec(X) \subseteq \mathbb{P}^N$ is (Zariski) closed subvar of dim ≤ 3.

Main pt: $\dim Sec(X) \leq 3$.

To see this, consider affine piece: $X_0 \subseteq \mathbb{C}^N$. Now consider

$$\begin{array}{ccc} (X_0 \times X_0 - \Delta) \times \mathbb{C} & \longrightarrow & \mathbb{C}^N \\ \psi & & \psi \\ (p_0, q_0) \times t & \longmapsto & t p_0 + (1-t) q_0 \end{array}$$

Source has dim $= 3$, and $Sec(X_0)$ is closure of image. ∎

Upshot:

If $X \subseteq \mathbb{P}^N$ w $N \geq 4$, we can repeatedly apply Claim till we get to

$$N = 3.$$

So now consider:

$$X \subseteq \mathbb{P}^3.$$

Claim: if we take general $O \in \mathbb{P}^3$, proj

$$\pi_O : X \longrightarrow \bar{X} \subseteq \mathbb{P}^2$$

gives the required map to $\mathbb{P}^2$.

Rough idea: need to show

$O \notin$ any tang line $\mathbb{T}_p X$ (argue as before)

O lies only on finitely many "nice" secant lines $\overline{p_i, q_i}$. (This takes a little more work) ∎

<u>Upshot</u>: Now we'll be able to use computations w. plane curves to study Riemann sfs.

Bezout's Thm:

- If $f(z) \in \mathbb{C}[z]$ is poly of deg d, then

$$f(z) = 0 \text{ has } d \text{ solns (counting multiplicities)}$$

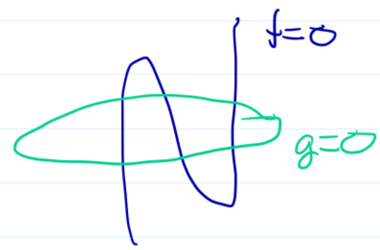
- Now suppose

$$f(z,w),\ g(z,w) \in \mathbb{C}[z,w]$$

are polys of degs d, e. <u>Ask</u>:

How many solns does (or do we expect) for the system:

$$f(z,w) = 0$$
$$g(z,w) = 0$$



<u>Bézout</u>: expect

$$d \cdot e = (\deg g) \cdot (\deg f) \text{ solns}$$

counting multiplicities, and "solns at ∞".

<u>Thm.</u> Let

$$X_d, Y_e \subseteq \mathbb{P}^2$$

be curves of deg d, e defined by homog. polys F_d, G_e

of degs d, e. Assume F, G have no common factors. Then

$$\#(X \cap Y) < \infty,$$

and $X \cap Y$ consists of $d \cdot e$ pts "counting multiplicities."

Rmk: In this statement, "curve" = "zeros of homog. poly."
We allow eg.

$$X_d = (\text{linear form } L)^d :$$

then X has line $L = 0$ as "d-fold multiple component."

Multiplicities

- Given $p \in \mathbb{P}^2$, let

$$\mathcal{O}_p = \mathcal{O}_p \mathbb{P}^2 = \text{germs of holo fns on } \mathbb{P}^2 \text{ at } p.$$

WLOG, can assume $p = 0 \in \mathbb{C}^2 \subseteq \mathbb{P}^2$, and then

$$\mathcal{O}_p = \mathbb{C}\{z, w\} =_{\text{def}} \text{ring of convergent power series in 2 vars}$$

Yoga: this ring captures all holo info of $\mathbb{P}^2$ in an (arb small) classical nbd of $p = 0$.

(Just as good; could work w. $\hat{\mathcal{O}}_p = \mathbb{C}[\![z, w]\!]$ formal power series
or
$$\mathcal{O}_p^{zar} = \mathbb{C}[z, w]_{(z,w)} \quad : \text{alg. local ring.}$$)

· Now say $p \in X \cap Y$. Let

$$f, g \in \mathcal{O}_p \text{ be local eqns of } X, Y$$

(Think: $f, g \in \mathbb{C}\{z, w\}$). Consider ideal

$$(f, g) \subseteq \mathcal{O}_p$$

<u>Fact:</u>

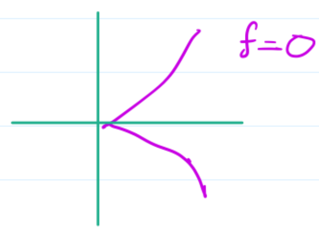
$$\dim_{\mathbb{C}} \mathcal{O}_p / (f, g) < \infty .$$

<u>Def:</u>

$$i_p(X, Y) = \dim_{\mathbb{C}} \mathcal{O}_p / (f, g)$$

<u>Exerc:</u> Get same answer from $\mathcal{O}, \hat{\mathcal{O}}, \mathcal{O}^{alg}$

<u>Ex.</u> $f = z^2 - w^3, \quad g = zw$



$\mathbb{C}$ basis for $\mathbb{C}\{z, w\}/(zw)$:

$$1, \begin{matrix} z & z^2 & z^3 & \cdots \\ w & w^2 & w^3 & \cdots \end{matrix}$$

Now mod out by $z^2 - w^3$. Then $w^3 \equiv z^2$ and

$$z^3 \equiv zw^3 \equiv 0, \quad w^4 \equiv wz^2 \equiv 0$$

So $\mathbb{C}$– basis for $\mathbb{C}\{z, w\}/(z^2 - w^3, zw)$ is

$$1, \begin{matrix} z, & z^2 \\ w & w^2, \end{matrix} \quad : \quad \text{so}$$

$$\boxed{i_p(f, g) = 5}$$

<u>Def:</u> Say X, Y meet <u>transversely</u> at p if

X and Y non-sing at p, and

$$T_p X + T_p Y = T_p \mathbb{P}^2$$

<u>Ex.</u> If X & Y meet transv. at p, then

$$i_p(X, Y) = 1$$

$x + x^3 y + y^2$

y

(and conversely).

<u>Pf.</u> Suppose f, g are local eqns of the two curves at $p = 0 \in \mathbb{C}^2$. Can suppose

$$f = z + HOT(z, w)$$
$$g = w + HOT(z, w)$$

<u>Claim:</u> can find biholo change of coords to new coords u, v st

$$f = u, \quad g = v.$$

(<u>Pf</u>: By inverse fn thm, map $(z, w) \longrightarrow (f(z,w), g(z,w)) = (u, v)$ invert. at origin),

So

$$i_p(f, g) = \mathbb{C}\{u, v\}/(u, v) = 1$$

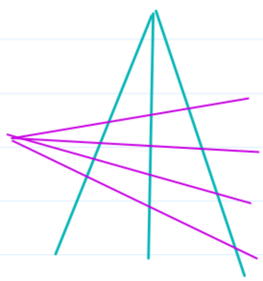
Converse for you.

Definitive version of B's thm is

<u>Thm:</u> Assume X_d, Y_e have no common comps. Then

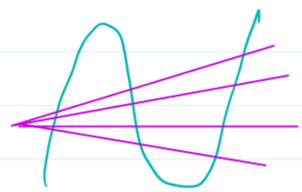
$$(*) \qquad \sum_{p \in X \cap Y} i_p(X, Y) = d \cdot e$$

Rmk: Essential content of Thm is that LHS of $(*)$ only depends on d, e. Once you know this you can reduce to special case, eg



$X = d$ lines
$Y = e$ lines

or



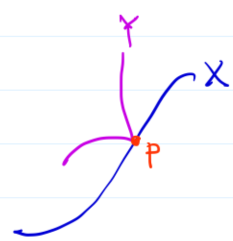
$Y = e$ lines

We'll prove Thm in special case when X is non-sing. (HW: when X has ord dps)

Lemma: In sit of Thm, assume X non-sing (ie an emb. RS). Let

$$G = \text{homog poly defining } Y$$

$$g = \text{local affine eqn defining } Y \text{ in nbd of } p.$$



View

$$g|X \text{ as holo fn on } X.$$

Then

$$i_p(X, Y) = \operatorname{ord}_p(g|X).$$

Pf: Let f be local eqn for X. Since X is non-sing, $\exists$ local analy parametrizn for X near p. i.e

$$X =_{loc.} \{(z, \phi(z))\}_{z \in \Delta} \quad \text{(say)}$$

w.

$$f(z, \phi(z)) \equiv 0.$$

Then

$$\mathcal{O}_p/(f) \xrightarrow[\sim]{\varphi^*} \mathbb{C}\{z\},$$

and

$$\mathcal{O}_p/(f,g) \cong \mathbb{C}\{z\}/(g(z,\phi(z))),$$

and the dim of this is $\mathrm{ord}_p(g|X)$.

<u>Pf of Bezout when X smooth.</u> Let

$$Y, Y' \subseteq \mathbb{P}^2$$

be two curves of deg e, defined by homog polys G_e, G_e' having no comps in common w X. By Rmk above suffices to show

$$\sum i_p(X,Y) = \sum i_p(X,Y').$$

For this, consider $\phi = G/G'$. This is mero fn on $\mathbb{P}^2$ (check!), and we consider $\phi|X$. Then by Lemma:

$$i_p(\phi|X) = \mathrm{ord}_p(g|X) - \mathrm{ord}_p(g'|X) \qquad g, g' \text{ local equs for } G, G'$$

$$\underset{\text{Lemma}}{=} i_p(X,Y) - i_p(X,Y').$$

But mero fn has same no of zeroes as poles, so

$$\sum i_p(\phi|X) = 0. \quad \text{QED}$$

Genus Formula:

<u>Thm</u>: Let $X \subsetneq \mathbb{P}^2$ be a non-sing curve of deg d. Then

$$g(X) = \binom{d-1}{2}.$$

Equiv.

$$2g(X) - 2 = d(d-3)$$

<u>Ex</u>:

d	2	3	4	5	6
g	0	1	3	6	10

etc.

<u>Rmk</u>: • All curves of $g=1$ are plane cubics
• "Most" curves of genus 3 are plane quartics
• For $d \geq 5$, plane curves of deg d are "increasingly special" among curves of genus $\binom{d-1}{2}$.

<u>Sketch of 1st Pf</u>:

<u>Recall</u>: If ω is any mero 1-form on X, then

$$(2g-2) = \sum_{x \in X} \mathrm{ord}_x(\omega).$$

Recall also: given

$$P = P(z_0, z_1, z_2) \text{ homog poly deg } d-3 \ (d \geq 3)$$

have

$$\omega_P =_{loc} p(x,y) \frac{dx}{\partial f / \partial y}.$$

Claim:

$$\mathrm{ord}_x(\omega_p) = i_x(X, (P=0)).$$

Idea:

$$\frac{dx}{\partial f/\partial y}\Big|X \text{ is holo and non-van in } \mathbb{C}^2,$$

so

$$\mathrm{ord}_x\left(p\,\frac{dx}{\partial f/\partial y}\right) = \mathrm{ord}_x(p|X).$$

By Lemma, this is $i_x(X, P=0)$.

So by Bezout,

$$2g-2 = d\cdot(d-3), \text{ as required.}$$

Sketch of Alternative Approach

- Say $X = \{F=0\}$, $F(0,1,0) \neq 0$
- Project from $[0,1,0]$ to get

$$\pi : X \longrightarrow \mathbb{P}^1, \quad \pi([X,Y,Z]) = [X,Z]$$

- Have

$$\deg(\pi) = d, \text{ so}$$

$$(2g-2) + 2d = \mathrm{ram}(\pi).$$

- Let $R = \{\frac{\partial F}{\partial Y}\}$: curve of deg $d-1$.

Claim:

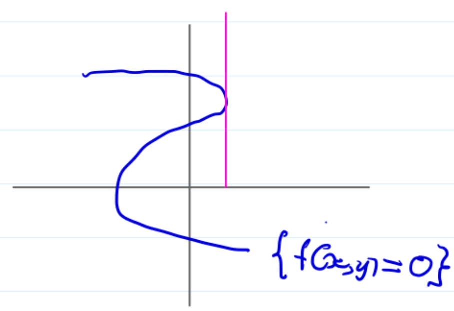
$$\mathrm{ram}_\pi(x) = i_x(X, R).$$

Granting claim, Bezout gives

$$(2g-2) + 2d = d(d-1), \text{ as required.}$$

Idea: Pass to affine coords $x = X/Z$, $y = Y/Z$. Then π is

$$(x, y) \longmapsto x$$



π ramifies at $(a,b) \in X$

$\Updownarrow$

tang vertical at (a,b)

$\Updownarrow$

$\frac{\partial f}{\partial y}(a,b) = 0.$

Moreover by taking local param, see multiplicities agree

Rmk: Suppose we use

$$\mu : X \longrightarrow \overline{X} \subseteq \mathbb{P}^2$$

to realize X as a plane curve of deg d w. δ ordinary double pts. Then

&

$$\boxed{\begin{aligned} g(X) &= \binom{d-1}{2} - \delta \\ 2g(X) - 2 &= d(d-3) - 2\delta \end{aligned}}$$

III. Riemann–Roch & Applications

Recall: when we studied ell fns we considered the vector spaces

$$V_k = \left\{ \text{ell fns } f \;\middle|\; \begin{array}{l} f \text{ has pole of order} \leq k \text{ on } \Lambda \\ f \text{ holo off } \mathbb{C}-\Lambda \end{array} \right\}$$

We showed:

$$\dim V_k = k \qquad (\text{when } k \geq 1).$$

This already implied that $\wp, \wp'$ must satisfy a diff eqn relating $(\wp')^2$ & $\wp^3$.

Analogous statement on arb RS is Riemann–Roch thm, which computes the dim of the vector spaces of mero fns w. bounded poles.

Divisors –

Divisors provide a language for discussing the analogues of the V_k.

$$X = \text{compact R.S.}$$

Def: A divisor on X is a finite formal $\mathbb{Z}$-linear comb of pt of X:

$$D = \sum_{P \in X} n_P P \qquad (n_P \in \mathbb{Z}, \text{ all but finitely many} = 0)$$

or

$$D = n_1 P_1 + \cdots + n_k P_k.$$

$\mathrm{Div}(X)$ = additive group of all such

(So $\mathrm{Div}(X)$ = free abelian group on pts of X)

<u>Degree</u> of divisor is

$$\deg(D) = \sum n_p \in \mathbb{Z}$$

D is <u>effective</u> if all $n_p \geq 0$: write $D \geq 0$

<u>Def</u>: Let f be a non-const mero fn on X. Define the divisor of f to be

$$(f) \underset{\text{AKA}}{=} \mathrm{div}(f) \underset{\text{def}}{=} \sum \mathrm{ord}_p(f) \cdot P$$

⊛ Divisor of rel fns form cf: <u>principal divisors</u>

Sim, for ω a mero 1-form, it's divisor is

$$\mathrm{div}(\omega) = \sum \mathrm{ord}_p(\omega) \cdot P$$

<u>Ex</u>. Have

$$\deg(\mathrm{div}(f)) = 0$$

$$\deg(\mathrm{div}(\omega)) = 2g-2.$$

The generalization of the spaces V_k for all fns are:

<u>Def</u>. Fix divisor D on X. Define

$$\mathcal{L}(D) = \{\text{mero } f \mid D + \mathrm{div}(f) \geq 0\}$$

By convention, we include 0-fn $\in \mathcal{L}(D)$, so $\mathcal{L}(D)$ a v.s. Use: $\mathrm{ord}_p(f+g) \geq \min(\mathrm{ord}_p(f), \mathrm{ord}_p(g))$

Unpacking: Say

$$D = n_1 P_1 + \quad + n_k P_k, \quad \text{all } n_i > 0.$$

$$f \in \mathcal{L}(D) \iff \mathrm{div}(f) \geq -n_1 P_1 - \quad - n_k P_k$$

$\iff$ f has poles of order $\leq n_i$ at P_i, and no other poles

Now say

$$D = n_1 P_1 + \quad + n_k P_k - m_1 Q_1 - \quad - m_\ell Q_\ell$$

$$n_i, m_j > 0$$

Then

$$f \in \mathcal{L}(D) \iff \mathrm{div}(f) \geq -n_1 P_1 - \ldots - n_k P_k + m_1 Q_1 + \quad + m_\ell Q_\ell$$

$\iff$ f has poles of order at most n_i at P_i; no other poles; and f has zeroes of order $\geq m_j$ at Q_j

i.e.

$\mathcal{L}(D) =$ spaces of mero fns w bounded poles and required zeroes

Rmk: $D \geqslant 0 \iff$ const fn $1 \in \mathcal{L}(D)$.

Ex. $X = \mathbb{C}/\Lambda$, $O \in X$: image of Λ. If

$$D = k \cdot O, \text{ then } \mathcal{L}(D) \cong V_k.$$

Ex. $X = \mathbb{C} \cup \infty = \mathbb{P}^1$, $D = k \cdot \infty$.

$$\mathcal{L}(D) \cong \{\text{polys } p(z) \text{ of deg} \leqslant k\}$$

Riemann-Roch Problem: Compute (or estimate)

$$\ell(D) =_{\text{def}} \dim \mathcal{L}(D).$$

Prop. $\ell(D) \leqslant \deg D + 1$.

Pf: HW.

Linear Equivalence:

Def: Two divisors D_1, D_2 are linearly equivalent if

$$D_1 - D_2 = \text{div}(f) \text{ some mero fn } f.$$

Notation: $D_1 \equiv D_2$.

NB: $D_1 \equiv D_2 \Rightarrow \deg(D_1) = \deg(D_2)$

Def.

$$\mathcal{Cl}(X) = \mathrm{Pic}(X) = \mathrm{Div}(X)/\equiv$$

Sim:

$$\mathrm{Pic}^0(X) = \mathrm{Div}^0(X)/\equiv$$

divisors of deg 0,

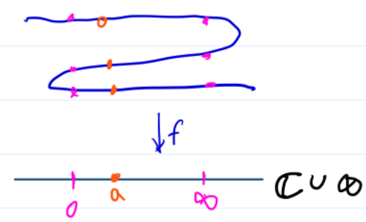
Ex. Say $f \in \mathbb{C}(X)$ non-const mero fn. View f as defining

$$f : X \to \mathbb{P}^1 \quad \text{: say deg } d$$

Write

$$\mathrm{div}(f) = \underbrace{D_0}_{\text{divisor of 0's}} - \underbrace{D_\infty}_{\text{divisor of poles}} \quad :$$

: eff divisors of deg d.



So $D_0 \equiv D_\infty$.

More gen, given any $a \in \mathbb{C}$, let

$$D_a = f^*(a) = \sum_{P \in f^{-1}(a)} e_f(P) \cdot P \quad :$$

Then

$$D_a \equiv D_0 \quad \text{all } a \in \mathbb{C},$$

(Reason: $\mathrm{div}(f-a) = D_a - D_\infty$)

Conversely: Suppose D, D' eff divisors w disjoint support s.t.

$$D \equiv D' ; \qquad \text{say} \quad D - D' = \mathrm{div}(f)$$

View.

$$f : X \longrightarrow \mathbb{P}^1, \quad \text{then}$$

$$D = f^*(0), \quad D' = f^*(\infty)$$

<u>Def:</u> Consider holo

$$\varphi : X \longrightarrow \mathbb{P}^r, \quad \varphi(x) \text{ not in hyperplane.}$$

Given any hyperplane $H \subseteq \mathbb{P}^r$, can define

$$\varphi^*(H) \underset{\text{AKA}}{=} X \cdot_\varphi H \ : \text{eff divisor on } X$$

(Exerc: $\varphi^*(H)$ supported on $\varphi^{-1}(H)$, multiplicities via local eqn.)

