INDUCTION AND PIGEONHOLE

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Problem 1. Prove for all positive numbers n the identity

$$\frac{1}{n+1} + \cdots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n}.$$

Problem 2. Let n be a positive integer. Show that

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} < \frac{3}{2}.$$

Problem 3. Let $x_1, ..., x_n, y_1, ..., y_m$ be some positive integers, with $n, m > 1$. Assume that $x_1 + \cdots + x_n = y_1 + \cdots + y_m < mn$. Prove that in the expression

$$x_1 + \cdots + x_n = y_1 + \cdots + y_m$$

you can suppress some but not all of the terms and still have equality.

Problem 4. Prove that the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geqslant 1$ satisfies the identity

$$F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3.$$

Problem 5. The vertices of a convex polygon are colored by at least three colors such that no two consecutive vertices have the same color. Prove that you can dissect the polygon into triangles by diagonals that do not cross and whose endpoints have different colors.

Problem 6. A sequence of m positive integers contains exactly n distict terms. Prove that if $2^n \leqslant m$ then there exists a block of consecutive terms whose product is a perfect square.

Problem 7. Let p be a prime number and a, b, c integers such that a and b are not divisible by p. Prove that $ax^2 + by^2 \equiv c \bmod p$ has integer solutions.

Problem 8. Show that there is a positive term of the Fibonacci sequence that is divisible by 1000.

Problem 9. Let m be a positive integer. Prove that among any $2m + 1$ distinct integers of absolute value less than $2m - 1$ there exist three whose sum is equal to 0.

Problem 10. Let $P_1, ..., P_{2m}$ be a permutation of the vertices of a regular polygon. Prove that the closed polygonal line $P_1P_2...P_{2m}$ contains a pair of parallel segments.

Problem 11. The points of the plane are colored with one of a finite number of colors. Prove that some rectangle has all vertices the same color.

Problem 12. Inside the unit square lie several circles, the sum of whose circumferences is at least 10. Prove that infinitely many lines intersect at least four circles.