NUMBER THEORY

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Problem 1. Three infinite arithmetic progressions are given, whose terms are positive integers. Assuming that each of the numbers $1, 2, 3, 4, 5, 6, 7, 8$ appears in one of the arithmetic progressions, prove that 1980 appears in one of the arithmetic progressions.

Problem 2. Show that no positive integers x, y, z can satisfy the equation $x^2 + 10y^2 = 3z^2$.

Problem 3. For p and q coprime positive integers prove the reciprocity law

$$\left\lfloor \frac{p}{q} \right\rfloor + \left\lfloor \frac{2p}{q} \right\rfloor + \cdots + \left\lfloor \frac{(q-1)p}{q} \right\rfloor = \left\lfloor \frac{q}{p} \right\rfloor + \left\lfloor \frac{2q}{p} \right\rfloor + \cdots + \left\lfloor \frac{(p-1)q}{p} \right\rfloor.$$

Problem 4. Prove that for any real number x and for any positive integer n,

$$\lfloor nx \rfloor \geqslant \frac{\lfloor x \rfloor}{1} + \frac{\lfloor 2x \rfloor}{2} + \cdots + \frac{\lfloor nx \rfloor}{n}.$$

Problem 5. Show that for each positive integer n,

$$n! = \prod_{i=1}^{n} LCM(1, 2, ..., \lfloor n/i \rfloor).$$

Problem 6. Prove that if n is a positive integer that is divisible by at least two primes, then there exists an n-gon with all angles equal and with side lengths the numbers $1, 2, ..., n$ in some order.

Problem 7. Prove that for every positive integer n,

$$\sum_{k|n} \phi(k) = n.$$

Problem 8. Prove that for every n, there exist n consecutive integers each of which is divisible by two different primes.

Problem 9. Prove that there exists a positive integer k such that $k \cdot 2^n + 1$ is composite for every positive integer n.

Problem 10. A lattice point $(x, y) \in \mathbb{Z}^2$ is visible from the origin if x and y are coprime. Prove that for any positive integer n there exists a lattice point (a, b) whose distance from every visible point is greater than n.

Problem 11. Find a solution to the Diophantine equation

$$x^2 - (m^2 + 1)y^2 = 1$$

where m is a positive integer.