PRACTICE PUTNAM

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Problem 1. Let $I_m = \int_0^{2\pi} \cos x \cos 2x \cdots \cos mx dx$. For which integers m, $1 \leqslant m \leqslant 10$ is $I_m \neq 0$.

Problem 2. Evaluate $\int_0^\infty t^{-1/2} e^{-1985(t+t^{-1})} dt$. You may assume that $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$.

Problem 3. Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \leqslant i \leqslant r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^r \operatorname{tr}(M_i) = 0$, where $\operatorname{tr}(A)$ is the trace. Prove that $\sum_{i=1}^r M_i$ is the $n \times n$ zero matrix.

Problem 4. For a positive real number r, let $G(r)$ be the minimum value of $|r - \sqrt{m^2 + 2n^2}|$ taken over integers m, n. Prove or disprove the assertion that $\lim_{r \to \infty} G(r)$ exists and is equal to 0.

Problem 5. For each positive integer n, let $a(n)$ be the number of 0s in the base three representation of n. For which positive real numbers x does the series

$$\sum_{n=1}^\infty \frac{x^{a(n)}}{n^3}$$

converge?

Problem 6. Let F be a field in which $1 + 1 \neq 0$. Show that the set of solutions to the equation $x^2 + y^2 = 1$ with x and y in F is given by $(x, y) = (1, 0)$ and

$$(x, y) = \left(\frac{r^2 - 1}{r^2 + 1}, \frac{2r}{r^2 + 1} \right)$$

where r runs over elements of F for which $r^2 \neq -1$.

Problem 7. Prove that if $\sum_{n=1}^\infty a_n$ is a convergent sequence of positive real numbers, then so is $\sum_{n=1}^\infty (a_n)^{n/(n+1)}$.

Problem 8. Two real numbers x and y are chosen at random in the interval $(0, 1)$ with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in form $r + s\pi$ where r and s are rational.

Problem 9. Suppose that a sequence $a_1, a_2, a_3, ...$ satisfies $0 < a_n \leqslant a_{2n} + a_{2n+1}$ for all $n \geqslant 1$. Prove that the series $\sum a_n$ diverges.

Problem 10. If p is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by p^2.