MATH 322, SPRING 2019 FINAL, PRACTICE PROBLEMS

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Problem 1. Let $\alpha : (\theta, \phi) \mapsto (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$ be the spherical coordinate map. Let $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$. Calculate $\alpha^*\omega$.

Solution. We have

$$\begin{aligned}
\alpha^* dx &= -\sin\theta\sin\phi d\theta + \cos\theta\cos\phi d\phi \\
\alpha^* dy &= \cos\theta\sin\phi d\theta + \sin\theta\cos\phi d\phi \\
\alpha^* dz &= -\sin\phi d\phi.
\end{aligned}$$

Hence

$$\begin{aligned}
\alpha^*\omega = &(\cos\theta\sin\phi)(\cos\theta\sin\phi d\theta + \sin\theta\cos\phi d\phi) \wedge (-\sin\phi d\phi) \\
&+ \sin\theta\sin\phi(-\sin\phi d\phi) \wedge (-\sin\theta\sin\phi d\theta + \cos\theta\cos\phi d\phi) \\
&+ \cos\phi(-\sin\theta\sin\phi d\theta + \cos\theta\cos\phi d\phi) \wedge (\cos\theta\sin\phi d\theta + \sin\theta\cos\phi d\phi) \\
&= -\cos^2\theta\sin^3\phi d\theta \wedge d\phi - \sin^2\theta\sin^3\phi d\theta \wedge d\phi \\
&- (\sin^2\theta\cos^2\phi\sin\phi + \cos^2\theta\cos^2\phi\sin\phi)d\theta \wedge d\phi \\
&= -\sin\phi d\theta \wedge d\phi.
\end{aligned}$$

Problem 2. Let $\omega = e^{xyz}dx \wedge dy + x^3ydx \wedge dz + \sin(z)dx \wedge dy$. Calculate $d\omega$.

Solution. We have $d(e^{xyz}) = yze^{xyz}dx + xze^{xyz}dy + xye^{xyz}dz$, $d(x^3y) = 3x^2ydx + x^3dy$ and $d\sin z = \cos zdz$. Hence

$$\begin{aligned}
d\omega &= d(e^{xyz})dx \wedge dy + d(x^3y)dx \wedge dz + d(\sin z)dx \wedge dy \\
&= xye^{xyz}dz \wedge dx \wedge dy + x^3dy \wedge dx \wedge dz + \cos zdz \wedge dx \wedge dy \\
&= (xye^{xyz} - x^3 + \cos z)dx \wedge dy \wedge dz.
\end{aligned}$$

Problem 3. Let C be the right circular cylinder $\{x^2 + y^2 \leq 1, 0 \leq z \leq 5\}$, given the usual orientation in $\mathbb{R}^3$. Let

$$\omega = e^z dx \wedge dy.$$

Calculate $\int_{\partial C} \omega$.

Solution. We have $d\omega = e^z dx \wedge dy \wedge dz$. By Stoke's theorem

$$\int_{\partial C} \omega = \int_C d\omega = \int_{x^2+y^2\leq 1, 0\leq z\leq 5} e^z = \pi(e^5 - 1).$$

Problem 4. Let $\omega = \sum_{i=1}^n x_i dx_i$. Calculate $\int_C \omega$ where C is the oriented curve

$$C = \left\{ \cos(2\pi t), \cos(4\pi t), ..., \cos(2n\pi t) : 0 \leq t \leq \frac{1}{4} \right\}.$$

Solution. Let $\theta = \frac{1}{2}\sum_{i=1}^n x_i^2$ so that $d\theta = \omega$. Parameterize C by

$$\underline{r}(t) = \begin{pmatrix} \cos(2\pi t) \\ \cos(4\pi t) \\ \vdots \\ \cos(2n\pi t) \end{pmatrix}, \qquad 0 \leq t \leq \frac{1}{4}.$$

Hence,

$$\int_C d\theta = \theta\left(\underline{r}\left(\frac{1}{4}\right)\right) - \theta(\underline{r}(0)).$$

Since $\cos(\frac{2\pi k}{4})^2$ is 1 if k is even and 0 if k is odd. Hence $\theta\left(\underline{r}\left(\frac{1}{4}\right)\right) = \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor$ while $\theta(\underline{r}(0)) = \frac{n}{2}$, so that the integral is

$$\int_C d\theta = \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor - \frac{n}{2}.$$

Problem 5. Let $\mathbb{T}$ be the torus in $\mathbb{R}^4$, $\mathbb{T} = \{x_1^2 + x_2^2 = a^2, x_3^2 + x_4^2 = b^2\}$. Prove that $\mathbb{T}$ is orientable, and calculate its volume.

