SUMS AND INTEGRALS

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Problem 1. Evaluate the improper integral

$$\int_0^1 \frac{\log(1-x)}{x} dx.$$

Problem 2. Show that if $r_1, ..., r_n$ are non-negative reals and $x_1, ..., x_n$ are real numbers

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \min(r_i, r_j) x_i x_j.$$

Problem 3. Compute

$$\lim_{n\to\infty} \left[\frac{1}{\sqrt{4n^2 - 1^2}} + \frac{1}{\sqrt{4n^2 - 2^2}} + \cdots + \frac{1}{\sqrt{4n^2 - n^2}} \right].$$

Problem 4. Compute

$$\lim_{n\to\infty} \frac{1}{n} \sum_{a=1}^{n} \sum_{b=1}^{n} \frac{a}{a^2 + b^2}.$$

Problem 5. Show that

$$\int_0^1 x^{-x} dx = \sum_{n\geqslant 1} n^{-n}.$$

Problem 6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(x) \geqslant 1$ for all x. Suppose that

$$f(x)f(2x)...f(nx) \leqslant 2018n^{2019}$$

for every positive integer n and all $x \in \mathbb{R}$. Must f be constant?

Problem 7. Show that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. Calculate the volume of the $n-1$ unit sphere in $\mathbb{R}^n$.

Problem 8. A rectangle in $\mathbb{R}^2$ is called *great* if either its width or height is an integer. Prove that if a rectangle X can be dissected into great rectangles, then the rectangle itself is great. Does this generalize to higher dimensions?

Problem 9. Prove that

$$\lim_{n\to\infty} \left(\prod_{k=0}^{n} \binom{n}{k} \right)^{\frac{1}{n(n+1)}} = \sqrt{e}.$$

Problem 10. Evaluate

$$\lim_{x\to 1^-} \prod_{n\geqslant 0} \left(\frac{1 + x^{n+1}}{1 + x^n} \right)^{x^n}.$$