SEQUENCES AND SERIES

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Problem 1. Define the sequence $(a_n)_{n\geqslant 0}$ by $a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 6$ and

$$a_{n+4} = 2a_{n+3} + a_{n+2} - 2a_{n+1} - a_n, \qquad n \geqslant 0.$$

Prove that n divides a_n for all $n \geqslant 1$.

Problem 2. The sequence $a_0, a_1, a_2, ...$ satisfies

$$a_{m+n} + a_{m-n} = \frac{1}{2}(a_{2m} + a_{2n})$$

for all non-negative m, n with $m \geqslant n$. If $a_1 = 1$ find a_n.

Problem 3. Find the general term of the sequence defined by $x_0 = 3$, $x_1 = 4$ and, for $n \geqslant 2$,

$$(n+1)(n+2)x_n = 4(n+1)(n+3)x_{n-1} - 4(n+2)(n+3)x_{n-2}.$$

Problem 4. Let $(x_n)_{n\geqslant 0}$ be a sequence defined by $x_{n+1} = ax_n + bx_{n-1}$, with $x_0 = 0$. Show that the expression $x_n^2 - x_{n-1}x_{n+1}$ depends only on b and x_1, not on a.

Problem 5. Compute

$$\lim_{n\to\infty} \left|\sin\left(\pi\sqrt{n^2+n+1}\right)\right|.$$

Problem 6. Let k be a positive integer and μ a positive real number. Prove that

$$\lim_{n\to\infty} \binom{n}{k} \left(\frac{\mu}{n}\right)^k \left(1 - \frac{\mu}{n}\right)^{n-k} = \frac{\mu^k}{e^\mu k!}.$$

Problem 7. Show that if the series $\sum a_n$ converges, where $(a_n)_n$ is decreasing, then $\lim_n na_n = 0$.

Problem 8. Prove that for $n \geqslant 2$, the equation $x^n + x - 1$ has a unique root in the interval $[0, 1]$. If x_n denotes this root, prove the sequence $(x_n)_n$ is convergent and find its limit.

Problem 9. Let $f : [a, b] \to [a, b]$ be an increasing function. Show there is $\xi \in [a, b]$ with $f(\xi) = \xi$.

Problem 10. Let $a_1, a_2, a_3, ...$ be non-negative numbers. Prove that $\sum_{n=1}^\infty a_n < \infty$ implies $\sum_{n=1}^\infty \sqrt{a_{n+1}a_n} < \infty$.

Problem 11. Does the series $\sum \sin \pi\sqrt{n^2+1}$ converge?

Problem 12. For a non-negative integer k, define $S_k(n) = 1^k + 2^k + \cdots + n^k$. Prove that

$$1 + \sum_{k=0}^{r-1} \binom{r}{k} S_k(n) = (n+1)^r.$$

Problem 13. Prove the identity

$$\sum_{k=1}^{n} (k^2+1)k! = n(n+1)!.$$