Solution. Parametrize $\mathbb{T}$ with patches $(a\cos s, a\sin s, b\cos t, b\sin t)$ with s, t in either $(-\frac{\pi}{2}, \frac{3\pi}{2})$ or $(-\frac{3\pi}{2}, \frac{\pi}{2})$. The patches overlap positively, since they are equal on the overlap. If α is one of the patches,

$$D\alpha = \begin{pmatrix} -a\sin s & 0 \\ a\cos s & 0 \\ 0 & -b\sin t \\ 0 & b\cos t \end{pmatrix}.$$

Thus
$$V(D\alpha) = \left(\det(D\alpha^t D\alpha)\right)^{\frac{1}{2}} = |ab|.$$
Hence the volume is $4\pi^2|ab|$.

Problem 6. Let $\operatorname{tr} : \mathrm{Mat}_{n\times n} \to \mathbb{R}$, $\operatorname{tr} M = \sum_i M_{ii}$ be the trace map. Let O_n be the orthogonal group of $n \times n$ matrices. Calculate the moments
$$m_1 = \frac{\int_{O_n} \operatorname{tr}(M) dV}{\int_{O_n} 1 dV}$$
$$m_2 = \frac{\int_{O_n} (\operatorname{tr} M)^2 dV}{\int_{O_n} 1 dV}.$$

Solution. Since the volume integral is invariant under translation in the orthogonal group, it is invariant under permuting the coordinates, and under multiplying a row or column by -1. We have
$$\int_{O_n} \operatorname{tr}(M) dV = \int_{O_n} \sum_i M_{ii} dV = n \int_{O_n} M_{11} dV = 0.$$
Thus $m_1 = 0$.

Also,
$$\begin{aligned}\int_{O_n} \operatorname{tr}(M)^2 dV &= \int_{O_n} (nM_{11}^2 + (n^2-n)M_{11}M_{22}) dV \\ &= \int_{O_n} nM_{11}^2 dV \\ &= \int_{O_n} \sum_{i=1}^n M_{1i}^2 dV \\ &= \int_{O_n} 1 dV.\end{aligned}$$
Hence $m_2 = 1$.

Problem 7. Let $S^{n-1} = \{\underline{x} \in \mathbb{R}^n : \|\underline{x}\| = 1\}$. Let O_n be the orthogonal group of $n \times n$ matrices M_n which satisfy $M_n^t M_n = I_n$.

a. Given $\underline{x} \in S^{n-1}$, let $O_{\underline{x}}$ be the orthogonal matrix which rotates $\underline{e}_1$ to $\underline{x}$ in the $(\underline{e}_1, \underline{x})$ plane while preserving their orientation, while leaving the orthogonal complement of this plane fixed. Give a matrix representation of $O_{\underline{x}}$.

b. Identify the $(n-1)\times(n-1)$ orthogonal group O_{n-1} with matrices in O_n whose first row and column are $\underline{e}_1$. Prove that each matrix M_n in O_n has a unique representation as $O_{\underline{x}} M_{n-1}$ with $\underline{x} \in S^{n-1}$ and $M_{n-1} \in O_{n-1}$.

c. Using the described coordinates, or otherwise, calculate the volume of O_n. It may be helpful to use that the volume on O_n is invariant under left and right translation, which was proved in the problems for Midterm 2. This reduces to the case that $\underline{x}$ is a neighborhood of $\underline{e}_1$.

Solution.

a. Given unit vector $\underline{x} \neq \underline{e}_1$, write $\underline{x} = x_1 \underline{e}_1 + \sqrt{1-x_1^2}\,\underline{\tilde{x}}$ where $\underline{\tilde{x}} = \frac{\tilde{x}}{\sqrt{1-x_1^2}}$. Let $\cos\theta = x_1$, $\sin\theta = \sqrt{1-x_1^2}$. Rotation by θ in the $\underline{e}_1$, $\underline{\tilde{x}}$ plane maps

$$\underline{e}_1 \mapsto x_1\underline{e}_1 + \sqrt{1-x_1^2}\,\underline{\tilde{x}},$$
$$\underline{\tilde{x}} \mapsto -\sqrt{1-x_1^2}\,\underline{e}_1 + x_1\underline{\tilde{x}}.$$

