COMBINATORICS

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Problem 1. Determine the number of permutations $a_1, a_2, ..., a_{2004}$ of the numbers $1, 2, ..., 2004$ for which

$$|a_1 - 1| = |a_2 - 2| = \cdots = |a_{2004} - 2004| > 0.$$

Problem 2. In how many regions do n spheres divide the 3 dimensional space if any two intersect along a circle, no three intersect along a circle, and no four intersect at one point?

Problem 3. Prove that

$$\binom{2k}{k} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} (2\sin\theta)^{2k} d\theta.$$

Problem 4. Let F_n be the Fibonacci sequence $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$ for $n \geqslant 2$. Prove

$$F_1\binom{n}{1} + F_2\binom{n}{2} + \cdots + F_n\binom{n}{n} = F_{2n}.$$

Problem 5. Prove the analogue of the binomial formula

$$[x+y]_n = \sum_{k=0}^{n}\binom{n}{k}[x]_k[y]_{n-k}$$

where $[x]_n = x(x-1)(x-2)\cdots(x-n+1)$.

Problem 6. Compute the sums

$$\sum_{k=1}^{n} k\binom{n}{k}, \qquad \sum_{k=1}^{n}\frac{1}{k+1}\binom{n}{k}.$$

Problem 7. For a positive integer n, denote by $S(n)$ the number of choices of signs '+' or '−' such that $\pm 1 \pm 2 \pm \cdots \pm n = 0$. Prove that

$$S(n) = \frac{2^{n-1}}{\pi}\int_0^{2\pi} \cos t \cos 2t \cdots \cos nt dt.$$

Problem 8. Prove the combinatorial identity

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

Problem 9. Prove that the number of nonnegative integer solutions to the equation

$$x_1 + x_2 + \cdots + x_m = n$$

is $\binom{m+n-1}{m-1}$.

Problem 10. A permutation σ of a set S is called a derangement if it does not have fixed points, i.e. $\sigma(x) \neq x$ for all x. Find the number of derangements of $\{1, 2, 3, ..., n\}$.