INEQUALITIES

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Problem 1. Let $a_1, a_2, ..., a_n$ be real numbers such that $a_1 + \cdots + a_n \geqslant n^2$ and $a_1^2 + \cdots + a_n^2 \leqslant n^3 + 1$. Prove that $n - 1 \leqslant a_k \leqslant n + 1$ for all k.

Problem 2. Let $a_1, a_2, ..., a_n$ be distinct real numbers. Find the maximum of

$$a_1 a_{\sigma(1)} + a_2 a_{\sigma(2)} + \cdots + a_n a_{\sigma(n)}$$

over all permutations σ of $\{1, 2, ..., n\}$.

Problem 3. Prove that the finite sequence $a_0, a_1, ..., a_n$ is a geometric progression if and only if

$$(a_0 a_1 + a_1 a_2 + \cdots + a_{n-1} a_n)^2 = (a_0^2 + \cdots + a_{n-1}^2)(a_1^2 + \cdots + a_n^2).$$

Problem 4. Consider the positive real numbers $x_1, ..., x_n$ with $x_1 x 2 \cdots x_n = 1$. Prove that

$$\frac{1}{n-1+x_1} + \frac{1}{n-1+x_2} + \cdots + \frac{1}{n-1+x_n} \leqslant 1.$$

Problem 5. Let $x_1, x_2, ..., x_n$, $n \geqslant 2$ be positive numbers such that $x_1 + \cdots + x_n = 1$. Prove that

$$\left(1 + \frac{1}{x_1}\right) \cdots \left(1 + \frac{1}{x_n}\right) \geqslant (n+1)^n.$$

Problem 6. Let $x_1, x_2, ..., x_n$ be n real numbers such that $0 < x_j \leqslant \frac{1}{2}$, for $1 \leqslant j \leqslant n$. Prove the inequality

$$\frac{\prod_{j=1}^n x_j}{\left(\sum_j x_j\right)^n} \leqslant \frac{\prod_{j=1}^n (1 - x_j)}{\left(\sum_j (1 - x_j)\right)^n}.$$

Problem 7. Let $n > 2$ be an integer, and let $x_1, x_2, ..., x_n$ be positive numbers with the sum equal to 1. Prove that

$$\prod_{i=1}^n \left(1 + \frac{1}{x_i}\right) \geqslant \prod_{i=1}^n \left(\frac{n - x_i}{1 - x_i}\right).$$

Problem 8. Show that for non-negative a_k, b_k, $1 \leqslant k \leqslant n$, one has

$$\left(\prod_{k=1}^n a_k\right)^{\frac{1}{n}} + \left(\prod_{k=1}^n b_k\right)^{\frac{1}{n}} \leqslant \left(\prod_{k=1}^n (a_k + b_k)\right)^{\frac{1}{n}}.$$

Problem 9. Show that if $f : [a, b] \to \mathbb{R}$ is convex and $a < x < b$, then one has

$$\frac{f(x) - f(a)}{x - a} \leqslant \frac{f(b) - f(a)}{b - a} \leqslant \frac{f(b) - f(x)}{b - x}.$$

Problem 10. Show that $\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$ implies $\frac{\sin x}{x} = \prod_{k=1}^\infty \cos \frac{x}{2^k}$.

Problem 11. Show that for complex numbers $z_1, z_2, ..., z_n$ one has

$$\frac{1}{\pi} \sum_{j=1}^n |z_j| \leqslant \max_{I \subset \{1, 2, ..., n\}} \left|\sum_{j \in I} z_j\right|.$$