MULTIVARIABLE CALCULUS

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Problem 1. Prove that if the function $u(x,t)$ satisfies the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x}, \qquad (x,t) \in \mathbb{R}^2$$

then so does the function

$$v(x,t) = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}} u(xt^{-1}, -t^{-1}), \qquad x \in \mathbb{R}, t > 0.$$

Problem 2. Determine the maximum and minimum of $\cos A + \cos B + \cos C$ when A, B, C are the angles of a triangle.

Problem 3. Of all triangles circumscribed about a given circle, find the one with the smallest area.

Problem 4. Find the integral of the function $f(x,y,z) = \frac{x^4+2y^4}{x^4+4y^4+z^4}$ over the unit ball $\{(x,y,z) : x^2+y^2+z^2 \leqslant 1\}$.

Problem 5. Show that for $s > 0$,

$$\int_0^\infty e^{-sx} x^{-1} \sin x dx = \arctan(s^{-1}).$$

Problem 6. Show that for $a, b > 0$,

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = \ln \frac{b}{a}.$$

Problem 7. Compute the flux of the vector field $\underline{F}(x,y,z) = x(e^{xy} - e^{zx})\underline{i} + y(e^{yz} - e^{xy})\underline{j} + z(e^{zx} - e^{yz})\underline{k}$ across the upper hemisphere of the unit sphere.

Problem 8. Compute

$$\oint_C y^2 dx + z^2 dy + x^2 dz,$$

where C is the Viviani curve, defined as the intersection of the sphere $x^2 + y^2 + z^2 = a^2$ with the cylinder $x^2 + y^2 = ax$.

Problem 9. Let f and g be differentiable functions on the real line satisfying the equation $(f^2 + g^2)f' + (fg)g' = 0$. Prove that f is bounded.

Problem 10. Show that all solutions to the differential equation $y'' + e^x y = 0$ remain bounded as $x \to \infty$.

Problem 11. Let f be a real-valued continuous nonnegative function on $[0,1]$ such that

$$f(t)^2 \leqslant 1 + 2\int_0^t f(s) ds$$

for all $t \in [0,1]$. Show that $f(t) \leqslant 1 + t$ for every $t \in [0,1]$.