LINEAR ALGEBRA

ROBERT HOUGH

Problem 1. Do there exist $n \times n$ matrices A and B such that $AB - BA = I_n$?

Problem 2. Compute the nth power of $m \times m$ matrix

$$J_m(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}.$$

Problem 3. Let $(F_n)_n$ be the Fibonacci sequence. Prove, using determinants the identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.

Problem 4. Prove the formula for the determinant of a circulant matrix

$$\det \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_n & x_1 & x_2 & \cdots & x_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_3 & x_4 & x_5 & \cdots & x_2 \\ x_2 & x_3 & x_4 & \cdots & x_1 \end{pmatrix} = (-1)^{n-1} \prod_{j=0}^{n-1} \left(\sum_{k=1}^{n} \zeta^{jk} x_k \right),$$

where $\zeta = e^{2\pi i/n}$.

Problem 5. Let A be an $n \times n$ matrix such that $A + A^t = 0$. Prove that $\det(I + \lambda A^2) \geqslant 0$ for all $\lambda \in \mathbb{R}$.

Problem 6. Let $A = (a_{ij})_{ij}$ be an $n \times n$ matrix such that $\sum_{j=1}^{n} |a_{ij}| < 1$ for each i. Prove $I_n - A$ is invertible.

Problem 7. Let A and B be $n \times n$ matrices such that there are non-zero real numbers a, b with $AB = aA + bB$. Prove that A and B commute.

Problem 8. Let $P(x) = x^n + x^{n-1} + \cdots + 1$. Find the remainder when $P(x^{n+1})$ is divided by $P(x)$.

Problem 9. Let n be a positive integer and $P(x)$ an nth degree polynomial with complex coefficients such that $P(0)$, $P(1)$,... $P(n)$ are all integers. Prove that the polynomial $n!P(x)$ has integer coefficients.

Problem 10. A linear map A on n-dimensional vector space V is an involution if $A^2 = I$. Prove that for every involution A there is a basis of V consisting of eigenvectors of A and find the maximum number of distinct pairwise commuting involutions.

Problem 11. Let $x_1, ..., x_n$ be differentiable functions of a single variable t, satisfying $\frac{dx}{dt} = Ax$ where A is an $n \times n$ matrix with positive entries. Suppose that for all i, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, ..., x_n$ necessarily linearly independent?