Projection in the orthogonal plane is given by $I - \underline{e}_1\underline{e}_1^t - \underline{\tilde{x}}\underline{\tilde{x}}^t$. Hence a general vector $\underline{v}$ is mapped to

$$\langle \underline{v}, \underline{e}_1\rangle \left(x_1\underline{e}_1 + \sqrt{1-x_1^2}\,\underline{\tilde{x}}\right) + \langle \underline{v}, \underline{\tilde{x}}\rangle \left(-\sqrt{1-x_1^2}\,\underline{e}_1 + x_1\underline{\tilde{x}}\right) + \\ + (I - \underline{e}_1\underline{e}_1^t - \underline{\tilde{x}}\underline{\tilde{x}}^t)\underline{v}.$$

Hence, letting $\underline{x}'$ be the last $n-1$ components of $\underline{x}$ as a vector in $\mathbb{R}^{n-1}$,

$$\begin{aligned} O_{\underline{x}} &= x_1\underline{e}_1\underline{e}_1^t + \sqrt{1-x_1^2}\,\underline{\tilde{x}}\underline{e}_1^t - \sqrt{1-x_1^2}\,\underline{e}_1\underline{\tilde{x}}^t + x_1\underline{\tilde{x}}\underline{\tilde{x}}^t + I - \underline{e}_1\underline{e}_1^t - \underline{\tilde{x}}\underline{\tilde{x}}^t \\ &= \begin{pmatrix} x_1 & -(\underline{x}')^t \\ \underline{x}' & I_{n-1} - \frac{\underline{x}'(\underline{x}')^t}{1+x_1} \end{pmatrix}. \end{aligned}$$

b. Let $\underline{x}$ be the first column of $M_n \in O_n$. Then $O_{\underline{x}}^{-1} M_n$ has first column and first row $\underline{e}_1$, hence is a matrix in O_{n-1} embedded in O_n as described. This proves that the map is onto. The uniqueness follows since $O_{\underline{x}} M_{n-1}$ has first column equal to $\underline{x}$.

c. Let $S(\underline{x}, \delta)$ be the set of matrices in O_n whose first column has distance from $\underline{x}$ at most δ. Then $O_{\underline{x}} \cdot S(\underline{e}_1, \delta) = S(\underline{x}, \delta)$. Since the volume form on O_n is invariant under left translation, $\operatorname{Vol} S(\underline{x}, \delta) = \operatorname{Vol} S(\underline{e}_1, \delta)$ for all $\underline{x} \in S^{n-1}$ and hence is equal to a constant c times the volume of $\{\underline{y} \in S^{n-1} : d(\underline{y}, \underline{e}_1) \leq \delta\}$. It follows that the volume of O_n is $c \operatorname{Vol}(S^{n-1}) = c\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$.

To calculate the constant c, let δ be coordinates on M_{n-1} in a neighborhood of the identity, identified with $\delta = 0$, and parametrize S^{n-1} near $\underline{e}_1$ by $\underline{x} = \begin{pmatrix} \sqrt{1-\epsilon_1^2 - \cdots - \epsilon_{n-1}^2} \\ \epsilon_1 \\ \vdots \\ \epsilon_{n-1} \end{pmatrix}$. Thus the coordinate chart is given by

$$M = \begin{pmatrix} \sqrt{1-\epsilon_1^2 - \cdots - \epsilon_{n-1}^2} & -\epsilon_1 \ \cdots \ -\epsilon_{n-1} \\ \begin{matrix} \epsilon_1 \\ \vdots \\ \epsilon_{n-1} \end{matrix} & I_{n-1} - \frac{\epsilon\epsilon^t}{1+\sqrt{1-\epsilon_1^2-\cdots-\epsilon_{n-1}^2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & M_{n-1}(\delta) \end{pmatrix}.$$

This coordinate chart can be translated arbitrarily on the right by a matrix from M_{n-1} without changing the first column, so that to calculate the volume, it suffices to determine the volume form when $\delta = 0$ and for ϵ in a neighborhood of the identity.