Then

$$\varphi^*(H_1) \equiv \varphi^*(H_2)$$

all hyperplanes. $\left(\varphi^*(H_1) - \varphi^*(H_2) = \operatorname{div.}\left({H_1}/{H_2} \,\middle|\, X \right) \right)$

<u>Ex</u> Suppose $X = \mathbb{P}^1$. Then

$$D \equiv D' \iff \deg D = \deg D',$$

So

$$\mathcal{Cl}(\mathbb{P}^1) = \mathbb{Z},$$

<u>Ex.</u> Suppose $X = \mathbb{C}/\Lambda$ is ell curve. Using group law on X, have a canonical map

$$\begin{array}{ccc} u: \mathrm{Div}^0(X) & \longrightarrow & X \\ \Psi & & \Psi \\ \sum n_p \cdot [P] & \longmapsto & \sum n_p P \end{array}$$

formal sum as divisor

sum in group law

Abel's Thm: $\ker(u) = \mathrm{Prin}(X)$.

So:

$$\boxed{Cl^0(X) \cong X}$$ as groups.

(Eventually we'll extend this to arb comp. RS)

<u>Ex</u> Let ω_1, ω_2 be mero 1-forms. Then

$$\mathrm{div}(\omega_1) \equiv \mathrm{div}(\omega_2).$$

If $K = \mathrm{div}(\omega)$, then

$$\mathcal{L}(K) \cong \{\text{holo 1-forms}\}.$$

<u>Pf:</u> We saw that $\omega_1 = \varphi\, \omega_2$ for some $\varphi \in \mathbb{C}(X)$, so

$$\mathrm{div}(\omega_1) = \mathrm{div}(\varphi) + \mathrm{div}(\omega_2)$$

Now suppose $f \in \mathcal{L}(\mathrm{div}(\omega))$. Then

$$\mathrm{div}(f\omega) = \mathrm{div}(f) + \mathrm{div}(\omega) \geq 0,$$

so $f\omega$ holo 1-form. Converse similar.

Prop. Suppose $D \equiv D'$. Then

$$\mathcal{L}(D) \cong \mathcal{L}(D').$$

Pf. Say $D - D' = \mathrm{div}(\varphi)$, $\varphi \in \mathbb{C}(X)$. Then

$$\mathrm{div}(f) + D \geq 0 \iff \mathrm{div}(f) + \mathrm{div}(\varphi) + D' \geq 0$$
$$\iff \mathrm{div}(f\varphi) + D' \geq 0.$$

So get isom:

$$\mathcal{L}(D) \longrightarrow \mathcal{L}(D'), \quad f \longmapsto f \cdot \varphi.$$

Linear series

Def: Let D be any divisor on X. Define

$$|D| = \{\text{eff. divisors } D' \mid D' \equiv D\}$$
$$=_{\text{exerc}} \{\mathrm{div}(f) + D \mid f \in \mathcal{L}(D)\}$$

⊗ "complete linear series" or system assoc to D

Ex $K = \mathrm{div}(\omega)$: $|K| = \{\text{all divisors of holo 1-forms}\}$

Exerc (add to HW). Say $D \geq 0$, choose basis

$$1, f_1, \ldots, f_r \in \mathcal{L}(D).$$

Assume that the poles of $f_1, \ldots, f_r$ don't have any common pt. Consider

$$\varphi: X \longrightarrow \mathbb{P}^r, \quad x \longmapsto \text{"}[1, f_1(x), \ldots, f_r(x)]\text{"}.$$

Then $D' \in |D| \iff$

$$D' = X \cdot_{\varphi} H' \quad \text{some hyperplane } H' \subseteq \mathbb{P}^r.$$

<u>Aside ~ sheafy point of view.</u>

Divisor D on X $\leftrightarrow$ lb $\mathcal{O}_X(D)$ on X w mero section s' s.t. $\mathrm{div}(s) = D$.

$$\mathcal{L}(D) = \Gamma(X, \mathcal{O}_X(D))$$

$$D \equiv D' \iff \mathcal{O}_X(D) \cong \mathcal{O}_X(D')$$

$$|D| = \{ \mathrm{div}(s) \mid s \in \Gamma(X, \mathcal{O}_X(D)) \}$$

Map $\varphi : X \longrightarrow \mathbb{P}^r$ corresp to basis $1, f_1, \ldots, f_r \in \mathcal{L}(D)$

$$\updownarrow$$

$\varphi : X \longrightarrow \mathbb{P}^r$, $\varphi(x) = [s_0(x), \ldots, s_r(x)]$, $s_0, \ldots, s_r \in \Gamma(X, \mathcal{O}_X(D))$ a basis.

Our next goal — which will take a while — is to prove

<u>Riemann's Thm:</u> Consider

X = compact RS (proj alg curve)

g = genus (X)

D = divisor on X, $d = \deg(D)$.

Then

$$\dim \mathcal{L}(D) \geqslant d + 1 - g$$

<u>Rmk:</u> Will see later on that equality holds when $d > 2g-2$.

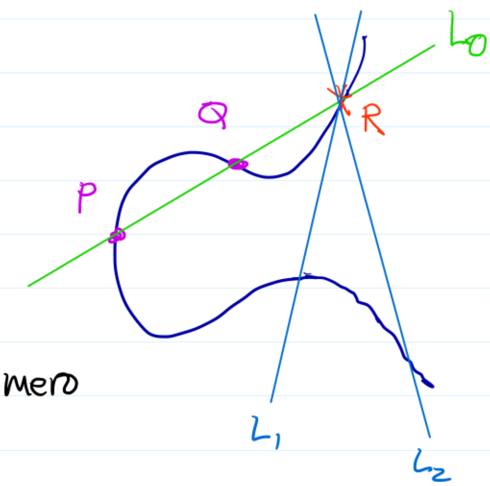
Ex. $X = \mathbb{P}^1$, $D = d\cdot(\infty)$. Then

$$\mathcal{L}(D) = \text{polys deg} \leq d \ : \ \dim\ d+1$$

Ex. $X = \mathbb{C}/\Lambda$, $D = d\cdot[0]$. Then

$$\mathcal{L}(D) \cong V_d \ : \ \dim d \text{ when } d \geq 1.$$

Idea will be to use geometry of plane curves. Let me start w toy example to explain idea.



Ex. Consider

$$C \subseteq \mathbb{P}^2 \text{ plane cubic}$$

(so $g=1$). Take $D = P+Q$. Let's try to construct geometrically two mero fns

$$\varphi_1, \varphi_2 \in \mathcal{L}(P+Q).$$

Plan: look for $\varphi_i = \frac{L_i}{L_0}\Big|C$, L_i linear forms

φ_1, φ_2 allowed to have poles at P, Q,

so take

L_0 = linear form defining line thru P, Q.

Problem: L_0 meets C at third pt R, and φ_i not allowed to have pole at R.

Soln: Take L_1, L_2 eqns of lines passing thru R.

Then
$$\mathrm{ord}_R\left(L_i/L_0 \,|\, C\right) = 0$$
(if L_i not tang to C at R), so good. We use

<u>Lemma:</u>
$$\dim_{\mathbb{C}}\left\{\begin{matrix}\text{linear forms}\\ L \text{ on } \mathbb{P}^2\end{matrix} \;\middle|\; \begin{matrix}L \text{ vanishes at}\\ \text{fixed pt } R\end{matrix}\right\} = 2.$$

So we take L_1, L_2 to be a basis.

<u>Plan:</u> Given arbitrary X, realize X as plane curve w nodes, proceed similarly.

<u>Main differences:</u> (i). Look for φ that are ratios of homog polys of large degree (not deg 1)

(ii) Heavier book-keeping.

To get going: need to understand something about the vector spaces of homog polys vanishing at given pts.

Linear Systems of Plane Curves

Consider
$$V_d = \left\{\begin{matrix}\text{all homog. polys}\\ \text{deg } d \text{ in 3 vars}\end{matrix}\right\} \quad (= S^d V_1)$$

This is cx vector space,

$$\dim_{\mathbb{C}} V_d = \binom{d+2}{2}.$$

Can view $\mathbb{P}(V_d)$ as space of all plane curves of deg d

Ex All plane conics form a $\mathbb{P}^5$:

- General conic is

$$Q = aX^2 + bXY + cY^2 + dYZ + eZ^2 + fXZ$$

$$\updownarrow$$

$$[a, b, \dots, f] \in \mathbb{P}^5$$

Lemma: Fix $p \in \mathbb{P}^2$, and put

$$V_d(p) = \{ F \mid F(p) = 0 \};$$

Then $V_d(p)$ is a codim 1 linear subspace in V_d.

Pf Say $p = [a, b, c]$,

$$F = \sum t_{ijk} X^i Y^j Z^k$$

(so $[t_{ijk}]$ are coords of F in $\mathbb{P}V_d$). Then

$$F(p) = 0 \Leftrightarrow \sum t_{ijk} \cdot (a^i b^j c^k) = 0 \quad (*)$$

and $(*)$ is linear cond in t_{ijk}.

(Challenge question: Describe set of all hyperplanes $V_d(p) \subseteq V_d$

Cor: Given any pts

$$p_1, \dots, p_r \in \mathbb{P}^2,$$

the set

$$V_d(p_1, \dots, p_r) =_{def} \{F \mid F(p_i) = 0 \text{ all } i\}$$

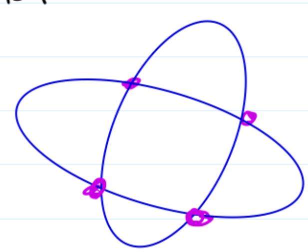
is linear subspace of V_d of codim $\leq r$.

Rmk: The actual codim can depend on the config of the p_i.

Eg take $d=2$ (conics) $r=4$ (thru 4 pts). Check:

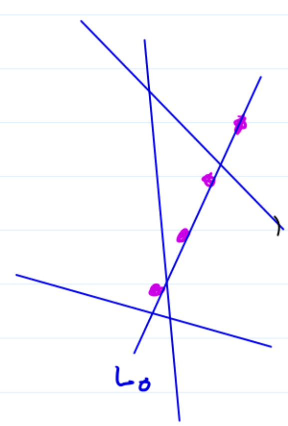
- If the 4 pts are non collinear, then

$$\dim V_d(p_1, \ \ , p_4) = 2$$



- If the four pts are collinear, then

$$\dim V_d(\ \ ) = 3, \quad \text{codim} = 3$$



Reason: if $L_0 = 0$ is line thru P_i, then

$$XL_0, YL_0, ZL_0 \in V_d(P_1, \ \ , P_4).$$

(Fancy viewpoint: if $Z = \{P_1, \ , P_r\}$ question is whether

$$H^1(\mathbb{P}^2, \mathcal{I}_{Z/\mathbb{P}^2}(d)) = 0 \text{ or } \neq 0)$$

Now recall Riemann's Thm:

Thm: Consider

X = sm proj curve, genus g.
D = divisor on X, deg $= d$

Then

$$\dim \mathcal{L}(D) \geq d + 1 - g.$$

Simplifying assumption:

D effective, $D = P_1 + \cdots + P_d$ (distinct)

(Assuming effectivity just reduces layer of notation. Dealing w repeated pts involves looking at polys w. given derivs, doesn't really change anything.)

(1°) Realize X as a plane curve w nodes:

$$\phi : X \longrightarrow \overline{X} \subseteq \mathbb{P}^2$$

w

$\overline{X}$ = plane curve of deg f : $F=0$
w. only ord nodes

$\cup$

Δ = nodes of $\overline{X}$.

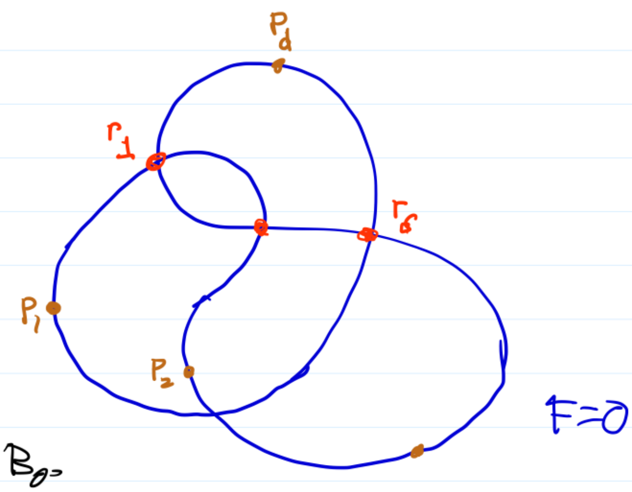
$\# \Delta = \delta$ Assume no P_i occurs in Δ.

$\Delta = \{ r_1, \ , r_\delta \}$

Recall

$$g = \frac{(f-1)(f-2)}{2} - \delta$$

Recall: we'll looking for mero fns that are allowed to have poles at P_i, nowhere else.



(2). Choose (general) homog. poly $B = B_{\theta}$ of deg

$$e \gg 0,$$

passing thru

$$P_1, \ , P_d \quad \& \ \Delta, \qquad (*),$$

(not containing $\overline{X}$ as comp). This will be denom of our mero fns.

<u>Observe:</u>

$(B=0)$ meets $\overline{X}$ in other pts as well, besides $(*)$. Call these R.

How many pts in R?

Note: $i_{node}(\overline{X}, B) \geq 2$ $\;(=2$ in general$)$

So by Bezout!

$$f \cdot e = \underbrace{d + 2\delta}_{D \,\&\, \Delta} + \# R$$

So

$$\boxed{\# R = fe - 2\delta - d}$$

(3°). We're looking to constr elts in $\mathcal{L}(D)$ via

$\varphi^*\left(\frac{A}{B_0}\right)$, A another homog poly of deg e,

so we need to force A to vanish on R & Δ

Let

$$V = V_e(R, \Delta) = \left\{ \begin{array}{l} \text{homog polys of} \\ \text{deg } e \text{ van at } R, \Delta \end{array} \right\}.$$

Have map

$$\begin{array}{ccc} V & \xrightarrow{\rho} & \mathcal{L}(D) \\ \Psi & & \Psi \\ A & \longmapsto & \varphi^*\left(\frac{A}{B_0}\right) \end{array}$$

Note:

(*) $$\dim V \geq \binom{e+2}{2} - \underbrace{(fe - 2\delta - d)}_{\# R} - \underbrace{\delta}_{\# \Delta}$$

$$= \binom{e+2}{2} - fe + \delta + d$$

However — ρ is not injective!!

In fact:

$$\varphi^*\left(\frac{A}{B_0}\right) = 0 \iff A|X_0 \equiv 0$$
$$\iff F \mid A.$$

Ie.

$$\ker\rho = \{A \mid A = F\bar{A},\ \deg\bar{A} = e-f\}$$
$$\cong V_{e-f}.$$

Upshot: have exact seq:

$$0 \longrightarrow V_{e-f} \longrightarrow V_e(R,\Delta) \xrightarrow{\rho} \mathcal{L}(D),$$

so

$$\dim \mathcal{L}(D) \geqslant \dim V_e(R,\Delta) - \dim V_{e-f}$$

$$\geqslant \underbrace{\left(\binom{e+2}{2} - fe + d + \delta\right) - \binom{e-f+2}{2}}_{\text{call this } M},$$

(4°). Now simplify. Find:

$$M = d + \delta + 1 - \binom{f-1}{2} = d + 1 - g.$$

Done!

Riemann–Roch

Thm. Consider:

X = sm proj curve (compact R.S)
$g = \text{genus}(X)$

$K = K_X = \text{div}(\omega)$: a canonical divisor
(so $\deg K = 2g-2$)

D = any divisor on X of deg d

Then:

$$\boxed{\dim \mathcal{L}(D) = d + 1 - g + \dim \mathcal{L}(K-D)}$$

Rmk: If $D \geq 0$, then

$$\mathcal{L}(K-D) \cong \{\text{holo diffs vanishing on } D\}$$

Notation: $\ell(D) = \dim \mathcal{L}(D)$.

Rmk/Lemma: $\ell(K) = \dim\{\text{holo diffs}\} \geq g$.

Pf. Realize X as plane curve of deg f w. δ nodes. Then

$$\left\{\begin{matrix}\text{curves of deg } f-3 \\ \text{van on nodes}\end{matrix}\right\} \hookrightarrow \{\text{holo diffs}\}$$

$\dim \geq \binom{f-1}{2} - \delta$
$= g$.

(RR will say that $\ell(K) = g$.)

Recall - we already know (Riemann's Thm)

$$\ell(D) \geqslant d+1-g.$$

Main Lemma: Let D be an effective divisor of deg. d. Then

$$\ell(D) \leqslant d+1-g+\ell(K-D)$$

Idea: Via residue thm, holo diffs give obstructions to existence of fns in $\mathcal{L}(D)$!!

Proof of Lemma: Assume

$$D = P_1 + \quad + P_d \quad \text{(distinct pts, for simplicity)}.$$

(1°). Fix basis

$$\omega_1, \ldots, \omega_s \in \{\text{holo diffs}\}.$$

(As we've seen: $s \geqslant g$). Let

$$f_1, \quad , f_k \in \mathcal{L}(P_1 + \cdots + P_d) \text{ be a basis.}$$ so $k = \ell(D)$

Then $f_\alpha \omega_j$ a mero 1-form, w poles at most at P_1, P_d. Have

$$\boxed{(*) \quad \sum_{p \in X} \operatorname{res}(f_\alpha \omega_j) = 0 \quad \text{all } \alpha, j.}$$

We want to unwind what this says.

(2°). For rest of this proof, fix local coords

$$z_1, \quad , z_d \quad \text{centered at } P_1, \quad , P_d.$$

Write locally

- $$\omega_i = \phi_{ij}(z_j)\, dz_j \quad (\text{near } P_j),$$

so

$$\text{"}\omega_i(P_j)\text{"} = \phi_{ij}(0)$$

- $$f_\alpha = \frac{\beta^j_\alpha}{z_j} + \text{holo}(z_j) \quad (\text{near } z_j)$$

for some $\beta^j_\alpha \in \mathbb{C}$.

So:

$$\boxed{\begin{aligned} \operatorname{res}_{P_j}(f_\alpha \omega_i) &= \phi_{ij}(0) \cdot \beta^j_\alpha \\ &= \omega_i(P_j) \cdot \beta^j_\alpha \end{aligned}}$$

(3°). Now consider the $s \times d$ matrix:

$$B = \begin{array}{cc} \begin{bmatrix} \omega_1(P_1) & \cdots & \omega_1(P_d) \\ \omega_2(P_1) & \cdots & \omega_2(P_d) \\ & \cdots & \\ \omega_s(P_1) & \cdots & \omega_s(P_d) \end{bmatrix} & s \\ d & \end{array}$$

View B as defining

$$B : \mathbb{C}^d \longrightarrow \mathbb{C}^s$$

Now for each $1 \le \alpha \le k$, let

$$\vec{v}_\alpha = \begin{pmatrix} \beta^1_\alpha \\ \\ \beta^d_\alpha \end{pmatrix} = \text{"vector of polar parts of } f_\alpha\text{"}$$

As noted,

$$\mathrm{Res}_{P_j}(f_\alpha \omega_i) = \omega_i(P_j)\, \beta^\alpha_j ,$$

so residue Thm says:

$$(**) \qquad B \cdot \vec{v}_\alpha = 0 \quad \text{for} \quad 1 \le \alpha \le k.$$

Note: $\vec{v}_\alpha = 0 \iff f_\alpha$ has no poles
$\iff f_\alpha$ is const.

