

LINEAR ALGEBRA

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Problem 1. Let M_n be the $(2n+1) \times (2n+1)$ for which

$$(M_n)_{ij} = \begin{cases} 0 & i = j \\ 1 & i - j \equiv 1, 2, \dots, n \pmod{2n+1} \\ -1 & i - j \equiv n+1, \dots, 2n \pmod{2n+1} \end{cases}.$$

Find the rank of M_n .

Problem 2. If A and B are square matrices such that $ABAB = 0$, does it follow that $BABA = 0$?

Problem 3. Let D_n denote the value of the $(n-1) \times (n-1)$ determinant

$$\det \begin{pmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{pmatrix}.$$

Is the set $\{D_n/n!\}_{n \geq 2}$ bounded?

Problem 4. Let A be the $n \times n$ matrix with $A_{jk} = \cos(2\pi(j+k)/n)$. Find the determinant of $I + A$.

Problem 5. Let A be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1 with equal probability. Find the expected value of $\det(A - A^t)$.

Problem 6. Suppose the differentiable functions $x_1(t), \dots, x_n(t)$ satisfy the equation $dx/dt = Ax$ where $A = (a_{ij})$ has $a_{ij} > 0$. Suppose for all i that $x_i(t) \rightarrow 0$ as $t \rightarrow \infty$. Are the functions necessarily linearly dependent?

Problem 7. Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \leq i \leq r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^r \text{tr}(M_i) = 0$. Prove that $\sum_{i=1}^r M_i = 0$.

Problem 8. Let A be a symmetric 0-1 matrix with ones on the diagonal, viewed as a matrix over the field of 2 elements. Prove that the vector of all 1s is in its span.

Problem 9. Let p be a prime, and let $A = (a_{ij})_{i,j=0}^{p-1}$ be the $p \times p$ matrix defined by

$$a_{ij} = \binom{i+j}{i}, \quad 0 \leq i, j \leq p-1.$$

Show that $A^3 \equiv I \pmod{p}$, where I denotes the identity matrix.

Problem 10. Let $A = A(n)$ be the $n \times n$ real matrix given by

$$A_{ij} = \begin{cases} 1 & |j-i| = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Let $V_n(x) = \det(xI - A)$ so $V_0(x) = 1$, $V_1(x) = x$, $V_2(x) = x^2 - 1$, $V_3(x) = x^3 - 2x$. Show that $V_{n+1}(x) = xV_n(x) - V_{n-1}(x)$, $n \geq 1$.