GEOMETRY AND TRIGONOMETRY

ROBERT HOUGH

Problem 1. Given three vectors $\underline{a}, \underline{b}, \underline{c}$, define

$$\underline{x} = (\underline{b} \cdot \underline{c})\underline{a} - (\underline{c} \cdot \underline{a})\underline{b}, \underline{y} = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}, \underline{z} = (\underline{b} \cdot \underline{a})\underline{c} - (\underline{b} \cdot \underline{c})\underline{a}.$$

Prove that if $\underline{a}, \underline{b}, \underline{c}$ form a triangle, so do $\underline{x}, \underline{y}, \underline{z}$.

Problem 2. Given two triangles ABC and $A'B'C'$ with the same centroid, prove that we can construct a triangle with sides equal to the segments AA', BB', CC'.

Problem 3. Let M be a point in the plane of triangle ABC. Prove that the centroids of the triangles MAB, MAC, MBC form a triangle similar to ABC.

Problem 4. Let $A_0, A_1, ..., A_{n-1}$ be the vertices of a regular n-gon inscribed in the unit circle. Prove $A_0A_1 \cdot A_0A_2 \cdot \cdots \cdot A_0A_{n-1} = n$.

Problem 5. On the axis of a parabola consider two fixed points at equal distance from the focus. Prove that the difference of the squares of the distances from these points to an arbitrary tangent to the parabola is constant.

Problem 6. Two convex polygons are placed one inside the other. Prove that the perimeter of the inner polygon is less than the perimeter of the outer one.

Problem 7. There are n line segments in the plane with sum of lengths equal to 1. Prove that there exists a straight line such that the projections onto the straight line have total length $\frac{2}{\pi}$.

Problem 8. Show that if each of the three main diagonals of a hexagon divides the hexagon into two parts of equal areas, then the three diagonals are concurrent.

Problem 9. Show that the trigonometric equation $\sin(\cos x) = \cos(\sin x)$ has no solutions.

Problem 10. Compute the integral $\int \sqrt{\frac{1-x}{1+x}} dx$, $x \in (-1, 1)$.

Problem 11. Prove the identity

$$1 - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots = 2^{\frac{n}{2}} \cos \frac{n\pi}{4}, \qquad n \geqslant 1.$$

Problem 12. Prove that

$$\frac{1}{\sin 45^\circ \sin 46^\circ} + \cdots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin 1^\circ}.$$

Problem 13. Compute the product

$$(1 - \cot 1^\circ)(1 - \cot 2^\circ) \cdots (1 - \cot 44^\circ).$$