So:

$$\frac{\mathcal{L}(D)}{(\text{consts})} \hookrightarrow \ker(B),$$

(We don't at the moment know that equality holds, since there may be vectors $\vec{v}$ that satisfy $B \cdot \vec{v} = 0$ without $\vec{v}$ being actual vector of polar part of $f \in \mathcal{L}(D)$)

i.e.

$$\ell(D) \le 1 + \dim \ker(B).$$

(4°). So now we need to study rk of B.

$$\dim \ker(B) = \underset{\substack{\uparrow \\ \text{dim of} \\ \text{source}}}{d} - \underset{\substack{\uparrow \\ \text{dim of} \\ \text{image}}}{s} + \dim \left\{ \begin{array}{l} \text{linear relations among} \\ \text{rows of } B \end{array} \right\}$$

<u>Note:</u> linear relation among rows of B

$$\Updownarrow$$

holo diff van at $P_1, \ldots, P_d$!!

(Rechoose basis of holo diffs so that first g vanish at $P_1, \ldots, P_d$ and no lin comb of remaining $s-g$ do.)

i.e.

$$\dim \ker B = d - s + \ell(K-D).$$

so.

$$\begin{aligned} \ell(D) &\leq 1 + \dim \ker B \\ &= d + 1 - s + \ell(K-D) \\ &\leq d + 1 - g + \ell(K-D) \qquad (\text{since } s \geqslant g). \end{aligned}$$

QED!

<u>Rmk:</u> Once we know RR, it will follow from this argument that $\vec{v}$ is vector of pr parts of some $f \in \mathcal{L}(D) \iff$

$$B \cdot v = 0.$$

Cor of Pf. $s = g$, ie

$$\dim\{\text{holo diffs}\} = g.$$

Pf Say $D = P_1 + \cdots + P_d$. We know

$$d+1-g \underset{\substack{\text{Riem.}\\ \text{Thm}}}{\leq} \ell(D) \leq d+1-s+\ell(K-D),$$

Take $d > 2g-2$, so $\ell(K-D) = 0$. Then

$$d+1-g \leq d+1-s,$$

so

$$s \leq g.$$

But we already know $s \geq g$. QED

Lemma 2: Let D be effective divisor of deg d.
Then

$$\ell(D) = d+1-g+\ell(K-D)$$

Pf. Case 1: $\ell(K-D) = 0$: OK by R's Thm & Main Lemma

Case 2: $\ell(K-D) \neq 0$,

Then $\mathcal{L}(K-D) \neq 0$, ie $K-D \equiv E$ effective.

$$\deg(E) = 2g-2-d.$$

Now apply Main Lemma to E:

$$\ell(E) = \ell(K-D) \leq (2g-2-d) + 1 - g + \ell(K-E)$$

$$= g-1-d + \ell(D)$$

ie

$$\ell(D) \geq d + 1 - g + \ell(K-D),$$

so equality holds by Main Lemma.

<u>Proof of RR</u>– Need to show:

$$\ell(D) = d + 1 + \ell(K-D).$$

- Say $\ell(D) > 0$: then $D \equiv D' \geq 0$, so done by previous Lemma.
- Say $\ell(K-D) > 0$: Then $K-D \equiv E \geq 0$, and RR follows as above by applying previous Lemma to E.
- So reduced to the case:

$$\ell(D) = \ell(K-D) = 0.$$

Apply Riemann's Thm to D, K-D:

$$0 \geq d + 1 - g, \text{ ie. } d \leq g-1$$

$$0 \geq (2g-2-d) + 1 - g, \text{ ie. } d \geq g-1$$

So if $\ell(D) = \ell(K-D) = 0$, then $d = g-1$ and RR true by inspection!

<u>QED.</u>

Sheafy Interpretation: $D \rightsquigarrow$ lb $\mathcal{O}_X(D)$

$$\ell(D) = \dim \Gamma(X, \mathcal{O}_X(D)) = h^0(X, \mathcal{O}_X(D))$$

Serre duality: $H^1(X, \mathcal{O}_X(D))$ dual to $H^0(X, \mathcal{O}_X(K-D))$, ie.

$$\ell(K-D) = h^1(X, \mathcal{O}_X(D)).$$

So RR says:

$$h^0(X, \mathcal{O}_X(D)) - h^1(X, \mathcal{O}_X(D)) = d+1-g.$$

The importance of RR comes from its applns to:

Linear series

Def: Let D be any divisor on X. Define

$$|D| = \{\text{eff. divisors } D' \mid D' \equiv D\}$$

$$=_{\text{exerc}} \{\text{div}(f) + D \mid f \in \mathcal{L}(D)\}$$

⊗ "complete linear series" or system assoc to D

Ex. Say D effective, $\ell(D) = 2$. Choose basis

$$1, f \in \mathcal{L}(D),$$

view f as map

$$X \xrightarrow[\substack{f\\ \parallel \\ \varphi}]{} \mathbb{P}^1$$

Assume that $(f)_\infty = D$. Then

$$D = f^*(\infty)$$

and the divisors $D' \in |D|$ are precisely the preimages of pts on $\mathbb{P}^1$. $(\text{div}(f-\alpha) = f^*(\alpha) - D)$

⊗ Note that

$D' \in |D|$

$\updownarrow$

$f \in \mathcal{L}(D)$ (up to scalar)

$(D' \leftrightarrow D + \text{div}(f))$

Give $|D|$ str of proj space

$|D| = \mathbb{P}(\mathcal{L}(D))$

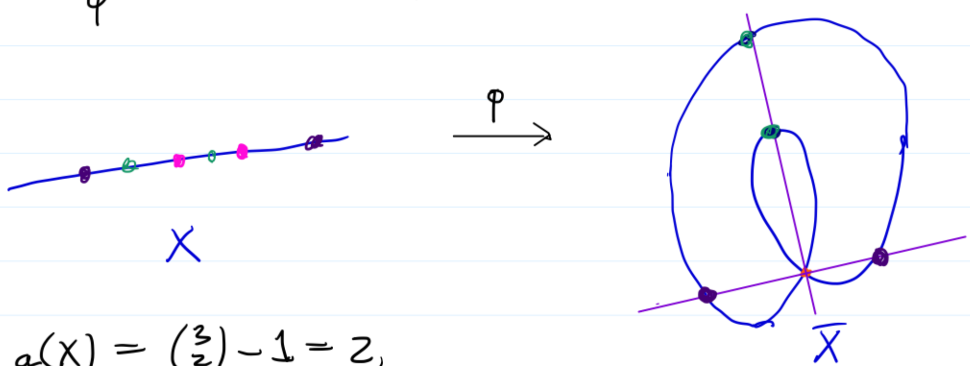
Write $\dim |D| = \dim \mathbb{P}(\ ) = \ell(D) - 1$

⊛ Sheafy: $|D| = \{\mathrm{div}(s) \mid s \in \Gamma(X, \mathcal{O}_X(D))\}$

Ex. K = canonical divisor.

$$|K| = \{\mathrm{div}(\omega) \mid \omega = \text{holo 1-form}\} \;:\; \text{so } |K| = \mathbb{P}\{\text{holo 1-forms}\}$$

Eg. $X \xrightarrow[\varphi]{} \overline{X} \subseteq \mathbb{P}^2$ deg 4 w. 1 node.



So $g(X) = \binom{3}{2} - 1 = 2$,

and $|K| = \{\text{divisors on } X \text{ residual to } \overline{X} \cdot \text{line thru node}\}$

Linear Series and Maps to $\mathbb{P}^n$:

Def. D a divisor on X. Say $|D|$ is basept free (bpf) if following holds:

$$\text{For every } P \in X, \ \exists \ D' = D'_P \in |D| \text{ s.t. } p \notin \operatorname{Supp} D'$$

$$\Updownarrow$$

$$\forall\, p \in X, \ \exists \ f = f_p \in \mathcal{L}(D) \text{ s.t.}$$

(*) $$\operatorname{ord}_p(f_p) + \operatorname{ord}_p D = 0$$

Assume $D \geq 0$, say $D = n_1P_1 + \; + n_rP_r \quad n_i > 0$. Then

$$(*) \Leftrightarrow \exists f_i \text{ s.t. } f_i \in \mathcal{L}(D),\ \operatorname{ord}_{P_i}(f_i) = -n_i$$

Key Constr: Assume $|D|$ bpf, eg $D \geq 0$. Choose basis

$$1, f_1, \ , f_r \in \mathcal{L}(D)$$

Define:

$$\varphi : X \longrightarrow \mathbb{P}^r, \text{ via } \varphi(z) = \text{"}[1, f_1(z), \ , f_r(z)]\text{"}$$

Then

$$D' \in |D| \iff \exists \text{ hyperplane } H' \subseteq \mathbb{P}^r \text{ s.t. } D' = X \cdot_{\varphi} H'$$

ie. Divisors in bpf linear series are hyperplane sections of X under $\varphi : X \longrightarrow \mathbb{P}^r$.

(Pf is on HW).

So: if D bpf, then

$$|D| = \mathbb{P}^{r*} = \{\text{proj space of hyperplanes in } \mathbb{P}^r\}$$

NB: $\mathcal{L}(D-P) \subseteq \mathcal{L}(D)$
$\quad\quad = \{f \mid D + \mathrm{div}(f) \ni p\}$

Prop: $|D|$ is bpf $\iff$ $\circledast$

$$\ell(D-P) = \ell(D) - 1 \quad \text{all } P \in X.$$

Pf: $|D|$ bpf $\iff$ $\forall\, p \in X, \exists$

$$D' \in |D| \text{ s.t. } P \notin \mathrm{Supp}(D')$$

If $D' = D + \mathrm{div}(f_0)$, then $f_0 \in \mathcal{L}(D)$, $f_0 \notin \mathcal{L}(D-P)$.

Now say $|D|$ bpf. Then get

$$\varphi_{|D|} : X \longrightarrow \mathbb{P}^r$$

Def. Say $|D|$ <u>very ample</u> if φ_D an embedding.

<u>**Prop:**</u> Assume $|D|$ bpf. $\varphi_{|D|}$ an embedding

$$\Updownarrow$$

$\forall\, P, Q \in X$ (including $P = Q$),

$$\ell(D-P-Q) = \ell(D) - 2.$$

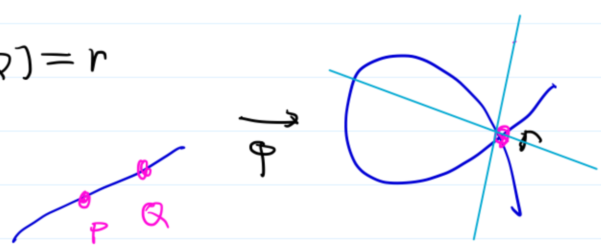
<u>Pf:</u> $\varphi_{|D|}$ an embedding

$$\Updownarrow$$

(i) $\varphi_D : X \longrightarrow \mathbb{P}^r$ is 1-1

(ii) $d\varphi_D \neq 0$ at every pt.

Suppose $\varphi_{|D|}(P) = \varphi_{|D|}(Q) = r$
Then



$\forall\, H \subseteq \mathbb{P}^n$ hyperpls thru r

$\varphi^*(H)$ vanishes at P & Q

ie only <u>one</u> condition for $D' \in |D|$ to contain both P & Q
i.e.

$$\ell(D-P-Q) = \ell(D) - 1$$

Sim, suppose $d\varphi = 0$ at P,
Then if H passes thru P,
$\mathrm{ord}_P(\varphi^* H) \geq 2$. ...

Prop: If

- $\deg D \geq 2g$, then D bpf, defines $\varphi: X \longrightarrow \mathbb{P}^{d-g}$ $(d = \deg X)$
- $\deg D \geq 2g+1$, then $\varphi_{|D|}$ an embedding.

Pf: HW. (Use RR)

Ex: $g(X) = 1$, $\deg D = 3$. Then

$$\varphi_{|D|} : X \hookrightarrow \mathbb{P}^2 \text{ as a plane cubic}$$

Most important case of this is:

Canonical Mapping:

Notation: $H^{1,0}(X) = \{\text{holo } 1\text{-forms}\}$: so $\dim_{\mathbb{C}} H^{1,0} = g$

Recall: $H^{1,0}(X) \cong \mathcal{L}(K)$, K = canon divisor.

$$|K| = \{ \operatorname{div} \omega \mid \omega \in H^{1,0}(X) \}$$

Prop: Assume $g \geq 1$. Then $|K|$ is bpf, ie. $\forall\, p \in X$, $\exists$

$$\omega \in H^{1,0}(X) \text{ st. } \omega(P) \neq 0.$$

Pf. We need to show $\ell(K-P) < \ell(K) = g$. Use RR:

$$\begin{aligned} \ell(K-P) &= \deg(K-P) + 1 - g + \ell(K-(K-P)) \\ &= 2g-3+1-g + \ell(P) \\ &= g-2+\ell(P) \end{aligned}$$

Now $\ell(P) \geq 1$ since $P \geq 0$. If $\ell(D) \geq 2$, then $\exists$

non-const fn $f \in \mathcal{L}(P)$. This defines

$$f : X \to \mathbb{P}^1$$

of deg 1, so $X \simeq \mathbb{P}^1$ & $g(X) = 0$. #

Assume henceforth: $g(X) \geq 2$.

So we have

$$\varphi_{|K|} : X \longrightarrow \mathbb{P}^{g-1} \quad \text{(canonical mapping)}$$

This is canonically defined up to linear change of coords on $\mathbb{P}^{g-1}$

Alternative interpr:

- Choose basis: $\omega_1, \ \ , \omega_g \in H^{1,0}(X)$. Then

$$\varphi_{|K|}(x) = [\omega_1(x), \ldots, \omega_g(x)] \in \mathbb{P}^{g-1}$$

(HW).

Thm. Assume $g \geq 2$. Then φ_K fails to be an embedding

$$\Updownarrow$$

X is hyperelliptic.

Pf. Need to show:

$\exists\, P, Q \in X$ s.t. $\ell(K-P-Q) = g-1$ $\iff$ X hyperell

So say $\ell(K-P-Q) = g-1$. Use RR

$$g-1 = (2g-2-2) + 1 - g + \ell(P+Q)$$

ie.

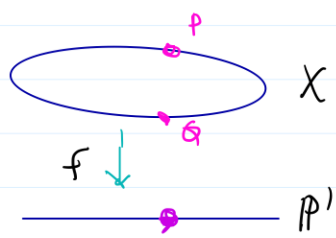
$$g-1 = g-3 + \ell(P+Q)$$

ie.

$$\ell(P+Q) = 2.$$

Take non-const $f \in \mathcal{L}(P+Q)$. Defines map

$$f : X \to \mathbb{P}^1 \quad \deg 2$$



So f hyperelliptic. Converse similar. $\square$

<u>Upshot</u>: If X non-hyperell. of genus g, $\exists$ canonical embedding

$$X \subseteq \mathbb{P}^{g-1} \text{ as a } \underbrace{\text{curve of deg } 2g-2}$$

means general hyperpl meets X at $2g-2$ pts.

<u>Curves of Small genus</u>:

We can use these considerations to analyze all curves of low genus.

<u>$g=0$</u>: If $g(X) = 0$, then $X \simeq \mathbb{P}^1$

(Pf: If $g(X)=0$, $\ell(P) = 2$ by RR, get $f : X \to \mathbb{P}^1$ deg 1)

<u>g = 1</u>: $g(X) = 1 \iff X \subseteq \mathbb{P}^2$ plane cubic.

(Later: $X \cong \mathbb{C}/\Lambda$).

<u>$g(X) = 2$</u>:

<u>Prop:</u> Every curve of genus $g = 2$ ie. admits a deg 2 cover

$$\varphi : X \to \mathbb{P}^1$$

<u>Pf.</u> In fact, $\varphi = \varphi_K$ is the canonical mapping. For

$$\deg(K) = 2\cdot 2 - 2 = 2, \quad \dim H^{1,0} = g = 2,$$

so:

$\omega_1, \omega_2 \in H^{1,0}$ a basis gives

$$X \to \mathbb{P}^1, \quad x \mapsto [\omega_1(x), \omega_2(x)]$$

deg 2.

<u>Digression</u>— Canonical map for hyperell. curves

Say X is hyperell curve of genus g. View X as compact of

$$X_0 = \{\, y^2 = f_{2g+2}(x) \,\} \subseteq \mathbb{C}^2.$$

So have

$X \xrightarrow{\pi} \mathbb{P}^1$ which is compact of $X_0 \to \mathbb{C}$, $(x, y) \mapsto x$.

Let

$\eta = \frac{dx}{y} \big| X_0$: this extends to holo 1-form on X.

More gen, let

$$\omega_1 = \eta, \quad \omega_2 = x\cdot \eta, \quad \dots, \quad \omega_g = x^{g-1}\cdot \eta$$

We've seen that

$\omega_1, \quad, \omega_g$ extend to holo 1-forms on X,

But $\dim H^{1,0}(X) = g$, so:

$$\boxed{\omega_1, \quad, \omega_g \text{ a basis of } H^{1,0}(X)}$$

We see: each ω is of form $\omega = \pi^*(p(x)) \cdot \eta$.

With a little thought, this implies:

<u>Prop.</u> Canon. mapping of hyperell curve X of genus g factors as

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\varphi_{|K|}"] & \\ \mathbb{P}^1 \arrow[r, hook] & \mathbb{P}^{g-1} \end{tikzcd}$$

$$x \longmapsto [1, x, x^2, \quad, x^{g-1}]$$

<u>Cor:</u> The double cover $X \to \mathbb{P}^1$ is uniquely defined, and its $2g+2$ ramif pts are intrinsically defined

Similarly, the $2g+2$ branch pts of π are canon defined up to changes of coords on $\mathbb{P}^1$.

<u>Vague Consequence:</u> The set of isom classes of hyperell curves of genus g has dim $= 2g-1$.

<u>$g(X) = 3$</u>

<u>Prop</u>. Let X be a non-hyperelliptic curve of genus 3. Then

$$X \cong \{\text{smooth plane curve of deg } 4\}$$

Moreover, any such curve is canonical model of non-h.e. RS of genus 3.

<u>Pf</u>: We consider canon embedding.

$$\varphi : X \hookrightarrow \mathbb{P}^2 = \mathbb{P}^{g-1}$$

Have $\deg(K_X) = 2g-2 = 4$, "so" X is smooth plane quartic.

Conversely, suppose

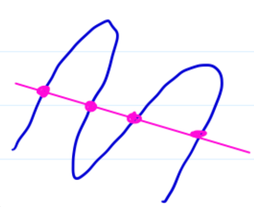
$$C \subseteq \mathbb{P}^2$$

is plane quartic. Have $g(C) = \binom{4-1}{2} = 3$. Moreover

$$H^{1,0}(C) \cong \left\{ \begin{matrix} \text{curves of deg} \\ d-3=1 \end{matrix} \right\},$$

ie.

$$K_C \equiv (C \cdot H)$$



Since

$$\dim |K| = 2 = \dim \left| \begin{matrix} \text{linear} \\ \text{series cut out} \\ \text{by lines} \end{matrix} \right|,$$

see embedding in $\mathbb{P}^2$ is canon embedding.

<u>(Vague) Example</u>: "How many" genus 3 curves are there?

