INDUCTION AND PIGEONHOLE

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Problem 1. Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, ..., 100$. Prove that there must be two distinct integers in A whose sum is 104.

Problem 2. Show that if there are n people at a party, then two of them know the same number of people there.

Problem 3. Five points lie in an equilateral triangle of size 1. Show that two of the points lie no farther than 1/2 apart.

Problem 4. A lattice point in the plane is a point (x, y) such that both x and y are integers. Find the smallest number n such that given n lattice points in the plane, there exist two whose midpoint is also a lattice point.

Problem 5. Show that for all positive integers n, $n^5/5 + n^4/2 + n^3/3 - n/30$ is an integer.

Problem 6. Show that $1 + 1/\sqrt{2} + 1/\sqrt{3} + \cdots + 1/\sqrt{n} < 2\sqrt{n}$.

Problem 7. Prove that from a set of ten distinct two-digit numbers, it is possible to select two disjoint subsets whose members have the same sum.

Problem 8. Let $A_1, A_2, ..., A_{1066}$ be subsets of a finite set X such that $|A_i| > \frac{1}{2}|X|$ for $1 \leqslant i \leqslant 1066$. Prove that there exist ten elements $x_1, ..., x_{10}$ of X such that every A_i contains at least one of $x_1, ..., x_{10}$.

Problem 9. Given any $n + 2$ integers, show that there exist two of them whose sum, or else whose difference, is divisible by $2n$.

Problem 10. Let $a_1 < \cdots < a_n, b_1 > \cdots > b_n$, and $\{a_1, ..., a_n, b_1, ..., b_n\} = \{1, 2, ..., 2n\}$. Show that $\sum_{i=1}^{n} |a_i - b_i| = n^2$.

Problem 11. For a partition π of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π, π' and numbers $x \neq y$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$.

Problem 12. A collection of subsets of $\{1, 2, ..., n\}$ has the property that each pair of subsets has at least one element in common. Prove that there are at most 2^{n-1} subsets in the collection.