HIDDEN INDEPENDENCE

ROBERT HOUGH

Problem 1. Slips of paper with the numbers from 1 to 99 are placed in a hat. Five numbers are randomly drawn out of the hat one at a time (without replacement). What is the probability that the numbers are chosen in increasing order?

Problem 2. In how many ways can a positive integer n be written as a sum of positive integers, taking order into account?

Problem 3. How many 8×8 matrices of 0's and 1's are there, such that every row and column contains an odd number of ones?

Problem 4. Let p be a prime number and $1 \leqslant k \leqslant p-1$. How many k-element subsets $\{a_1, ..., a_k\}$ of $\{1, 2, ..., p\}$ are there such that $a_1 + \cdots + a_k \equiv 0 \bmod p$?

Problem 5. Let π be a random permutation of $1, 2, ..., n$. Fix a positive integer $1 \leqslant k \leqslant n$. What is the probability that in the disjoint cycle decomposition of π, the length of the cycle containing 1 is k?

Problem 6. Choose n real numbers $x_1, ..., x_n$ uniformly and independently from the interval $[0, 1]$. What is the expected value of $\min x_i$, the minimum of $x_1, ..., x_n$.

Problem 7.

a. Let m and n be non-negative integers. Evaluate the integral
$$B(m, n) = \int_0^1 x^m (1-x)^n dx,$$
by interpretting this as a probability.

b. Let R be the region consisting of all triples (x, y, z) of non-negative real numbers with $x + y + z \leqslant 1$. Let $w = 1 - x - y - z$. Express
$$\iiint x^1 y^9 z^8 w^4 dx dy dz$$
in the form $a!b!c!d!/n!$.

Problem 8. Let π be a random permutation of $1, 2, ..., n$. What is the probability that 1 and 2 are in the same cycle?

Problem 9. Passengers $P_1, ..., P_n$ enter a plane with n seats. Each passenger has a different assigned seat. The first passenger sits in the wrong seat. After that, each passenger either sits in their assigned seat if unoccupied, or chooses a random available seat otherwise. What is the probability the last passenger sits in his assigned seat?

Problem 10. Let n balls be thrown uniformly and independently into n bins. What is the expected number of empty bins?