Let $V_4 = \{\text{homog polys deg } 4\}$: $\dim V_4 = \binom{4+2}{2} = 15$

So

$$\mathbb{P}(V_4) = \left\{\begin{matrix}\text{plane curves}\\ \text{of deg } 4\end{matrix}\right\} = \mathbb{P}^{14}$$

Singular curves param by hypsf $\Delta_4 \subseteq \mathbb{P}^{14}$, so

$$\left\{\begin{matrix}\text{smooth plane}\\ \text{curves deg } 4\end{matrix}\right\} = \underset{\text{Zariski open}}{U \subseteq} \mathbb{P}^{14} \qquad (U = \mathbb{P}^{14} - \Delta)$$

Now

$$\underset{\dim = 8.}{PGL(3)} = SL(3,\mathbb{C})/\text{scalars} \quad \text{acts by change of coords}$$

So expect that

$$\left\{\begin{matrix}\text{isom classes of non-h.e}\\ \text{curves of genus } g=3\end{matrix}\right\} \leftrightarrow \text{"} U/PGL(3) \text{"}$$

if this exists, it should have $\dim = 14 - 8 = 6$

<u>Problem</u>: can't willy-nilly take quotients of alg vars or cx mflds by group actions.

<u>Nonetheless</u>: there is a space $\mathcal{M}_g$ that parametrizes isom classes of RS's of genus g, and for $g \geq 2$:

$$\dim \mathcal{M}_g = 3g - 3.$$

<u>$g(X) = 4$</u>

Consider X non-hyperell, $g=4$. Now realized as

$$X \subseteq \mathbb{P}^3, \quad \deg X = 6.$$

Explicit description of X

Theorem: $X \subseteq \mathbb{P}^3$ is "complete intersection" of quadric & cubic, ie.

$$X = \{ Q_2 = F_3 = 0 \}$$

where $\deg Q = 2$, $\deg F = 3$. (Moreover, any such complete is canon curve of genus 4)

Pf (1°): Find quadric thru X.

- By constr of canon embedding:

$$V_1 = \left\{ \begin{matrix} \text{linear forms} \\ \text{on } \mathbb{P}^3 \end{matrix} \right\} \xrightarrow[\rho_1]{\text{restr}_X} \mathcal{L}(K_X)$$

- Follows that

$$\rho_m : V_m = \left\{ \begin{matrix} \text{forms of deg} \\ m \quad \text{on } \mathbb{P}^3 \end{matrix} \right\} \xrightarrow[\text{to } X]{\text{restr}} \mathcal{L}(mK_X)$$

- Note

$$\dim V_m = \binom{m+3}{3}.$$

- Consider ρ_2. By RR:

$$\begin{aligned} \dim \mathcal{L}(2K_X) &= 12 + 1 - 4 + \ell(\cancel{K_X} - 2K_X) \\ &= 9 \end{aligned}$$

$$\dim V_2 = 10.$$

- So

$$\ker(\rho_2 : \underset{\substack{\uparrow \\ \dim 10}}{V_2} \longrightarrow \underset{\substack{\uparrow \\ \dim 9}}{\mathcal{L}(2K)}) \neq 0$$

· Note: $\ker \rho_2 = \{Q \in V_2 \mid Q \text{ vanishes on } X\}$

$$= \{Q \in V_2 \mid X \subseteq \{Q=0\}\}$$

· Don't a priori know that ρ_2 is surj. But here can see

$$\dim \ker \rho_2 = 1,$$

because by Bezout thm, curve of deg 6 can't lie on 2 quadrics. (Hyperplane section would be 6 pts on 2 quadrics.)

· So $\exists !$ (up to scalars)

$$Q_2 \in \ker \rho_2.$$

(2°). Study cubics thru X. Consider

$$\rho_3 : \underset{\substack{\uparrow \\ \dim = 20.}}{V_3} \longrightarrow \underset{\substack{\uparrow \\ \dim = 15 \text{ (by RR)}}}{\mathcal{L}(3K)}$$

So

$$\dim \ker(\rho_3) \geqslant 5.$$

Seems like too many, but note: if $Z_0, \dots, Z_3$ are homog coords on $\mathbb{P}^3$, then automatically,

$$Z_0 Q_2, \quad , Z_3 Q_2 \in \ker(\rho_3).$$

So $\exists$

$$F_3 \in \ker(\rho_3), \quad F_3 \notin \operatorname{span}(Z_0 Q_2, \dots, Z_3 Q_2).$$

So

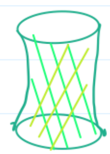
$$X \subseteq \{ Q_2 = F_3 = 0 \}$$

But by Bézout, RHS is curve of deg 6, and since $\deg X = 6$, get equality

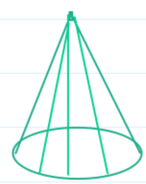
<u>(Challenge) Questions:</u>

(1°). The quadric sf $(Q=0)$ is uniquely determined by X. Now there are only two sorts of irred quadric sfs in $\mathbb{P}^3$:

- Non-sing quadr $\cong \mathbb{P}^1 \times \mathbb{P}^1$



- Cone over plane conic



So canonic model of every non-hyperell curve of genus $g=4$ lies either on smooth quadr or a quadric cone.

<u>Q</u>: How do you distinguish between these cases in terms of the intrinsic geometry of X?

(2°). Can you describe explicitly the canon model

$$X \subseteq \mathbb{P}^4 \quad (\deg 8)$$

of a non-hyperell RS of genus $g=5$?

Large genus $g = g(X) >> 0$

Consider canon model

$$X \subseteq \mathbb{P}^{g-1}, \quad \deg X = 2g-2$$

of non-hyperell RS of genus $g >> 0$

Ask: Can we similarly describe X explicitly by writing down its defining equations,

Petri: Once $g > 5$, X is "usually" cut out by quadric polys

(Exceptional cases: $\exists\, X \to \mathbb{P}^1$ deg 3, or $X \subseteq \mathbb{P}^2$ plane quintic)

Now let's do analysis as above

$$\rho_2 : \underbrace{V_2 = \left\{ \begin{smallmatrix} \text{polys deg } 2 \\ \text{in } \mathbb{P}^{g-1} \end{smallmatrix} \right\}}_{\dim = \binom{g+1}{2}} \longrightarrow \underbrace{\mathcal{L}(2K_X)}_{\dim = 3g-3}$$

In fact ρ_2 surj, so

$$\dim \ker(\rho_2) = \frac{g(g+1)}{2} - 3g + 3$$

$$= \frac{g^2 + g - 6g + 6}{2} = \binom{g-2}{2}$$

So

$$\left\{ \begin{smallmatrix} \text{quadrics} \\ \text{thru } X \end{smallmatrix} \right\} = \mathbb{C}^{\binom{g-2}{2}} \subseteq \mathbb{C}^{\binom{g+1}{2}} = \left\{ \begin{smallmatrix} \text{all} \\ \text{quadrics} \end{smallmatrix} \right\} = V_2$$

Issue: For almost all $W \subseteq V_2$ of codim $3g-3$, the quadrics in W won't have any common zeroes!

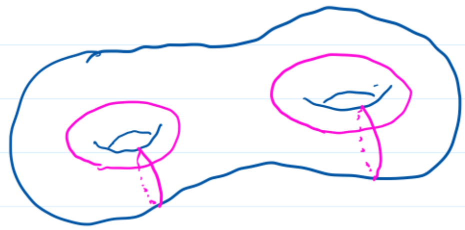
i.e. for $g >> 0$, {quadrs thru X} are very special subspaces of the space of all quadrics

Jacobians & Abel's Thm

$\circledast$

X = compact RS of genus g

Recall:



$$H_1(X,\mathbb{Z}) = \mathbb{Z}^{2g}$$

$$H^1(X) = H^1_{dR}(X;\mathbb{C})$$

$$\simeq \left\{\begin{array}{l}\text{closed } C^\infty \\ \mathbb{C}\text{-valued 1-forms} \\ (\eta \text{ s.t. } d\eta = 0)\end{array}\right\} \Big/ \left\{\begin{array}{l}\text{exact} \\ \text{1-forms}\end{array}\right\}$$

$$= \mathbb{C}^{2g}$$

Also:

(1). The bilinear map

$$H^1(X) \otimes H_1(X,\mathbb{C}) \longrightarrow \mathbb{C}$$

$$(\eta, \gamma) \longmapsto \int_\gamma \eta$$

is a perfect pairing.

(2). (Poincaré Duality). The (alternating) map

$$H^1(X) \otimes H^1(X) \longrightarrow \mathbb{C}$$

$$(\alpha, \beta) \longmapsto \int_X \alpha \wedge \beta$$

$\circledast$ Motivation: Let $X = \mathbb{C}/\Lambda$ be ell curve. Given X, what is intrinsic meaning of $\Lambda \subseteq \mathbb{C}$. Key observ. is that can view

$$\mathbb{Z}^2 = \Lambda = H_1(X,\mathbb{Z}),$$

and embedding $\Lambda \subseteq \mathbb{C}$ is

$$\Lambda = \left\{ \int_\gamma dz \;\middle|\; \gamma \in H_1 \right\}$$

i.e. we recover Λ via integrating holo 1 forms over homol. classes. Want to generalize to arb X.

is non-degen.

(3). $H^{1,0}(X) = \{\text{holo 1-forms}\}$ is $\mathbb{C}$-v.s. of $\dim = g$.

Recall also: holo 1-form is closed, so determines deR cohom class.

Def. (Anti-holo forms). If $\omega \in H^{1,0}(X)$ is locally given by

$$\omega = f(z)\,dz,$$

then define

$$\bar{\omega} \underset{\text{loc.}}{=} \overline{f(z)}\,d\bar{z}$$
$$= \bar{f}\cdot(dx - i\,dy).$$

$H^{0,1} = \{\text{anti-holo 1-forms}\}$: $\dim_{\mathbb{C}} H^{0,1} = g$

Ex. If ω is holo, then $d\bar{\omega} = 0$,

So have maps:

$$\begin{matrix} H^{1,0}(X) & \searrow & \\ & & H^1(X) \\ H^{0,1}(X) & \nearrow & \end{matrix}$$

Thm. (Hodge decomp.) Each of these maps is inj, and have direct sum decomposition:

$$H^{1,0} \oplus H^{0,1} = H^1(X,\mathbb{C})$$

More concretely, choose a basis

$$\omega_1, \dots, \omega_g \in H^{1,0}$$

Then

$$\begin{matrix} \omega_1, \dots, \omega_g \\ \overline{\omega}_1, \dots, \overline{\omega}_g \end{matrix} \quad \in \quad H^1(X, \mathbb{C})$$

is a basis.

Pf. Enough to show that the wedge product pairing among $\omega_i, \overline{\omega}_j$ is non-degen.

For reasons of type

$$\int_X \omega_i \wedge \omega_j = 0 \qquad \int \overline{\omega}_i \wedge \overline{\omega}_j = 0 \qquad \text{all } i, j.$$

(Check: $dz \wedge dz = 0$, $d\overline{z} \wedge d\overline{z} = 0$.) Let

$$a_{ij} = \int_X \omega_i \wedge \overline{\omega}_j$$

<u>Claim</u>: If $A = (a_{ij})$, then

$$iA \quad \text{a pos def Herm matrix}$$

Granting the claim, it follows that mx for cup prod wrt $\omega_1, \ , \omega_g, \overline{\omega}_1, \ , \overline{\omega}_g$ is

$$\begin{pmatrix} 0 & A \\ \bar{A} & 0 \end{pmatrix}, \quad \text{w. } iA > 0,$$

and this is non-degen.

<u>Pf of Claim:</u> Fix $0 \neq \omega \in H^{1,0}(X)$. Need to show:

$$i \int_X \omega \wedge \bar{\omega} > 0.$$

We compute in local coords:

$$\omega = f \cdot (dx + i\,dy)$$

$$\bar{\omega} = \bar{f} \cdot (dx - i\,dy)$$

$$\omega \wedge \bar{\omega} = |f|^2 \cdot (dx + i\,dy) \wedge (dx - i\,dy)$$

$$= |f|^2 \cdot (-2i \cdot dx \wedge dy)$$

(i.e. $dz \wedge d\bar{z} = -2i \cdot dx \wedge dy$). So

$$i \int \omega \wedge \bar{\omega} \underset{loc}{=} 2 \cdot \int_X |f|^2 \cdot dx \wedge dy > 0.$$

<u>Cor of Pf.</u> Define

$$\langle \omega, \eta \rangle = i \int_X \omega \wedge \bar{\eta}.$$

Then

$$\langle \, , \rangle \text{ a pos def Herm form on } H^{1,0}(X).$$

o o o

Now define a map

$$\begin{array}{rccc} \mathrm{per} : & H_1(X,\mathbb{Z}) & \longrightarrow & H^{1,0}(X)^* \cong \mathbb{C}^g \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & \gamma & \longmapsto & T_\gamma \end{array}$$

where

$$T_\gamma : H^{1,0}(X) \longrightarrow \mathbb{C}$$

is the linear functional given by integration over a fixed cycle γ:

$$T_\gamma(\omega) = \int_\gamma \omega .$$

<u>Notation:</u> often write $\int_\gamma \in H^{1,0}(X)^*$ instead of T_γ.

<u>Thm:</u> The homom per is injective and

$$\Lambda =_{def} \mathrm{im}(\mathrm{per}) \subseteq H^{1,0}(X)^*$$

is a <u>lattice</u>, ie. per takes a basis of $H_1(X,\mathbb{Z})$ to

$\mathbb{R}$-linearly indep elts of $H^{1,0}(X)^*$.

(ie. $\Lambda \cong \mathbb{Z}^{2g} \subseteq \mathbb{C}^g$ sitting as discrete sg.)

Defs $\Lambda \subseteq H^{1,0}(X)^*$ is called the period lattice of X. Quotient $H^{1,0}(X)^*/\Lambda$ is Jacobian of X:

$$\mathrm{Jac}(X) = H^{1,0}(X)^* / \Lambda$$

Note: $\mathrm{Jac}(X)$ is a complex Lie group

So $\mathrm{Jac}(X)$ is complex torus of (complex) dim $= g$.

Proof of Thm: Let $\gamma_1, \ldots, \gamma_{2g} \in H_1(X, \mathbb{Z})$ be a basis. Suppose

$$\lambda_1, \quad, \lambda_{2g} \in \mathbb{R} \quad \text{are real nos s.t.}$$

$$(*) \qquad \sum \lambda_i \cdot \mathrm{per}(\gamma_i) = 0 \in H^{1,0}(X)^*.$$

Need to show: $\lambda_1 = \quad = \lambda_{2g} = 0$.

By def of per, $(*) \iff$

$$(**) \qquad \sum \lambda_i \int_{\gamma_i} \omega = 0 \quad \text{all } \omega \in H^{1,0}(X)$$

But since $\lambda_i \in \mathbb{R}$, can conjugate $(**)$ to find

$$\sum \lambda_i \cdot \int_{\gamma_i} \bar{\omega} = 0 \quad \text{all } \bar{\omega} \in H^{0,1}(X),$$

(Check: $\overline{\int_\gamma \omega} = \int_\gamma \overline{\omega}$)

Ie.

$$\int_{\sum \lambda_i \gamma_i} \eta = 0 \quad \text{all } \eta \in H^1_{dR}(X, \mathbb{C})$$

So $\quad \sum \lambda_i \gamma_i = 0 \quad \in H_1(X, \mathbb{R})$.

In coords–

- Choose bases:

$$\omega_1, \ldots, \omega_g \in H^{1,0} \qquad \vec{\omega} = (\omega_1, \ldots, \omega_g)$$

$$\gamma_1, \ldots, \gamma_{2g} \in H_1(X, \mathbb{Z})$$

- Identify $H^{1,0}(X)^*$ w $\mathbb{C}^g$:

$$\text{per}: H_1(X, \mathbb{Z}) \longrightarrow H^{1,0}(X)^*$$

$$\gamma \longmapsto \int_\gamma \vec{\omega} = \left(\int_\gamma \omega_1, \ldots, \int_\gamma \omega_g \right)$$

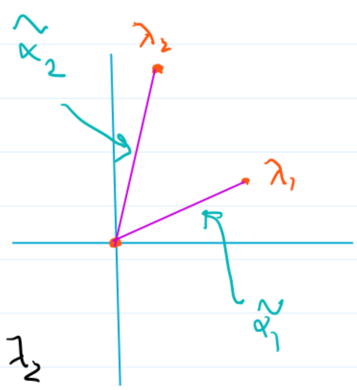
Period lattice gen by rows of the $2g \times g$ period matrix.

$$\begin{bmatrix} \int_{\gamma_1} \omega_1 & & \int_{\gamma_1} \omega_g \\ & \vdots & \\ \int_{\gamma_{2g}} \omega_1 & & \int_{\gamma_{2g}} \omega_g \end{bmatrix}.$$

Period matrix of X.

<u>Ex</u> $X = \mathbb{C}/\Lambda$ RS of genus 1,

$$\pi : \mathbb{C} \longrightarrow X$$



Let

$\lambda_1, \lambda_2 \in \Lambda$ be basis,

Let

$\tilde{\alpha}_1, \tilde{\alpha}_2$: paths fr 0 to λ_1, λ_2

$$\alpha_1 = \pi(\tilde{\alpha}_1), \quad \alpha_2 = \pi(\tilde{\alpha}_2):$$

so

$\alpha_1, \alpha_2 \in H_1(X, \mathbb{Z})$ a basis.

$\omega = dz \in H^{1,0}(X)$ a basis

Periods:

$$\int_{\alpha_1} \omega = \int_{\tilde{\alpha}_1} dz = \lambda_1$$

$$\int_{\alpha_2} \omega = \int_{\tilde{\alpha}_2} dz = \lambda_2$$

So

$$\text{Period Lattice} = \Lambda$$

$$\mathrm{Jac}(X) = X.$$

(Cor of This discussion: Any X of $g=1$ is $\mathbb{C}/\Lambda$.)

Rmk: Can view periods as change of basis data for two natural bases for H_1 or H^1

$$\gamma_1, \ldots, \gamma_{2g} \in H_1(X, \mathbb{Z}) : \quad \text{natural topological basis for } H_1(X, \mathbb{C})$$

$$\begin{matrix} \omega_1, & , \omega_g \\ \bar{\omega}_1, & , \bar{\omega}_g \end{matrix} \in H^1(X, \mathbb{C}) : \quad \text{natural bases reflecting complex structure}$$

Jacobian is offspring of marriage between these two

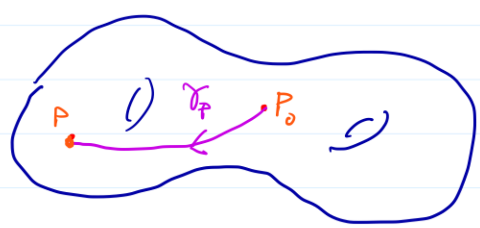
$$\begin{array}{ccc} (\text{topology}) & \text{———} & (\text{complex structure}) \\ & | & \\ & \mathrm{Jac}(X) & \end{array}$$

Abel-Jacobi Map

Fix a base-pt $P_0 \in X$. We define a map

$$u : X \longrightarrow \mathrm{Jac}(X)$$

as follows.



Given $P \in X$, let γ_P be a path from P_0 to P.

For $\omega \in H^{1,0}(X)$, consider

$$\int_{p_0}^{p} \omega = \int_{\gamma_p} \omega \in \mathbb{C}.$$

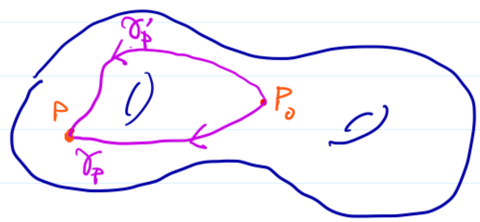
For fixed p, γ_p this gives a fn

$$\int_{p_0}^{p} \in H^{1,0}(X)^*.$$

Note that this depends on choice of γ_p.

