INVARIANTS

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Problem 1. Let X be a subset of the positive integers with the property that the sum of any two not necessarily distinct elements in X is again in X. Suppose that $\{a_1, a_2, ..., a_n\}$ is the set of all positive integers not in X. Prove that $a_1 + a_2 + \cdots + a_n \leqslant n^2$.

Problem 2. The entries of a matrix are real numbers of absolute value less than or equal to 1, and the sum of the elements in each column is 0. Prove that you can permute the elements of each column in such a way that the sum of the elements in each row has absolute value less than or equal to 2.

Problem 3. An ordered triple of numbers is given. It is permitted to change two of them from a, b to $\frac{a+b}{\sqrt{2}}, \frac{a-b}{\sqrt{2}}$. Is it possible to move from $(1, \sqrt{2}, 1+\sqrt{2})$ to $(2, \sqrt{2}, 1/\sqrt{2})$?

Problem 4. There is a heap of 1001 stones on a table. You are allowed to perform the following operation: you choose one of the heaps containing more than one stone, throw away a stone from the heap, then divide it into two smaller heaps. Is it possible to reach a situation in which all of the heaps on the table contain exactly 3 stones by performing this operation?

Problem 5. On an arbitrarily large chessboard consider a generalized knight that can jump p squares in one direction and q in the other, $p, q > 0$. Show that the knight can return to its initial position only after an even number of moves.

Problem 6. The number 99...99 having 1997 nines is written on a blackboard. Each minute a number written on the blackboard is factored into two factors, each factor is increased or decreased by 2, and the resulting two numbers are written. Is it possible that at some point all of the numbers on the blackboard are equal to 9?

Problem 7. A real number is written in each square of an $n \times n$ chessboard. We can perform the operation of changing all signs in a row or column. Prove that by performing this operation a finite number of times we can produce a new table with row and column sums non-negative.

Problem 8. Consider the integer lattice in the plane with one pebble placed at the origin. We play a game in which at each step a pebble is removed and two new pebbles are placed at two neighboring nodes, provided those nodes are unoccupied. Prove that at any time there will be a pebble at distance at most 5 from the origin.

Problem 9. Let there be given 9 lattice points in three-dimensional Euclidean space. Show that there is a lattice point on the interior of the line segment joining two of these points.

Problem 10. Let $a_1, ..., a_{2n+1}$ be a set of integers such that if any one of them is removed, the remaining $2n$ integers can be divided into two sets of n integers with equal sum. Prove $a_1 = a_2 = \cdots = a_{2n+1}$.