PROBABILITY

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Problem 1. An unfair coin (probability p of showing heads) is tossed n times. What is the probability that the number of heads will be even?

Problem 2. Two persons agreed to meet in a definite place between noon and one o'clock. If either person arrives before the other, they wait 15 minutes. Calculate the probability the meeting occurs.

Problem 3. Real numbers are chosen at random from the interval $[0, 1]$. If after choosing the nth number the sum of the numbers so chosen first exceeds 1, show that the expected value for n is e.

Problem 4. Let α and β be given positive real numbers with $\alpha < \beta$. If two points are selected at random from a straight line segment of length β, what is the probability that the distance between them is at least α?

Problem 5. Two real numbers x and y are chosen at random in the interval $(0, 1)$ with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$ where r and s are rational.

Problem 6. Three points are selected uniformly at random on the circumference of a circle. What is the probability that the triangle they form contains the center of the circle? What about four points on the surface of a sphere?

Problem 7. Let $(x_1, x_2, ..., x_n)$ be a point chosen at random from the n-dimensional region defined by $0 < x_1 < x_2 < ... < x_n < 1$. Let f be a continuous function on $[0, 1]$ with $f(1) = 0$. Set $x_0 = 0$ and $x_{n+1} = 1$. Show that the expected value of the Riemann sum

$$\sum_{i=0}^{n} (x_{i+1} - x_i) f(x_{i+1})$$

is $\int_0^1 f(t)P(t)dt$, where P is a polynomial of degree n, independent of f, with $0 \leqslant P(t) \leqslant 1$.

Problem 8. Let p_n be the probability that $c+d$ is a perfect square when the integers c and d are selected independently at random from the set $\{1, 2, ..., n\}$. Show that $\lim_{n\to\infty}(p_n\sqrt{n})$ exists and can be expressed in the form $r(\sqrt{s} - t)$ where s and t are integers and r is a rational number.

Problem 9. The points $1, 2, ..., 1000$ are paired up at random to form 500 intervals $[i, j]$. What is the probability that among these intervals is one which intersects all of the others?

Problem 10. A fair coin is flipped until the numbers of heads exceeds the number of tails. What is the expected number of flips?