However if γ_p, γ_p' are two such paths, then

$\gamma_p - \gamma_p'$ is a one-cycle on X,



so

$$\int_{\gamma_p - \gamma_p'} \omega = \text{period of } \omega \text{ over } \gamma \in H_1(X, \mathbb{Z}),$$

ie.

$$\int_{\gamma_p - \gamma_p'} \in \Lambda = \mathrm{im}\left(H_1(X,\mathbb{Z}) \longrightarrow H^{1,0}(X)^*\right)$$

So get well-defined elt

$$\int_{p_0}^{p} \in H^{1,0}(X)^*/\Lambda = \mathrm{Jac}(X).$$

Then define

$$u : X \longrightarrow \mathrm{Jac}(X)$$
$$P \longmapsto \int_{P_0}^{P}$$

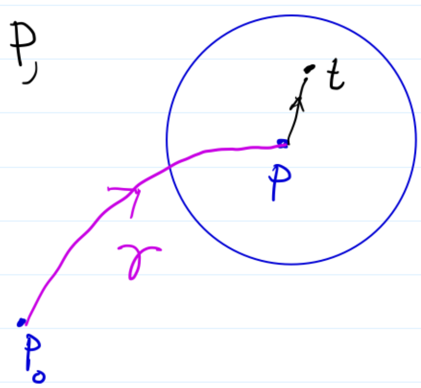
<u>Rmk:</u> If we choose a diff base-pt $P_0' \in X$, then the resulting AJ map

$$u' : X \longrightarrow \mathrm{Jac}(X), \quad P \longmapsto \int_{P_0'}^{P}$$

differs from u by transl by $\int_{P_0}^{P_0'} \in \mathrm{Jac}(X)$.

<u>Prop:</u> u is holo map of cx mflds.

<u>Pf:</u> Fix path γ from P_0 to P, and let



$z =$ local holo coord centered at P.

Write

$$\omega_i \underset{loc.}{=} f_i(z)\,dz$$

Assertion boils down to statement that indef integrals

$$a_i(t) = \int_P^t f_i(z)\,dz$$

are holo in t, which is clear. $\square$

Rmk. Will eventually see that u is embedding.

Amusing Rmk: Can view canonical mapping $\varphi_K : X \to \mathbb{P}^{g-1}$ as Gauss map assoc to u. Namely:

- Since $\mathrm{Jac}(X)$ is torus, transl gives canon identif
$$T_a \mathrm{Jac} = T_0 \mathrm{Jac} \quad \forall a \in J_1$$
- So for $P \in X$, have
$$du_P : T_P X \longrightarrow T_{u(P)} \mathrm{Jac} = T_0 \mathrm{Jac}$$
- Then
$$\begin{array}{c} \mathbb{P}(du_P) : \mathbb{P}(T_p X) \longrightarrow \mathbb{P}(T_0 \mathrm{Jac}) = \mathbb{P}^{g-1} \\ \quad\quad\quad\quad\quad \| \\ \quad\quad\quad\quad\quad P \end{array}$$
is canon mapping! (Pf: "$\frac{d}{dP} \int_{P_0}^{P} \omega_i = \omega_i(P)$")

So u "integrates" φ_K.

Given any $k \in \mathbb{Z}$, get map
$$\begin{array}{ccc} u_k : \mathrm{Div}^k(X) & \longrightarrow & \mathrm{Jac}(X), \\ \cup\!\!| & & \cup\!\!| \\ \sum n_p \cdot P & \longrightarrow & \sum n_p \cdot u(P) \end{array}$$

divisors of deg k

Note: The map

$$u_0 : \mathrm{Div}^0(X) \longrightarrow \mathrm{Jac}(X)$$

is completely canonical, i.e. doesn't depend on base-pt.
(Eg

$$u_0(P) - u_0(Q) = \int_{P_0}^{P} - \int_{P_0}^{Q} = \int_{Q}^{P} \quad)$$

Thm. (Easy half of Abel's Thm). Let

$$D = P_1 + \cdots + P_d$$
$$E = Q_1 + \cdots + Q_d$$

be two effective divisors on X of deg d, w. $D \equiv E$. Then

$$\sum u(P_i) = \sum u(Q_i) \quad \text{in } \mathrm{Jac}(X),$$

(i.e. $u(D-E) = 0$)

Pf. May suppose pts appearing in D, E are distinct.

- For simplicity, assume P_i, Q_j distinct.
- $D \equiv E \implies \exists$ mero $f \in \mathbb{C}(X)$ s.t.

$$\mathrm{div}(f) = D - E.$$

- View f as holo

$$f : X \longrightarrow \mathbb{P}^1$$

So

$$D = f^*([0])$$

$$E = f^*([\infty])$$

$$\begin{array}{c} X \\ \downarrow f \\ \mathbb{P}^1 \end{array}$$

Given $\lambda \in \mathbb{P}^1$, define

$$D_\lambda = f^*([\lambda]) = \sum_{f(p)=\lambda} e_f(p) \cdot [p].$$

Thus

$$D_0 = D, \quad D_\infty = E, \quad \deg(D_\lambda) = d \text{ all } \lambda.$$

<u>Key Lemma</u>: Consider the Abel-Jacobi image

$$u(\lambda) = u(D_\lambda) \in \mathrm{Jac}(X):$$

(Ie, writing formally $D_\lambda = P_1(\lambda) + \dots + P_d(\lambda)$, we set

$$u(\lambda) = \sum u(P_i(\lambda)). \;)$$

Then this defines a holomorphic mapping

$$u : \mathbb{P}^1 \longrightarrow \mathrm{Jac}(X).$$

(In other words, the Abel-Jacobi sums $\sum u(P_i(\lambda))$ vary holomorphically w λ.)

Then thm follows from a general fact:

<u>Prop:</u> Let $A = \mathbb{C}^g/\Lambda$ be a complex torus of dimension g. Then any holo mapping

$$u : \mathbb{P}^1 \longrightarrow A$$

is const.

In situation of Thm, this implies that

$$u : \mathbb{P}^1 \longrightarrow \mathrm{Jac}(X) \ , \quad \lambda \mapsto u(D_\lambda)$$

is const. In particular,

$$u(D) = u(D_0) = u(D_\infty) = u(E).$$

<u>Proof of Prop:</u> Consider diagram

$$\begin{tikzcd} & \mathbb{C}^g \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[ur, dashed, "\tilde{u}"] \arrow[r, "u"] & A = \mathbb{C}^g/\Lambda \end{tikzcd}$$

univ. covering

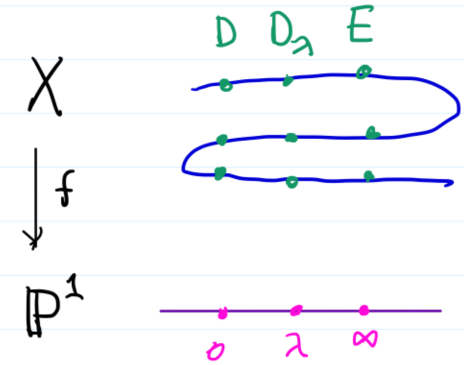
- Since $\mathbb{P}^1$ simply connected, u lifts to $\tilde{u} : \mathbb{P}^1 \longrightarrow \mathbb{C}^g$
- Since π a local isom and u holo, see that $\tilde{u}$ holo.
- But then coord fns on $\mathbb{C}^g$ are global holo fns on $\mathbb{P}^1$, hence const. $\square$

<u>Rmk:</u> Similarly, any holo map

$$u : \left(\begin{array}{c} \text{simply conn. compact} \\ \text{complex mfld} \end{array} \right) \longrightarrow A = \mathbb{C}^g/\Lambda$$

is const.

Sketch of Pf of Key Lemma.



- Suppose first that λ is not a branch pt of $f: X \to \mathbb{P}^1$

- Then f a top covering over nbd of λ, so $\exists$

$$\mathbb{P}^1 \supseteq V \ni \lambda \text{ small nbd,}$$

s.t.

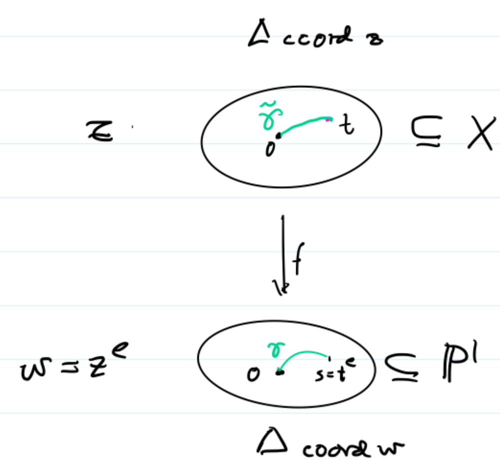
$$f^{-1}(V) = \coprod W_i, \quad W_i \text{ nbd of } P_i(\lambda) \text{ w. } W_i \to V \text{ an isom.}$$

So $u: V \longrightarrow \mathrm{Jac}(X)$ is just sum of $W_i \to \mathrm{Jac}(X)$, each of which is holo.

- Essential issue is to understand happens at ramif pt. So consider

$P \in X$ over $\lambda \in \mathbb{P}^1$ at which f looks locally like

$$f: \Delta_z \longrightarrow \Delta_w$$
$$z \longmapsto z^e = w$$



So z = local coord centered at P,

$w = z^e$ local coord at λ,

· Now let $\tilde{\gamma}$ be a path (eg a line seg.) from 0 to $t \in \Delta_z$.

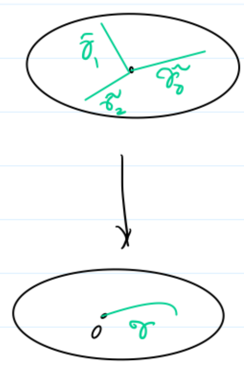
· Denote by $\gamma = f \circ \tilde{\gamma}$ its image in Δ_w, so

γ a path from 0 to $s = t^e$.

· Set $\delta = \exp(2\pi i/e)$. Then the e different lifts of γ are

$$\tilde{\gamma}_0 = \tilde{\gamma}, \quad \tilde{\gamma}_1 = \delta \cdot \tilde{\gamma}, \quad \ldots, \quad \tilde{\gamma}_{e-1} = \delta^{e-1} \cdot \tilde{\gamma}, \qquad (*)$$

so $\tilde{\gamma}_i$ a path from 0 to $\delta^i \cdot t$.



· Now fix any one-form η on X.

· Then the "η-component" of the AJ-images of the sum of the pts

$$t, \delta t, \ldots, \delta^{e-1} t \in X$$

is

$$(**) \qquad \sum_{k=0}^{e-1} \int_{\tilde{\gamma}_i} \eta, \quad .$$

so we need to show that this expression is a holo fn in $s = t^e$.

· We compute. Say $\eta \underset{loc}{=} \phi(z)\, dz$. Let

$\Phi(z)$ be a primitive of $\phi(z)$ in Δ,

so

$$\Phi'(z) = \phi(z), \quad \Phi(0) = 0.$$

Then

$$(**) = \sum_{k=0}^{e-1} \Phi(\delta^i t).$$

So assertion follows from:

<u>Lemma:</u> If $\Phi(z)$ is analy fn, then

$$\sum_{k=0}^{e-1} \Phi(\delta^i t) = \text{analy fn of } t^e.$$

<u>Pf</u> Plug into power series for Φ, and note that

$$\sum_{k=0}^{e-1} (\delta^i)^m = 0 \quad \text{unless} \quad m \equiv 0 \ (\text{mod } e).$$

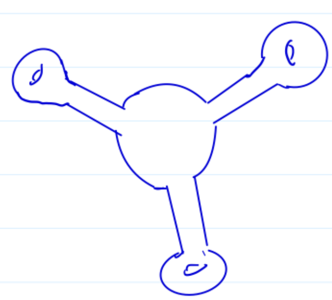
QED for Thm.

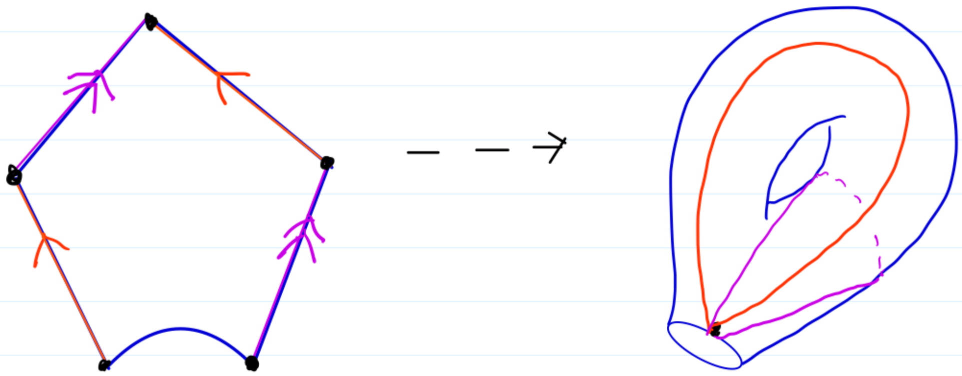
Relations Among Periods

Let

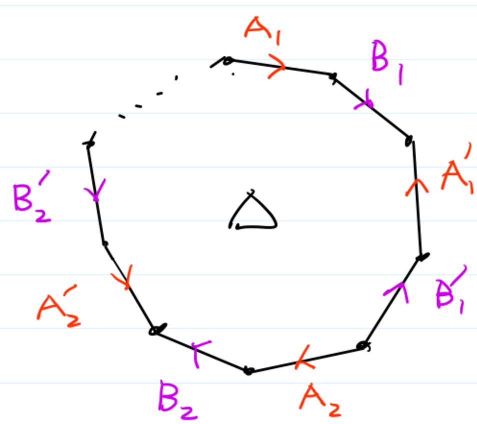
$$X = \text{R.S. genus } g.$$

Viewing X as sphere w g handles attached, can realize X topologically as $4g$-gon w sides identified in pairs:





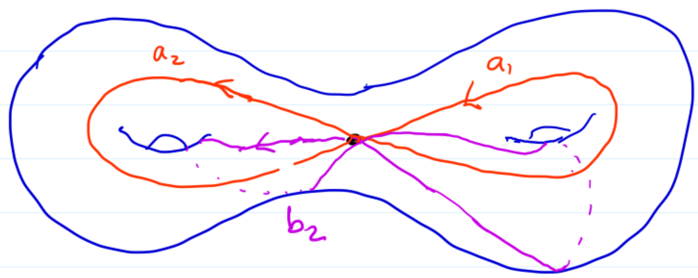
So X realized as $4g$-gon Δ w. these identifications



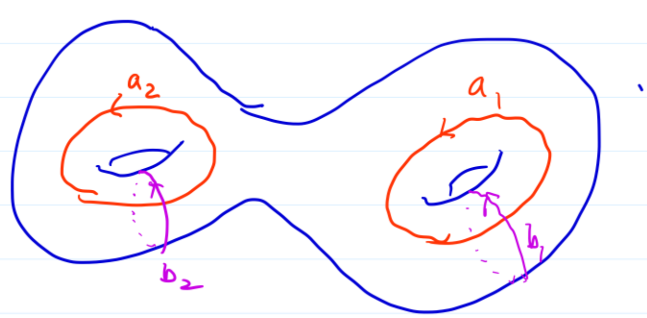
Let

a_i, b_i $(1 \leq i \leq g)$

be the curves on X determined by the edges of Δ.



These are homologous to



So viewed as (co)homology classes, these satisfy the intersection (cup product) relations

$$a_i \cdot b_i = 1 \quad (1 \leq i \leq g)$$

$$a_i \cdot a_j = b_i \cdot b_j = 0 \quad \text{all } i, j.$$

(Ie they form a symplectic basis for $H_1(X, \mathbb{Z})$ wrt the intersection from (ie. cup product).)

Note also:

$\exists$ (non-compact) simply conn. RS $\tilde{X}$ (viz the univ. cover of X), plus embedding

$$\Delta \hookrightarrow \tilde{X}$$

s.t. composition

$$\Delta \hookrightarrow \tilde{X} \hookrightarrow X$$

is holo.

Moreover, given finitely many pts $P_i \in X$, can assume (by translating Δ in $\tilde{X}$) that the P_i lie in $\mathrm{int}(\Delta)$.

(Ie. Δ will play role of period parallelogram in $g=1$)

Now consider:

σ = closed C^∞ (or holo or mero) form on X

(If σ mero, assume no poles on a_i or b_i)

<u>Def.</u> ("A-periods & B-periods") Set

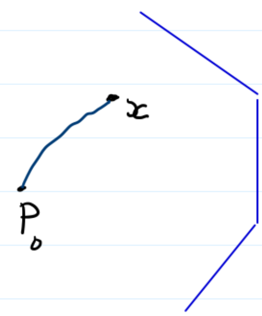
$$A_i(\sigma) = \int_{a_i} \sigma, \quad B_i(\sigma) = \int_{b_i} \sigma.$$

Note that σ lifts to form on $\tilde{X} \supseteq \Delta$, which we again call σ.

Fix

$$P_0 \in \mathrm{int}(\Delta),$$

say,



$\sigma = C^\infty$ closed form on X (eg $\sigma =$ holo).

For $x \in \Delta$, define

$$f_\sigma(x) = \int_{P_0}^{x} \sigma \qquad (\text{indep of path since } d\sigma = 0)$$

This is single valued fn on Δ which is holo if σ is. Have

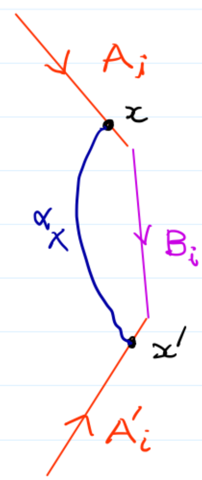
$$d f_\sigma = \sigma.$$

<u>Prop</u> Let σ, τ be C^∞ closed 1-forms on X. Then

$$\boxed{\int_{\partial\Delta} f_\sigma \cdot \tau = \sum_{i=1}^{g} \left(A_i(\sigma) B_i(\tau) - A_i(\tau) B_i(\sigma) \right)}$$

Rmk: Pf will show that statement continues to hold if τ mero w no poles on $\partial\Delta$.

Pf. Given $x \in A_i$, let $x' \in A_i'$ be corresp point. Let



$$\alpha_x = \text{path in } \Delta \text{ fr } x \text{ to } x'.$$

Then

$$f_\sigma(x) - f_\sigma(x') = \int_{P_0}^{x} \sigma - \int_{P_0}^{x'} \sigma$$

$$= \int_{x'}^{x} \sigma$$

Now $\alpha_x \sim b_i$, so $\displaystyle\int_{\alpha_x} \sigma = \int_{b_i} \sigma = B_i(\sigma)$,

i.e.

$$f_\sigma(x) - f_\sigma(x') = -B_i(\sigma) \quad \forall x \in a_i$$

Sim, for $y \in B_i$ w "partner" $y' \in B_i'$, have

$$f_\sigma(y) - f_\sigma(y') = B_i(\sigma).$$

Now since τ is 1-form on X, it takes same values on A_i & A_i', and B_i & B_i'. So

$$\int_{\partial\Delta} f_\sigma \tau = \sum_{i=1}^{g} \left(\int_{A_i} - \int_{A_i'} + \int_{B_i} - \int_{B_i'} \right) f_\sigma \cdot \tau$$

$$= \sum_{i=1}^{g} \left(\int_{x \in A_i} (f_\sigma(x) - f_\sigma(x'))\tau + \int_{y \in B_i} (f_\sigma(y) - f_\sigma(y')) \cdot \tau \right)$$

$$= \sum_{i=1}^{g} \left(\int_{A_i} -B_i(\sigma)\tau + \int_{B_i} A_i(\sigma)\tau \right)$$

$$= \sum_{i=1}^{g} \left(A_i(\sigma) B_i(\tau) - A_i(\tau) \cdot B_i(\sigma) \right).$$

<u>Prop</u> Suppose ω is any non-zero holo 1 form on X. Then

$$\operatorname{Im}\left(\sum_{i=1}^{g} A_i(\omega) \overline{B_i(\omega)} \right) < 0$$

and

$$\operatorname{Im}\left(\sum \overline{A_i(\omega)} \cdot B_i(\omega)\right) > 0$$

Pf. The two statements are equiv, so we focus on the first.

