REAL ANALYSIS

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Problem 1. Does $\lim_{x\to\frac{\pi}{2}}(\sin x)^{\frac{1}{\cos x}}$ exist?

Problem 2. Let $f : (0,\infty) \to (0,\infty)$ be an increasing function with $\lim_{t\to\infty}\frac{f(2t)}{f(t)} = 1$. Prove that $\lim_{t\to\infty}\frac{f(mt)}{f(t)} = 1$ for any $m > 0$.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property $\lim_{h\to 0^+}\frac{f(x+2h)-f(x+h)}{h} = 0$ for all $x \in \mathbb{R}$. Prove that f is a constant.

Problem 4. Give an example of a continuous function on an interval which is nowhere differentiable.

Problem 5. For a non-zero real number x prove $e^x > 1 + x$.

Problem 6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. For $x \in \mathbb{R}$ define

$$g(x) = f(x)\int_0^x f(t)dt.$$

Show that if g is a non-increasing function then f is identically 0.

Problem 7. For $x \geqslant 2$ prove the inequality

$$(x+1)\cos\frac{\pi}{x+1} - x\cos\frac{\pi}{x} > 1.$$

Problem 8. Let $x_1, ..., x_n$ be real numbers. Find the real numbers a that minimize the expression

$$|a - x_1| + |a - x_2| + \cdots + |a - x_n|.$$

Same question for

$$|a - x_1|^2 + \cdots + |a - x_n|^2.$$

Problem 9. Let f be a real-valued continuous function on $\mathbb{R}$ satisfying for all $x \in \mathbb{R}$ and $h > 0$,

$$f(x) \leqslant \frac{1}{2h}\int_{x-h}^{x+h} f(y)dy.$$

Prove f is convex and its maximum on any closed interval is assumed at an endpoint.

Problem 10. Compute the integral

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}dx.$$

Problem 11. Compute the integral

$$\int_0^\infty \frac{\ln x}{x^2+a^2}dx$$

where $a > 0$ is a constant.

Problem 12. Give an example of a function $f : (2, \infty) \to (0, \infty)$ with the property that

$$\int_2^\infty f^p(x) dx$$

is finite if and only if $p \in [2, \infty)$.

Problem 13. Compute

$$\lim_{n\to\infty} \left(\frac{2^{\frac{1}{n}}}{n+1} + \frac{2^{\frac{2}{n}}}{n+\frac{1}{2}} + \cdots + \frac{2^{\frac{n}{n}}}{n+\frac{1}{n}} \right).$$