Recall first that if ω is any non-zero holo form, then

$$\operatorname{Im} \int \omega \wedge \bar{\omega} < 0$$

(If $\omega = f(z)dz$, then $\omega \wedge \bar{\omega} = -2i \cdot |f|^2 \cdot dx\,dy$) We will apply previous Prop w.

$$\sigma = \omega, \qquad \tau = \bar{\omega}.$$

Note first that by Stokes' Thm,

$$\int_{\partial\Delta} f_\omega \cdot \bar{\omega} = \int_{\Delta} d(f_\omega \cdot \bar{\omega})$$

$$= \int_{\Delta} (df_\omega \wedge \bar{\omega} + f_\omega \wedge d\bar{\omega})$$

$$= \int_{\Delta} \omega \wedge \bar{\omega} \qquad \begin{pmatrix} df_\omega = \omega \\ d\bar{\omega} = 0 \text{ since } \bar{\omega} \in H^{0,1} \end{pmatrix}$$

So by Prop

$$\int_{\partial\Delta} f_\omega \cdot \bar{\omega} = \sum \left(A_i(\omega) B_i(\bar{\omega}) - A_i(\bar{\omega}) B_i(\omega) \right)$$

So

(*)

$$\operatorname{Im}\left(\sum A_i(\omega) B_i(\bar{\omega}) - A_i(\bar{\omega}) B_i(\omega) \right) < 0.$$

Now $\quad A_i(\bar{\omega}) = \overline{A_i(\omega)}$, $B_i(\bar{\omega}) = \overline{B_i(\omega)}$, so

$$\begin{gathered} A_i(\omega) B_i(\bar{\omega}) - A_i(\bar{\omega}) B_i(\omega) \\ \| \\ 2i \cdot \operatorname{Im}\left(A_i(\omega) B_i(\bar{\omega}) \right) \end{gathered}$$

So (*) $\Rightarrow$

$$\operatorname{Im}\left(\sum A_i(\omega) B_i(\bar{\omega}) \right) < 0$$

<u>Cor</u>: Let ω be holo 1-form on X. Then

$$A_i(\omega) = 0 \text{ all } i \iff \omega = 0$$

$$B_i(\omega) = 0 \text{ all } i \iff \omega = 0.$$

<u>Def</u>. Choose basis $\omega_1, \ldots, \omega_g \in H^{1,0}(X)$. Define $\mathbb{A}, \mathbb{B}$ to be the $g \times g$ A- and B- period mxs of X, ie.

$$\mathbb{A}_{ij} = \left(A_i(\omega_j) \right) = \int_{a_i} \omega_j$$

$$\mathbb{B}_{ij} = \left(B_i(\omega_j) \right) = \int_{b_i} \omega_j$$

Cor of Cor: A, B are non-singular.

Normalized Period Matrices

Since A is non-sing, $\exists$ basis

$$\omega_1, \dots, \omega_g \in H^{1,0}(X)$$

s.t.

$$\int_{a_j} \omega_i = \delta_{ij} \quad (1 \leq i \leq g)$$

This is normalized basis of 1-forms.

Corresp period matrix is then

$$\begin{pmatrix} I_g \\ Z \end{pmatrix}, \quad \text{where } Z = (z_{ij}) = \int_{b_j} \omega_i$$

So period lattice is

$$\Lambda = \mathbb{Z}^g + Z \cdot \mathbb{Z}^g \subseteq \mathbb{C}^g$$

Riemann Bilinear Relations –

Thm. (1). Z is symmetric, i.e. ${}^tZ = Z$

(2) $\operatorname{Im} Z$ is pos definite symm matrix: $\operatorname{Im} Z > 0$.

Ex. Say $g = 1$. Then (2) says $Z = (\tau)$, $\tau \in UHP$. So description of period lattice as

$$\Lambda = \mathbb{Z}^g + Z\cdot\mathbb{Z}^g$$

w.

$$^tZ = Z, \quad \operatorname{Im} Z > 0$$

is analogous to descr of lattice in $g=1$ as $\mathbb{Z}+\mathbb{Z}\cdot\tau$, $\tau\in UHP$.

Rmk (One of the deeper meanings of Thm). Let

$$\Lambda \subseteq \mathbb{C}^g \quad \text{be an arb lattice}$$

Can always choose coords st.

$$\Lambda = \Lambda_Z = \mathbb{Z}^g + Z\cdot\mathbb{Z}^g$$

for some $g\times g$ complex matrix Z. Let

$$T_\Lambda = \mathbb{C}^g/\Lambda_Z :$$

this is cx torus, but if $g\geq 2$, T_Λ usually can't be realized as a proj alg var: in fact for 'most' Z, T_Λ doesn't carry any non-const mero fns.

It turns out: Riemann relations

$$Z = {}^tZ, \qquad \operatorname{Im} Z > 0$$

are exactly conditions that guarantee that T_Λ admits a proj embedding. In partic,

$$\text{Thm} \Rightarrow \text{Jac(RS) can be realized as a proj var.}$$

We'll see one manifestation of this when we discuss the Riemann ϑ-fn.

<u>Proof of Thm.</u> Choose normalized basis of diffs $\omega_1, \ldots, \omega_g$ so

$$A_i(\omega_j) = \delta_{ij}.$$

Apply basic period relation to $f_{\omega_i} \cdot \omega_j$. Get

$$\int_{\partial\Delta} f_{\omega_i} \cdot \omega_j = \sum_{k=1}^{g} \Big(A_k(\omega_i)\, B_k(\omega_j) - A_k(\omega_j)\, B_k(\omega_i) \Big)$$

$$= B_i(\omega_j) - B_j(\omega_i)$$

$$= z_{ij} - z_{ji}.$$

But by Stokes:

$$\int_{\partial\Delta} f_{\omega_i} \cdot \omega_j = \int_{\Delta} d(f_{\omega_i} \cdot \omega_j)$$

$$= \int_{\Delta} \big(\underbrace{\omega_i \wedge \omega_j}_{=0} + \underbrace{f_{\omega_i}\, d\omega_j}_{=0} \big) \qquad d\omega_j = 0$$

So $\qquad z_{ij} = z_{ji}.$

Now need to show $\operatorname{Im}(Z) > 0$. Fix

$$\lambda_1, \ldots, \lambda_g \in \mathbb{R}.$$

· Need to show:

$$\sum_{\alpha,\beta} \lambda_\alpha \lambda_\beta \operatorname{Im}(B_\alpha(\omega_\beta)) \geqslant 0$$

Set

$$\omega = \sum \lambda_j \omega_i \text{.}$$

We know:

$$\operatorname{Im}\left(\sum_{k=1}^{g} \overline{A_k(\omega)} \cdot B_k(\omega)\right) > 0$$

$$\|$$

$$\operatorname{Im}\left(\sum_{k=1}^{g} \overline{A_k(\lambda_1\omega_1 + \cdots + \lambda_g\omega_g)} \cdot B_k(\lambda_1\omega_1 + \cdots + \lambda_g\omega_g)\right)$$

$$\|$$

$$\operatorname{Im}\left(\sum_{k=1}^{g} \left(\overline{\lambda}_k \cdot \sum_i \lambda_i B_k(\omega_i)\right)\right)$$

$$\| \quad \lambda_k \in \mathbb{R}$$

$$\operatorname{Im}\left(\sum_{k,i} \lambda_k \cdot \lambda_i B_k(\omega_i)\right)$$

$$\|$$

$$\sum \lambda_k \cdot \lambda_i \cdot \operatorname{Im}(B_k(\omega_i)). \qquad \text{QED.}$$

Abel's Thm

X = compact RS of genus g.

Thm: Let
$$D = \sum n_p P$$
be divisor of deg 0 s.t. $u(D)=0$, where
$$u : \mathrm{Div}^0(X) \longrightarrow \mathrm{Jac}(X)$$
is Abel-Jacobi map. Then $\exists$ $f \in \mathbb{C}(X)$ s.t.
$$\mathrm{div}(f) = D.$$

Rmk: Will see later (Jacobi inversion thm) that AJ map u is surj. Granting this, Abel's Thm $\Longrightarrow$
$$\mathcal{C}\ell^0(X) =_{\mathrm{def}} \mathrm{Div}^0(X)/\mathrm{Princ}(X)$$
$$\cong \mathrm{Jac}(X).$$

Will prove Thm in various steps & substeps.

Step 1. Write
$$D = n_1 P_1 + \dots + n_d P_d, \quad \sum n_i = 0.$$

Main Claim: $\exists$ mero diff η w simple poles at the P_i, holo elsewhere, st (a). $\operatorname{res}_{P_i}(\eta) = n_i$

(b) $\int_\gamma \eta \in 2\pi i \mathbb{Z}$ all $\gamma \in H_1(X, \mathbb{Z})$.

(Idea: $\eta = \frac{df}{f}$).

Granting main claim, consider:

$$f(x) =_{\text{def}} \exp\left(\int_{P_0}^{x} \eta\right), \quad x \in X - \{P_1, \ldots, P_d\},$$

where

$$\int_{P_0}^{x} = \int_{\substack{\text{path from} \\ P_0 \text{ to } x}}.$$

The integral $\int_{P_0}^{x} \eta$ isn't well-defined indep of path, but by (b) diff paths change $\int_{P_0}^{x} \eta$ by elt of $2\pi i \mathbb{Z}$, so $\exp\left(\int_{P_0}^{x} \eta\right)$ well defined.

Clearly f holo away from the P_i. What does it look like at P_i?

Locally near P_i, write

$$\eta = \frac{n_i}{z} + (\text{holo}) \qquad (z \text{ local coord at } P_i).$$

So

$$\int_{P_0}^{x} \eta = n_i \log(x) + (\text{holo})$$

and

so

$$f(x) = e^{\int_{P_0}^{x} \eta} = x^{n_i} \cdot (\text{non-zero holo}),$$

ie.

$$\mathrm{ord}_{P_i}(f) = n_i \quad , \text{ done}$$

So need to prove main claim.

Step 2: Say

$$D = \sum n_i P_i$$

is divisor of deg 0 (ie $\sum n_i = 0$). Then $\exists$ mero 1-form η w simple poles at P_i,

$$\mathrm{res}_{P_i}(\eta) = n_i, \qquad \eta \text{ holo off } \{P_i\}$$

Pf. Consider

$$W = \left\{ \begin{array}{l} \text{mero diffs w} \\ \text{simple poles only} \\ \text{at } \{P_i\} \end{array} \right\} \xrightarrow{\alpha} \mathbb{C}^d$$

$$\eta \longmapsto (\mathrm{res}_{P_1}(\eta), \dots, \mathrm{res}_{P_d}(\eta))$$

By residue thm,

$$\mathrm{Im}(\alpha) \subseteq \{(r_1, \dots, r_d) \mid \sum r_i = 0\}$$

$$\|_{\text{def}}$$

V : has dim $= d-1$.

Need to show

$$\mathrm{Im}(\alpha) = V.$$

We count dims.

$$\ker(\alpha) = H^{1,0}(X) \; ; \; \dim = g$$

$$W = \mathcal{L}(K_X + P_1 + \quad + P_d)$$

$$\dim W = (2g-2+d) + 1 - g + \ell(-P_1 - \cdots - P_d)$$
$$= d+g-1.$$

So

$$\dim \mathrm{im}(\alpha) = (d+g-1) - g = d-1$$
$$= \dim V.$$

<u>Step 3</u>. Suppose now the Abel–Jacobi image of D is zero:

$$(*) \qquad \sum n_i u(P_i) = 0 \in \mathrm{Jac}(X).$$

Choose η as in Step 2, so that

$$\mathrm{res}_{P_i}(\eta) = n_i$$

<u>Key point</u>: Show that by adding holo diff to η can arrange that

$$\int_{a_i} \eta, \int_{b_i} \eta \in 2\pi\sqrt{-1}\,\mathbb{Z}. \quad (1 \le i \le g)$$

· Fix normalized basis $w_1, \ldots, w_g \in H^{1,0}(X)$, so

$$\int_{a_i} \omega_j = \delta_{i,j}.$$

· Recall: if

$$z_{\alpha,l} = \int_{b_l} w_\alpha = B_l(\omega_\alpha),$$

then $z_{\alpha l} = z_{l,\alpha}$ (Symm of normalized period matrix Z)

· Substep 1: Replacing η by

$$\eta - \left(\sum_{l=1}^{g} \left(\int_{a_l} \eta \right) \cdot \omega_l \right),$$

can assume

$$\boxed{\int_{a_l} \eta = 0 \qquad \text{all} \quad 1 \leq l \leq g}$$

Substep 2: Basic period relation $\Longrightarrow$

$$\int_{\partial\Delta} f_{w_\alpha} \cdot \eta = \sum_{l=1}^{g} \left(A_l(\omega_\alpha) B_l(\eta) - A_l(\eta) B_l(\omega_\alpha) \right)$$

all A-periods of η vanish

(*) $$= B_\alpha(\eta)$$

Now apply classical residue thm to compute integral on the left.

We view the poles P_i of η as lying in $\mathrm{int}(\Delta)$. So:

$$\int_{\partial\Delta} f_{\omega_\alpha} \cdot \eta = (2\pi\sqrt{-1}) \cdot \sum_{k=1}^{d} f_{\omega_\alpha}(P_k) \cdot \mathrm{res}_{P_k}(\eta)$$

$$= (2\pi\sqrt{-1}) \sum n_k \cdot f_{\omega_\alpha}(P_k)$$

Putting together w (*), we find

$$(**) \quad \boxed{(2\pi\cdot\sqrt{-1}) \sum_{k=1}^{d} n_k \cdot \int_{P_0}^{P_k} \omega_\alpha = \int_{b_\alpha} \eta \qquad \text{all} \quad 1 \leq \alpha \leq g}$$

<u>Substep 3</u>: Hypothesis $\sum n_k \, u(P_k) = 0$ means

$\exists \; e_1, \ldots, e_g, f_1, \ldots, f_g \in \mathbb{Z}$ s.t.

$$\sum_{k=1}^{d} n_k \cdot \int_{P_0}^{P_k} = \sum_{i=1}^{g} \left(e_i \int_{a_i} + f_i \int_{b_i} \right)$$

as fnls on $H^{1,0}$.

So: for each $1 \leq \alpha \leq g$, have

$$\sum_{k=1}^{d} n_k \cdot \int_{P_0}^{P_k} \omega_\alpha = \sum_{i=1}^{g} e_i \int_{a_i} \omega_\alpha + \sum_{i=1}^{g} f_i \int_{b_i} \omega_\alpha$$

$$= e_\alpha + \sum_{i=1}^{g} f_i \int_{b_i} \omega_\alpha$$

$$\Big\} = \text{by Riemann Relat}$$

$$= e_\alpha + \sum_{i=1}^{g} f_i \int_{b_\alpha} \omega_i$$

So by (**), find

$$\int_{b_\alpha} \eta = (2\pi\sqrt{-1}) \cdot \left(e_\alpha + \sum_{i=1}^{g} f_i \int_{b_\alpha} \omega_i \right)$$

<u>Substep 4</u>: Set

$$\eta' = \eta - (2\pi\sqrt{-1}) \cdot \sum_{i=1}^{g} f_i \, \omega_i$$

Then for each $1 \leq \alpha \leq g$.

$$\int_{b_\alpha} \eta' = (2\pi\sqrt{-1}) e_\alpha \;, \quad \int_{a_\alpha} \eta = -2\pi\sqrt{-1} \cdot f_\alpha$$

$$\in (2\pi\sqrt{-1}) \cdot \mathbb{Z} \qquad \text{QED!!}$$

Recap:

(1). Define

$$\mathrm{Jac}(X) = H^{1,0}(X)^* / H_1(X,\mathbb{Z})$$

where $H_1(X,\mathbb{Z}) \hookrightarrow H^{1,0}(X)^*$ is $\gamma \mapsto \int_\gamma$.

$$\mathrm{Jac}(X) = \mathbb{C}^g/\Lambda = \text{complex torus of } \dim_{\mathbb{C}} = g.$$

(2). Define holo

$$u : X \longrightarrow \mathrm{Jac}(X) \quad \text{via} \quad u(P) = \int_{P_0}^{P} \in H^{1,0}(X)^*/H_1(X,\mathbb{Z})$$

Using group structure on Jac(X) this extends by linearity to

$$u : \mathrm{Div}^k(X) \longrightarrow \mathrm{Jac}(X)$$

When $k=0$, u indep of choice of base pt

(3). Let

$$a_i, b_i \in H_1(X,\mathbb{Z}) \quad (1 \le i \le g)$$

be "symplectic basis". Can choose "normalized basis" $\omega_1, \omega_g \in H^{1,0}$ s.t.

$$\int_{a_i} \omega_j = \delta_{ij}.$$

Set

$$Z = \left(\int_{b_i} \omega_j\right) \quad g \times g \text{ normalized period m}$$

Riemann Bilinear Relations

(i) ${}^t Z = Z$ (ii) $\mathrm{Im}(Z) > 0$

(4). Abel's Thm: Given divisor $D \in \mathrm{Div}^0(X)$, have

$$u(D) = 0 \in \mathrm{Jac}(X)$$
$$\Updownarrow$$
$$D = \mathrm{div}(f) \quad \text{some } f \in \mathbb{C}(X)$$

Structure of A-J Map

X = RS genus $g \geq 1$
Ψ
P_0 = base pt.,   giving $u : X \longrightarrow \mathrm{Jac}(X)$

Prop. u is an embedding, ie. X embeds into its Jacobian.

Pf. Need to show u is 1-1, and $du \neq 0$. First statement follows from Abel's Thm:

Suppose have $P_1 \neq P_2$ s.t. $u(P_1) = u(P_2)$. Then by Abel's Thm:

$$P_1 - P_2 = \mathrm{div}(f) \quad \text{some } f \in \mathbb{C}(X),$$

$\Rightarrow X \cong \mathbb{P}^1$.

For second, choose basis $\omega_1, , \omega_g \in H^{1,0}(X)$. Then up to scalars

$$\text{"} du(P) = (\omega_1(P), \ldots, \omega_g(P)) \text{"}$$

But this is $\neq 0$ since $|K_X|$ bpf.

Jacobi Inversion Thm -

Now consider

$$u : \mathrm{Div}^\circ(X) \longrightarrow \mathrm{Jac}(X) :$$

group homom

Thm. (Jacobi Inversion, I) Mapping u is surjective

(Cor : $\quad Cl^\circ(X) \cong \mathrm{Jac}(X)$)

Will actually prove a somewhat stronger statement. Given $d \geq 1$, consider

$$X^d = X \times \cdots \times X \quad (d\text{-times})$$

$$\tilde{u}_d : X^d \longrightarrow \mathrm{Jac}(X)$$

$$\tilde{u}_d(P_1, \ldots, P_d) = u(P_1 + \cdots + P_d) = \sum u(P_i)$$

Note:

$$\dim X^d = d, \quad \dim \mathrm{Jac} = g,$$

so might hope $\tilde{u}_d$ surjective for $d \geq g$. This is true

Thm (Jacobi Inversion, II)

$$\tilde{u}_g : X^g \longrightarrow \mathrm{Jac}(X)$$

surj (and hence $\tilde{u}_d$ surj for $d \geq g$).

<u>Cor:</u> $u: \mathrm{Div}^0(X) \longrightarrow \mathrm{Jac}(X)$ surj.

<u>Pf.</u> Given any $\xi \in \mathrm{Jac}(X)$, Jac. Inv. II $\Rightarrow \exists\ P_1, \ \ , P_g \in X$ s.t

$$\xi = \tilde{u}_g(P_1, \ \ , P_g).$$

But this means that

$$\xi = u(P_1 + \ \ + P_g - g \cdot P_0).$$

To prove J.I II, we will use a non-trivial (but hopefully believable) fact from analy / alg geom:

<u>Remmert Proper Mapping Thm:</u>

Let $f: X \longrightarrow Y$ be a proper holo mapping bet cx mflds. Then

$f(X) \subseteq Y$ is an analytic subset (locally defined by analy eqns)

Also need

Let $$f: X_0 \longrightarrow W_0$$

be surj proper mapping of cx mflds. Then for gen $w \in W_0$,

$$\dim f^{-1}(w) = \dim X_0 - \dim W_0.$$

Now let's apply these to

$$\tilde{u}_g : X^g \longrightarrow \mathrm{Jac}(X).$$

If $\tilde{u}_g$ not surj, then

$$W = \mathrm{Im}(\tilde{u}_g) \subsetneq \mathrm{Jac}(X)$$

proper analy subset, hence

$$\dim W \leq g-1.$$

So:

Every fibre of $\tilde{u}_g : X^g \longrightarrow W$ would have $\dim \geq 1$.

Now fix $P_1, \ldots, P_g \in X$. By Abel's Thm,

$$\tilde{u}_g(P_1, \ldots, P_g) = \tilde{u}_g(Q_1, \ldots, Q_g)$$

$$\Updownarrow$$

$$P_1 + \cdots + P_g \equiv Q_1 + \cdots + Q_g$$

Hence

$$\dim \tilde{u}_g^{-1}(\tilde{u}_g(P_1, \ldots, P_g)) \geq 1$$

$$\Updownarrow$$

$$\dim |P_1 + \cdots + P_g| \geq 1.$$

So Thm follows from

Lemma: $\exists \; P_1, \dots, P_g \in X$ s.t.

$$\ell(P_1 + \cdots + P_g) = 1, \quad \text{ie} \quad \dim |P_1 + \cdots + P_g| = 0,$$

Rmk: In fact this holds for "general" $P_1, \dots, P_g$, ie. all g-tuples in

$$X^g - \left(\begin{smallmatrix}\text{proper algy} \\ \text{subset}\end{smallmatrix}\right)$$

Pf. By RR,

$$\ell(P_1 + \ +P_g) = g+1-g + \ell(K - P_1 - \ - P_g).$$

So enough to show $\exists \; P_1, \dots, P_g$ so there is no holo 1-form van at $P_1, \ , P_g$. It suffices to take $P_1, \dots, P_g$ whose images under canon mapping

$$\varphi_{|K|} : X \longrightarrow \mathbb{P}^{g-1}. \qquad \square$$

This argument suggests that it is interesting to consider more generally the maps

$$\tilde{u}_d : X^d \longrightarrow \mathrm{Jac}(X)$$

$$\tilde{u}_d(P_1, \ , P_d) = u(P_1 + \ + P_d)$$

However, it is un-natural to choose an ordering of the pts. So we introduce the so-called <u>symmetric products</u> of X.

Fix $d \geq 1$, and consider the action of the symmetric group S_d on X^d by permuting the coords.

<u>Thm / Def.</u> The quotient

$$\mathrm{Sym}^d(X) \underset{\text{AKA}}{=} X_d \underset{\text{def}}{=} X^d / S_d$$ called d^{th} <u>symm prod</u> of X.

has a natural structure of a complex manifold (of dim $= d$),
So

X_d parameterizes all eff. divisors of deg d on X.

There is a holo mapping

$$\begin{array}{ccccc} u_d : & X_d & \longrightarrow & \mathrm{Jac}(X) \\ & \Psi & & \Psi \\ & D & \longrightarrow & u(D) \end{array}$$

<u>Key Idea of Pf</u> : The surprising pt, which is special to the case $\dim X = 1$, is that

$$X^d / S_d \text{ is a manifold} \qquad (*)$$

(S_d does not act freely along diagonals). For $(*)$, the crucial point is to understand what happens when $X = \mathbb{C}$. Here

$$\mathrm{Sym}^d(\mathbb{C}) = \mathbb{C}^d,$$

and the quotient map is:

$$\begin{array}{cccc} \pi : & \mathbb{C}^d & \longrightarrow & \mathbb{C}^d \\ & \Psi & & \Psi \\ & t = (t_1, \ldots, t_d) & \longrightarrow & (\sigma_1(t), \ldots, \sigma_d(t)), \end{array}$$

where $\sigma_i(t)$ are elem symm fns of $t_1, \ldots, t_d$.

(Geometrically, can view

$$\mathrm{Sym}^d(\mathbb{C}) = \{\text{all monic polys } X^d + a_1 X^{d-1} + \quad + a_d\}$$

and (up to some signs) π is map

$$(t_1, \quad , t_d) \longmapsto (X - t_1) \cdot \quad \cdots (X - t_d)$$

that takes d-tuple of cx numbers to poly w given d-tuple as roots).

<u>Exerc:</u> $\mathrm{Sym}^d(\mathbb{P}^1) = \mathbb{P}^d$

So now consider

$$u_d : X_d \longrightarrow \mathrm{Jac}(X).$$

Consider eff divisor D of deg d on X: abusing notation a little, we'll write

$$D \in X_d.$$

Then by Abel's Thm

$$\begin{aligned} u_d^{-1}(u_d(D)) &= \{D' \mid u_d(D') = u_d(D)\} \\ &= \{D' \mid D' \equiv D\} \\ &= |D|. \end{aligned}$$

i.e.

The fibres of u_d are the pts of X_d parametrizing complete linear series.

Previously we observed that there is natural way to give $|D|$ the structure of a proj space.

<u>Thm</u>: Every fibre of

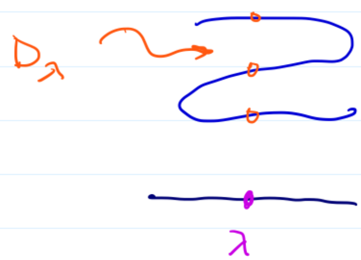
$$u_d : X_d \longrightarrow \mathrm{Jac}(X)$$

is isom to a projective space sitting as a submanifold of X_d. i.e. the identif

$$|D| = u_d^{-1}(u_d(D)) \subseteq X_d$$

is an embedding of $|D|$, with its structure of proj space, as submfld of X_d.

(Exerc: Try to convince yourself of this when $\dim |D| = 1$



$$\begin{array}{c} X \\ \downarrow \\ \mathbb{P}^1 \end{array} \quad ; \quad \begin{array}{c} \mathbb{P}^1 \longrightarrow X_d \\ \lambda \longmapsto D_\lambda \end{array} \text{ holo}$$

)

<u>Ex.</u> $d=2$. Have

$$u_2 : \underset{\text{sf}}{X_2} \longrightarrow \underset{\dim g}{\mathrm{Jac}(X)}$$

<u>Case 1</u>: X non-hyperell.

Then u_2 is 1-1 (by Abel), and in fact u_2 is an embedding.

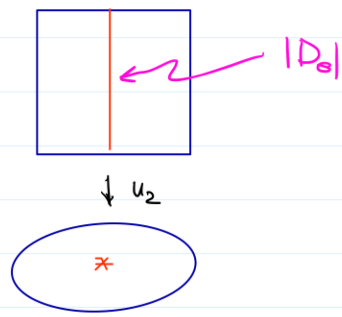
<u>Case 2:</u> X is hyperell; say $|D_0|$ is hyperell linear series

Then $|D_0| = \mathbb{P}^1 \subseteq X_2$,

and

u_2 maps $|D_0|$ to a point,

and is isom onto its image away from $|D_0|$.



<u>Challenge:</u> • What is the self-int number of $|D_0|$ in X_2?

• Show that if $g=2$, then $X_2 = Bl_{pt}(\mathrm{Jac})$

<u>Ex.</u> Suppose $d \geqslant g$. Then RR says

$$\dim |D| = d - g + \ell(K-D).$$

Geom interpr is as follows:

$$u_d : \underset{\text{dim } d}{X_d} \twoheadrightarrow \underset{\text{dim } g}{\mathrm{Jac}(X)}$$

"Most fibres" have dim $d-g$. Roch's term $\ell(K-D)$ measures the amount by which the dim of a given fibre jumps.

<u>Ex.</u> Assume $d \geqslant 2g-1$. Then

$$\dim |D| = d - g \quad \text{for } \underline{\text{every}}\ D.$$

In this case,

$$u_d : X_d \longrightarrow \mathrm{Jac}(X)$$

is a $\mathbb{P}^{d-g}$-bundle over Jac. In fact,

$$X_d = \mathbb{P}(E_d),$$

E_d a vb of rk $d+1-g$ on Jac

<u>Ex.</u> For $d < g$,

$$u_d : X_d \longrightarrow \mathrm{Jac}(X)$$

is gen 1-1 over its image.

$$W_d =_{\mathrm{df}} u_d(X_d) \subseteq \mathrm{Jac}(X)$$

is analy subvar of dim d parameterizing linear equiv classes of effective divisors of deg d.

Most interesting case:

$$W_{g-1} \subseteq \mathrm{Jac}(X).$$

this is <u>hypersurface</u>. Will explain how to write down its eqn.

<u>Riemann's Count</u> - can use this discussion to "compute"

$n_g =$ "dim of moduli space $\mathcal{M}_g$ parametrizing isom classes of RS's of genus g."

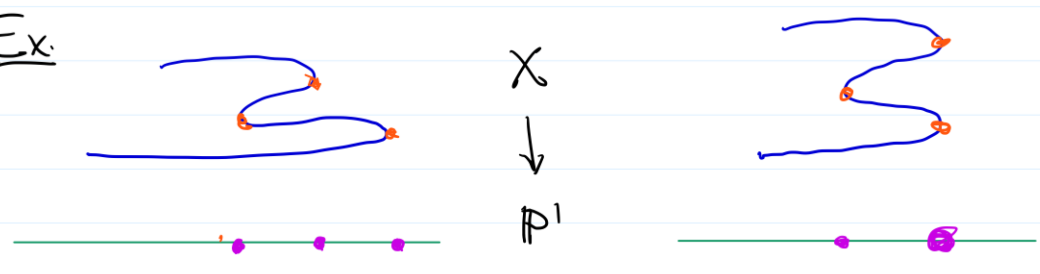
<u>Idea</u>: given X of genus g, consider branched coverings

$$\pi : X \to \mathbb{P}^1$$

of degree $d \gg 0$. We will compute the dim of $\{(X, \pi)\}$ in two diff ways.

<u>Def.</u> Covering π is <u>simple</u> if each ramif point has ram index $= 1$, and no two ram pts lie over same branch pt.

<u>Ex.</u>



simple   not simple

<u>Lemma</u>: Given divisor $B \subseteq \mathbb{P}^1$, $\exists$ only finitely many simple coverings $\pi : X \to \mathbb{P}^1$ of deg d w.

$$\mathrm{Br}(\pi) = B, \qquad (\# B = b)$$

<u>Sketch</u>: Have 1-1 correspondences:

Giving π $\longleftrightarrow$ Giving d-sheeted covering space $X^\circ \to \mathbb{P}^1 - B$ whose monodromy around each $b \in B$ is simple transp

$\updownarrow$

And this set is finite $\rightsquigarrow$ Giving transp $\tau_1, \ldots, \tau_b \in S_d$ w $\tau_1 \cdots \tau_b = 1$ / conjugacy

Note also that if $\pi : X \to \mathbb{P}^1$ has degree d, then

$$b = \deg(Br(\pi)) = (2g-2) + 2d,$$

Moreover, $Br(\pi) \in Sym^b(\mathbb{P}^1) \,/\, Aut(\mathbb{P}^1)$

$$\| $$

$$\mathbb{P}^b \,/\, Aut(\mathbb{P}^1) = \mathcal{B}$$

Now let

$$Z = Z_{d,g} = \text{"}\{(X,\pi) \mid g(X) = g,\ \pi : X \to \mathbb{P}^1 \text{ simple, deg } d\}\text{"}$$

Consider

$$\begin{tikzcd} & Z \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ \mathcal{M}_g & & \mathcal{B} \end{tikzcd} \qquad \begin{tikzcd} & (X,\pi) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ X & & Br(\pi) \end{tikzcd}$$

By Lemma, pr_2 gen surj, finite to one, so

$$\begin{aligned} \dim Z &= \dim B \\ &= b-3 \\ &= (2g-2) + 2d - 3 \end{aligned} \qquad (*)$$

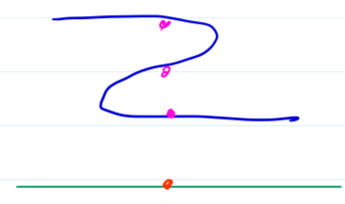
Now we need to compute dim of Z in terms of $m_g = \dim \mathcal{M}_g$. i.e need to compute

$$\dim pr_1^{-1}(X) = \dim\{\pi \mid \pi : X \to \mathbb{P}^1 \text{ simple covering}\}$$

Given X, what are data required to specify π? Need:

$$\begin{array}{c} \mathrm{Sym}^d(X) \\ \downarrow u_d \\ \mathrm{Jac}(X) \end{array}$$

- Linear equivalence class of divisor D of deg d
- 1-dim basept free lin series $\Lambda \subseteq |D|$



$$\left(D = \pi^*(pt),\ \Lambda = \{\pi^*(\lambda)\} \right)$$

We choose $d \gg 0$ st

$$u_d^{-1}([D]) \cong \mathbb{P}^{d-g} \quad \text{all } D.$$

So for fixed X, choice of $\pi : X \to \mathbb{P}^1$ is determined by gen choice of

$$pt\ [D] \in \mathrm{Jac}(X), \qquad \mathbb{P}^1 \subseteq \mathbb{P}^{d-g} = \{u^{-1}(D)\}$$

g dims worth of pts in Jac

$$\dim = \dim \mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^{d-g}) = 2(d-g-1)$$

So, all told

$$\begin{aligned} \dim pr_1^{-1}(X) &= g + 2(d-g-1) \\ &= 2d-g-2 \end{aligned}$$

So

$$\dim Z = m_g + 2d - g - 2. \qquad (**)$$

Comparing (*) & (**), we "find"

$$(2g-2) + 2d - 3 \;=\; m_g + 2d - g - 2$$

ie

$$\boxed{m_g = 3g-3.}$$

Ex. What is least degree d for which one expects "general" X of genus g to admit

$$\pi : X \longrightarrow \mathbb{P}^1 \quad \text{of degree } d?$$

- If $\deg \pi = d$, then $b = \deg Br(\pi) = 2g-2+2d$.
- "So," as above:

$$\dim\{(X,\pi)\} = b-3 = 2g-5+2d.$$

So need:

$$2g-5+2d \geqslant 3g-3 = \dim\{X\}$$

ie

$$\boxed{2d \geqslant g+2}$$

This turns out to be correct, but not trivial to prove.

Theta Functions & Theta Divisor

· Fix:

$$\Omega = g \times g \text{ complex matrix}$$

s.t.

(i). ${}^t\Omega = \Omega$

(ii) $\operatorname{Im} \Omega > 0$

Will define Riemann ϑ-fn $\vartheta(z, \Omega)$, an entire fn on $\mathbb{C}^g$.

Let:

$$\mathbb{C}^g \supseteq \Lambda_\Omega = \text{lattice generated by } \mathbb{Z}^g \text{ and rows of } \Omega.$$

$$A_\Omega = \mathbb{C}^g / \Lambda_\Omega : \text{ complex torus.}$$

Will see that $\{\vartheta = 0\}$ is invariant under translation by Λ_Ω, so defines divisor (hypersf)

$$\Theta = \{\vartheta = 0\} \subseteq A_\Omega.$$

· When Ω = normalized period mx of RS X, turns out that up to translation:

$$\Theta = W_{g-1} \subseteq \operatorname{Jac}(X)$$

· Will conclude w Torelli's Theorem:

R.S. X is determined up to isom by $(\operatorname{Jac}(X), \Theta)$.

Ref on ϑ-fns: Mumford, Tata Lectures on Theta, Vol I, Chapt. 2, esp §3.

· Fix Ω as above.

Prop / Def. The infinite series

$$\vartheta(\vec{z}, \Omega) \underset{\text{def}}{=} \sum_{\vec{n} \in \mathbb{Z}^g} \exp\left(\pi i \, {}^t\vec{n}\, \Omega \, \vec{n} + 2\pi i \, {}^t\vec{n} \cdot \vec{z}\right)$$

converges absolutely and uniformly on compact subsets of $\mathbb{C}^g$ to define an entire fn.

Pf. Since $\mathrm{Im}(\Omega)$ is pos def, $\exists$ const $C_1 > 0$ s.t.

$$\mathrm{Im}\left({}^t\vec{n}\, \Omega \, \vec{n}\right) \geqslant C_1 \cdot \sum_{i=1}^{g} n_i^2$$

(Take C_1 s.t. $\mathrm{Im}(\Omega) > C_1 \cdot I_g$). Now say

$$\max_i |\, \mathrm{Im}(z_i)| \leq \frac{C_2}{2\pi}$$

Then

$$|\exp\left(\pi i \, {}^t\vec{n}\, \Omega \, \vec{n} + 2\pi i \, {}^t\vec{n} \cdot \vec{z}\right)| \leqslant \exp\left(-\pi \cdot C_1 \sum n_i^2 + C_2 \cdot \sum |n_i|\right)$$

$$\leqslant C_3 \cdot \left(\sum_{n \geqslant 0} \exp(-\pi C_1 n^2 + C_2 n)\right)^g,$$

and this converges like $\int_0^\infty e^{-x^2} dx$.

Prop. ("Quasi-periodicity"): For fixed Ω and $\vec{m} \in \mathbb{Z}^g$, $\vartheta(\vec{z}) = \vartheta(\vec{z}, \Omega)$ satisfies

$$\vartheta(\vec{z} + \vec{m}) = \vartheta(\vec{z}) \quad (\text{periodic wrt } \mathbb{Z}^g)$$

$$\vartheta(\vec{z} + \Omega\vec{m}) = \exp(-\pi i\, {}^t\vec{m}\,\Omega\,\vec{m} - 2\pi i\, {}^t\vec{m}\cdot\vec{z}) \cdot \vartheta(\vec{z})$$

Pf. Compute term by term.

Cor 1. $\vartheta(\vec{z})$ is not identically zero.

Pf. ϑ is $\mathbb{Z}^g$-periodic and

$$\vartheta(z) = \sum e^{\pi i ({}^t\vec{n}\,\Omega\,\vec{n})}\, e^{2\pi i ({}^t\vec{n}\cdot\vec{z})}$$

is its Fourier expansion. Since coeffs $\neq 0$, $\vartheta \not\equiv 0$.

Cor 2. Given $\vec{z} \in \mathbb{C}^g$, $\vec{\lambda} \in \Lambda_\Omega$, have

$$\vartheta(\vec{z}) = 0 \iff \vartheta(\vec{z} + \vec{\lambda}) = 0.$$

In other words the zero locus

$$\{\vartheta = 0\} \subseteq \mathbb{C}^g$$

is invariant under transl by Λ_Ω, and so defines a divisor

$$\Theta = \{\vartheta = 0\} \subseteq A_\Omega = \mathbb{C}^g / \Lambda_\Omega,$$

called the Θ-divisor of A_Ω.

Rmk: Can use ϑ-fn and its cousins to define proj embeddings

$$A_\Omega \hookrightarrow \mathbb{P}^N.$$

Viz, for $\vec{a}, \vec{b} \in \mathbb{Q}^g$, define

$$\vartheta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(z, \Omega)$$

$$\|$$

$$\exp\left(\pi i\, {}^t\vec{a}\,\Omega\vec{b} + 2\pi i\, {}^t\vec{a}(\vec{z}+\vec{b})\right)\vartheta\left(\vec{z} + \Omega\vec{a} + \vec{b}, \Omega\right)$$

Thm of Lefschetz:

Let $L = 3\cdot\Lambda_\Omega \subseteq \mathbb{C}^g$ (so $\mathbb{C}^g/L = \mathbb{C}^g/\Lambda_\Omega$)

Then for $\vec{a}, \vec{b} \in \frac{1}{3}\mathbb{Z}^g/\mathbb{Z}^g$, the fns

$$\left[\ \ldots, \vartheta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\vec{z}, \Omega), \ \ldots\ \right]$$

define a proj emb

$$\mathbb{C}^g/L \hookrightarrow \mathbb{P}^N.$$

Theta Divisor of RS

Recall: given $g\times g$ cx mx Ω, w

$${}^t\Omega = \Omega, \qquad \operatorname{Im}\Omega > 0$$

we define

$$\vartheta(z) = \vartheta(z, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp\left(\pi i\, {}^t n \Omega n + 2\pi i\, {}^t n \cdot z\right) :$$

This is entire fn of $z \in \mathbb{C}^g$, and

$$\{\vartheta = 0\} \subseteq \mathbb{C}^g$$

is invariant under transl by

$$\Lambda_\Omega = \mathbb{Z}^g + \Omega \cdot \mathbb{Z}^g,$$

so defines divisor

$$\Theta \subseteq A_\Omega = \mathbb{C}^g / \Lambda_\Omega .$$

Note also: $\vartheta(z)$ is even fn of z.

Now say:

$$X = \text{R.S. genus } g$$

$$\Omega = \text{normalized period mx},$$

so

$$A_\Omega = \mathrm{Jac}(X).$$

Want to understand geometry of theta divisor

$$\Theta \subseteq \mathrm{Jac}(X).$$

Return to polygonal description of X as $4g$-gon Δ w sides identified.

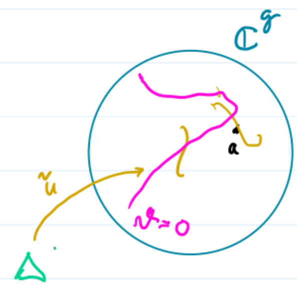
· Choose:

$$\vec{\omega} = (\omega_1, \ldots, \omega_g) : \quad \text{vector of normalized diffs}$$

Base pt $P_0 \in \Delta$

Then define

$$\tilde{u} : \Delta \longrightarrow \mathbb{C}^g, \quad \tilde{u}(P) = \int_{P_0}^{P} \vec{\omega}$$



Basic point is:

<u>Thm.</u> (Riemann). $\exists$ vector $\delta \in \mathbb{C}^g$ with the following property:

For any $a \in \mathbb{C}^g$, the function

$$f_a(P) = \vartheta\left(a + \int_{P_0}^{P} \omega\right)$$

either vanishes identically on Δ, or else has g zeroes

$$Q_1, \ldots, Q_g \in \Delta \quad \text{(counting multiplicities)}$$

s.t.

$$\sum_{i=1}^{g} \tilde{u}(Q_i) \equiv -a + \delta \quad (\text{mod } \Lambda)$$

For pf, see Mfd Tata, Chapt II, Thm 3.1. Two ingredients

- Residue calculation w. df/f on Δ
- Periodicity properties of ϑ.

ϑ is periodic around A-cycles, and we know how it transforms around B-cycles.

Geometric Interpretation

Consider:

$$u: X \longrightarrow \mathrm{Jac}(X) \qquad u(P) = \int_{P_0}^{P}$$

$$\cup\!\!|$$

$$\Theta =_{\text{def}} \{\vartheta = 0\}$$

Let

$$X_a = X + a \subseteq \mathrm{Jac} \quad : \text{transl of } X \text{ by } a$$

Thm says:

If $X_a \cap \Theta$ is finite, then

$X_a \cdot \Theta$ is divisor of deg g whose A-J sum is $-a+\delta$.

ie $\exists\, Q_1, , Q_g$ s.t

$Q_1 + + Q_g - gP_0 \equiv -a + \delta$

& $\forall_i \quad a + Q_i - P_0 \in \Theta$

($\therefore$ also $-a - Q_i + P_0 \in \Theta$)

RMK: For most $a \in \mathrm{Jac}$, $X_a \nsubseteq \Theta$ since we can choose a s.t. X_a passes thru an arb pt of Jac

Rmk: Can see Thm as an effective version of Jacobi inversion

Recall we have

$$\begin{array}{rccc} u_{g-1}: & X_{g-1} & \longrightarrow & \mathrm{Jac}(X) \\ & \Psi & & \Psi \\ & P_1 + \cdots + P_{g-1} & \longmapsto & \sum_{i=1}^{g-1} \int_{P_0}^{P_i} \end{array}$$

We defined

$$W_{g-1} = \mathrm{Im}\, u_{g-1} \subsetneq \mathrm{Jac}.$$

Thm:

$$W_{g-1} = \Theta + \delta,$$

ie. up to translation, Θ coincides w. W_{g-1}.

Pf. (1°). Claim: $W_{g-1} \subseteq \Theta + \delta$.

Pf. Enough to show general pt of LHS is $\in$ RHS.

· Choose general $P_1, \ldots, P_g \in X$ s.t. $\ell(P_1 + \cdots + P_g) = 1$. Apply R's Thm w.

$$a = \delta - (P_1 + \cdots + P_g - g P_0).$$

(This is also general pt $a \in \mathrm{Jac}$.). R's Thm says

$\exists\ Q_1, \dots, Q_g \in X$ s.t.

(i) $\quad P_0 - Q_i - a \in \Theta \quad$ for each i

(ii) $\quad Q_1 + \cdots + Q_g - gP_0 \equiv -a + \delta$

Look at (ii)

$$Q_1 + \quad + Q_g - gP_0 \equiv -(\delta - (P_1 + \cdot + P_g - gP_0)) + \delta$$

$$\equiv P_1 + \quad + P_g - gP_0$$

So:

$$Q_1 + \quad + Q_g \equiv P_1 + \cdots + P_g \quad \text{(as divisors on } X\text{)}$$

Since $\ell(P_1 + \quad + P_g) = 1$, this means

$$Q_1 + \cdots + Q_g = P_1 + \cdots + P_g.$$

Now plug this into (i), w $Q_i = P_i$:

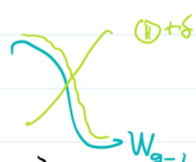
$$P_0 - P_i - (\delta - (P_1 + \quad + P_g) - gP_0) \in \Theta$$

$$P_0 + \quad + \hat{P_i} + \quad + P_{g-1} \in \Theta + \delta.$$

So:

$$W_{g-1} \subseteq \Theta + \delta. \qquad (*)$$

(2°). Both sides of $(*)$ are divisors, so to prove equality, suffices to show:



$$\#(X_a \cap W_{g-1}) = g = \#(X_a \cap (\Theta + \delta))$$

for general a (since the X_a cover Jac and have no common pts).

Since

$$X_a \underset{\text{homog}}{\sim} -X_a, \qquad \left(-X_a = \text{image of } X_a \text{ under mult by } -1\right)$$

it's in turn equiv to show:

$$\#\left(-X_a \cap W_{g-1}\right) = g \quad \text{for general } a \in \text{Jac},$$

Now say

$$\xi \in (-X_a) \cap W_{g-1}, \qquad \text{view } \xi \in Cl^0(X)$$

Take $a = P_1 + \ \ + P_g - gP_0$ for gen P_i. Then

$$\xi = a + (P_0 - x) \equiv Q_1 + \ \ + Q_{g-1} - (g-1)P_0 \quad \text{some } x, Q_1, \ , Q_{g-1} \in X$$

So

$$a \equiv Q_1 + \ \ + Q_{g-1} + x - gP_0$$

ie.

$$P_1 + \cdot + P_g \equiv x + Q_1 + \cdot + Q_{g-1} \quad \text{some } x, Q_i.$$

But then as before

$$P_1 + \ \ + P_g = x + Q_1 + \ \ + Q_{g-1},$$

So x is one of the P_i, so g solns. QED.

Often : one uses b.p. to identify

$$\text{Jac}(X) = Cl^{g-1}(X)$$

and one identifies

$$\Theta \text{ w } W_{g-1}$$

So

$$(\mathrm{Jac}, \Theta) = (Cl^{g-1}(X), W_{g-1}).$$

Under this identification: mult by -1 in Jac becomes involution

$$D \longmapsto K_X - D \quad \text{in } Cl^{g-1}(X).$$

Thm. (Riemann Sing. Thm): Fix $\xi \in Cl^{g-1}$

$$\xi \text{ is class of divisor of deg } g-1$$

Then

$$\mathrm{mult}_\xi(W_{g-1}) = \ell(\xi).$$

I.e if we identify Θ w W_{g-1}, $\mathrm{mult}_\xi(\Theta) = \ell(\xi)$. So

$$\xi \in \Theta \iff \ell(\xi) \geq 1 \iff \xi \text{ is effective.}$$

Prop. For any RS X,

$$g-4 \leq \dim \mathrm{Sing}(\Theta) \leq g-3$$

and $\dim \mathrm{Sing}\, \Theta = g-3 \iff X$ hyperell.

Ex. $g=4$. Then

$$\mathrm{Sing}(\Theta) = \{\xi \in Cl^3(X) \mid r(\xi) \geq 1\}$$

(i). X non hyperell, so $X = Q_2 \cap F_3 \subseteq \mathbb{P}^3$

$$\mathrm{Sing}(\Theta) \longleftrightarrow \text{rulings on } Q$$

(= 1 or 2 pts depending on whether Q is sing or not.)

(ii) X hyperell:

$$\mathrm{Sing}(\Theta) \cong X \quad (= g_2^1 + x)$$

<u>Schottky Problem</u>: Which abelian vars (A, Θ) are Jacobians?

Fact: For general (A, Θ), Θ is smooth

<u>Thm of Andreotti–Mayer</u>:

The locus of Jacobians is an irred comp of

$$\{(A, \Theta) \mid \dim \mathrm{Sing}\,\Theta \geq g-4\}$$

<u>Torelli's Thm.</u> (ref. ACGH, Ch. IV, §4)

In preparation, let me state

<u>Prop</u> (Geometric RR) Consider

$X \subseteq \mathbb{P}^{g-1}$: canon model of non-he curve of genus g

$D = P_1 + \cdots + P_d$ eff divisor of deg d.

$\overline{D} = \mathrm{Span}(D) \subseteq \mathbb{P}^{g-1}$: lin space spanned by D

Then

$$r(D) =_{\text{def}} \dim |D| = d-1-\dim \overline{D}$$

Ex. $d=3$. Expect 3 pts to span a $\mathbb{P}^2$. Prop says they span a line $\Leftrightarrow$ D a trigonal divisor.

Pf.

$$\dim \overline{D} = g-1-\dim(\text{linear spaces thm } D)$$

$$= g-1-\ell(K-D)$$

By RR:

$$\dim|D| = d-g+\ell(K-D)$$

$$= d-g+g-1-\dim\overline{D} \quad \square$$

Ex. $D = P_1+\dots+P_{g-1}$: Prop says

$$\dim \overline{D} < d-2 \iff \dim|D| \geq 1.$$

Now we turn to Torelli. Recall

$$\text{R.S. } X \text{ of genus } g \rightsquigarrow (\mathrm{Jac}(X), \Theta_X)$$

Thm: Suppose X, X' are R.S. s.t. $\exists$

$$\phi : \mathrm{Jac}(X) \xrightarrow{\cong} \mathrm{Jac}(X')$$
$$\phi^* \Theta_{X'} = \Theta_X \quad (\text{up to transl})$$

Then $X \cong X'$. (i.e. X is determined up to isom by its "polarized" Jacobian.)

Rmk: $(\mathrm{Jac}(X), \Theta_X)$ arise from X via Hodge theory (ie via decomp:

$$H^1(X,\mathbb{Z})\otimes\mathbb{C} = H^{1,0}\oplus H^{0,1}.)$$

Torelli says can recover X from this Hodge info. This is holy grail of Hodge-theoretic study of moduli.

To prove Thm, we will prove:

Thm* One can recover X from the data $(\mathrm{Jac}(X), \Theta_X)$.

Recall that we saw:

One has the identification

$$\begin{array}{ccc} \mathrm{Jac}(X) & = & C\ell^{g-1}(X) \\ \cup\!\! & & \cup\!\! \\ \Theta_X & = & W_{g-1} \end{array} \qquad \text{(up to transl)}$$

Then Thm* will follow fr:

Thm** One can recover X from the data

$$\mathrm{Jac}(X) = C\ell^{g-1}(X) \supseteq W_{g-1} = \Theta \text{ (up to transl)}$$

For simplicity, we'll assume X non-hyperell. We'll give

Andreotti's argument: the dual hypersurface of $X\subseteq\mathbb{P}^{g-1}$ is the branch divisor of the Gauss mapping of Θ_X.

Gauss mapping: Let

$$A = V/\Lambda \text{ be a complex torus, } \dim V = n$$

let $S \subseteq A$ be a hypsf. Assume for moment S is non-sing. Then for each $s \in S$, have

$$T_s S \subseteq T_s A \underset{\text{canon}}{=} T_0 A = V,$$

ie. $T_s S$ a hyperplane in V. This gives Gauss map:

$$\gamma : S \longrightarrow \mathbb{P}V^* = \text{proj space of hyperplanes in } V$$

Both sides have dim $= n-1$, so would typically expect γ to be generically finite covering.

What is S is singular? Let

$$S_{reg} = \{ \text{smooth pts of } S \}.$$

Then get

$$\gamma^0 : S_{reg} \longrightarrow \mathbb{P}V^*$$

At least in alg setting we can constr a "compactif."

$$\begin{tikzcd} \text{proj} \quad \overline{S} \arrow[dr, "\overline{\gamma}"] & \\ \cup\!\mid & \\ S_{reg} \arrow[r, "\gamma^0"'] & \mathbb{P}V^* \end{tikzcd}$$

by taking

$$\overline{S} = \left(\begin{matrix} \text{closure of grape of} \\ \gamma^0 \text{ in } \overline{S} \times \mathbb{P}V^* \end{matrix} \right).$$

Now let's apply this to

$$\Theta = W_{g-1} \subseteq \mathrm{Jac}(X) = H^{1,0}(X)^* / H_1(X,\mathbb{Z})$$

R's Sing Thm says $\Theta_{reg} = \{ [P_1 + \cdots + P_{g-1}] \mid \ell(P_1 + \cdots + P_{g-1}) = 1 \}$.

So Gauss map will be

$$\gamma : \Theta_{reg} \longrightarrow \mathbb{P} H^{1,0}(X) \quad (= \mathbb{P} H^{1,0}(X)^{**})$$

Can we guess what this map must be? Take

$$\xi = P_1 + \cdots + P_{g-1}, \quad \ell(P_1 + \cdots + P_{g-1}) = 1$$

Then

$$\gamma(\xi) = \mathbb{P} T_\xi \Theta \in \mathbb{P} H^{1,0}(X) = |K|.$$

Now by Geom RR,

$$\ell(P_1 + \cdots + P_{g-1}) = 1 \iff \mathrm{span}(P_1, \ldots, P_{g-1}) = (\text{hyperplane}) \subseteq \mathbb{P}^{g-1}$$

and

$$\gamma(\xi) = \begin{pmatrix} \text{hyperplane in } \mathbb{P}^{g-1} \\ \text{cutting out canon divisor} \end{pmatrix} \in |K|$$

<u>Lemma:</u> Let $\xi = D$ be a divisor of deg $g-1$ s.t. $\ell(D) = 1$. Then

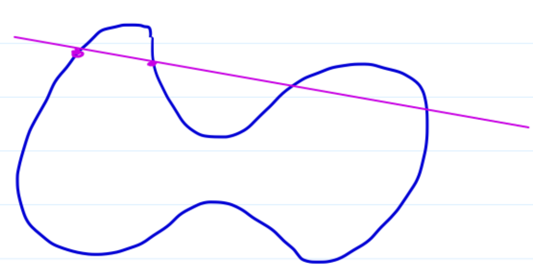
$$\gamma(D) = \qquad\qquad \subseteq \mathbb{P}^{g-1}$$

<u>Pf.</u> For you.

<u>Ex.</u> $g(X) = 3$, so $X \subseteq \mathbb{P}^2$ a smooth plane quartic. Then

$$\Theta \cong X_2 = \{\text{pairs of pts}\}$$

$$\gamma : \Theta \longrightarrow \mathbb{P}^{2*}, \quad (p,q) \mapsto \overline{pq}$$



So here

$$\gamma : X_2 \longrightarrow \mathbb{P}^{2*}$$

is covering of deg 6.

Sim:

> In arbitrary genus,
>
> $$\gamma : \bar{\Theta} \longrightarrow (\mathbb{P}^{g-1})^*$$
>
> is generically finite covering of $\deg = \binom{2g-2}{g-1}$

Go back to

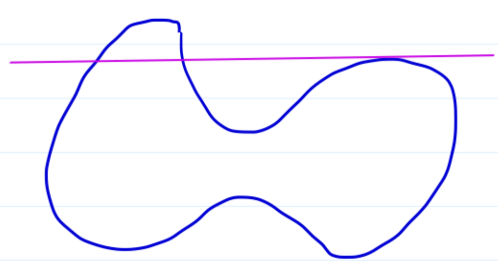
$$\gamma : X_2 \longrightarrow \mathbb{P}^{2*}$$

In gen, there is a codim 1 subset $B \subseteq \mathbb{P}^{2*}$ over which γ fails to be a covering space.

Here

$$B = \{\text{lines } \ell \subseteq \mathbb{P}^2 \mid \#(\ell \cap X) < 4\}$$

$$= \{\text{lines } \ell \subseteq \mathbb{P}^2 \mid \ell \text{ tangent to } X\}$$



The set of all such lines is called the <u>dual</u> of X:

$$X^* \subseteq \mathbb{P}^{2*}$$

Similarly,

> In arbitrary genus g, the branch divisor of
>
> $$\Theta \longrightarrow (\mathbb{P}^{g-1})^*$$
>
> is the set of all hyperplanes tangent to X:
>
> $$X^* \subseteq (\mathbb{P}^{g-1})^* \quad : \text{ dual hypsf to } X$$

(Technical aside: since $\operatorname{codim}_{\Theta}(\operatorname{Sing}\Theta)=3$, sing pts don't hurt)

Upshot so far:

$$(\mathrm{Jac}, \Theta) \rightsquigarrow \begin{array}{l}\text{dual variety } X^* \subset (\mathbb{P}^{g-1})^* \\ \text{of all hyperplanes tang to } X \text{ in } \mathbb{P}^{g-1}\end{array}$$

Torelli finally follows from

<u>Classical Duality Thm</u>: $(X^*)^* = X$