We have, at $\epsilon = \delta = 0$,

$$\frac{\partial M}{\partial \epsilon_i} = \begin{pmatrix} 0 & 0 & \cdots & -1 & \cdots & 0 \\ 0 & & & & & \\ \vdots & & & & & \\ 1 & & 0 & & 0 & \\ \vdots & & & & & \\ 0 & & & & & \end{pmatrix}, \qquad \frac{\partial M}{\partial \delta_i} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial M_{n-1}}{\partial \delta_i} \end{pmatrix}$$

in which the -1 occurs in the first row, $i+1$st slot, and the 1 occurs in the first column, $i+1$st row. Since

$$V(DM) = \left(\det DM^t DM\right)^{\frac{1}{2}}$$

and $DM^t DM$ has entries of type,

$$\sum_{i,j} \frac{\partial M_{i,j}}{\partial \epsilon_l} \frac{\partial M_{i,j}}{\partial \epsilon_m}, \; \sum_{i,j} \frac{\partial M_{i,j}}{\partial \delta_l} \frac{\partial M_{i,j}}{\partial \epsilon_m}, \; \sum_{i,j} \frac{\partial M_{i,j}}{\partial \delta_l} \frac{\partial M_{i,j}}{\partial \delta_m},$$

at $\epsilon = \delta = 0$ those cross terms with derivatives in δ and ϵ are 0, so that the matrix $DM^t DM$ has two blocks, corresponding to the ϵ and δ variables. The δ has determinant $V(DM_{n-1})^2$ on O_{n-1}, while the ϵ matrix is $2I_{n-1}$ and hence has determinant 2^{n-1}. Hence $V(DM) = 2^{\frac{n-1}{2}} V(DM_{n-1})$. Meanwhile, the chart

$$\underline{x} = \begin{pmatrix} \sqrt{1 - \epsilon_1^2 - \cdots - \epsilon_{n-1}^2} \\ \epsilon_1 \\ \vdots \\ \epsilon_{n-1} \end{pmatrix}$$

has

$$D\underline{x}(0) = \begin{pmatrix} 0 \\ I_{n-1} \end{pmatrix}$$

and hence has $V(D\underline{x}(0)) = 1$. Thus

$$V(DM(0,0)) = 2^{\frac{n-1}{2}} V(D\underline{x}(0)) V(DM_{n-1}(0)).$$

By right invariance of V under multiplication by elements of M_{n-1}, this is invariant under right multiplication by elements of M_{n-1}. By continuity of V, the constant is essentially unchanged in a neighborhood of $\underline{e}_1$ in S^{n-1}. It follows that

$$\mathrm{Vol}(O_n) = 2^{\frac{n-1}{2}} \mathrm{Vol}(S^{n-1}) \mathrm{Vol}(O_{n-1}).$$

Iterating this identity, together with the formula $\mathrm{Vol}(S^{n-1}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ obtains the formula

$$\mathrm{Vol}(O_n) = 2^{\frac{n(n-1)}{4}} \prod_{j=1}^{n} \frac{2\pi^{\frac{j}{2}}}{\Gamma\left(\frac{j}{2}\right)} = 2^{\frac{n(n+3)}{4}} \pi^{\frac{n(n+1)}{4}} \prod_{j=1}^{n} \frac{1}{\Gamma\left(\frac{j}{2}\right)}.$$

This formula was first obtained by Hurwitz.

Problem 8. Let $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be the unit circle. Let $\gamma, \delta : S^1 \to \mathbb{R}^3$ be non-intersecting smooth loops, so $\gamma(s) \neq \delta(t)$ for all s, t. Define $F_{\gamma,\delta}(s,t) = \gamma(s) - \delta(t)$. Let

$$\ell(\gamma, \delta) := \frac{1}{4\pi} \int_{S^1 \times S^1} F^*_{\gamma,\delta} \left(\frac{xdy \wedge dz + ydz \wedge dx + zdy \wedge dz}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right)$$

be the linking number. Prove

a. $\ell(\gamma, \delta)$ is unchanged if γ and δ are continously deformed without intersecting.
b. $\ell(\gamma, \delta) \in \mathbb{Z}$.
c. If $\gamma(s) = (\cos s, \sin s, 0)$ for $s \in [0, 2\pi]$ and $\delta(t) = (-1 + \frac{1}{2}\cos t, 0, \frac{1}{2}\sin t)$ for $t \in [0, 2\pi]$, then $\ell(\gamma, \delta) = 1$.

Solution.

a. Let

$$\theta_3 = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2+y^2+z^2)^{\frac{3}{2}}}.$$

It is straightforward to check that $d\theta_3 = 0$. The fact that $\ell(\gamma, \delta)$ is unchanged under homotopy follows from the Lemma from Lecture on the change of the integral of a closed form under homotopy.

b. Perform a homotopy which translates δ continuously to a large translate T after which δ and γ are separated by a hyperplane. In this case the curve $\gamma(t) - \delta(s)$ is separated from 0 by a hyperplane, so the resulting surface is contractible. Hence, after translation, the integral is 0. The integral varies continuously except at points where the paths intersect. At a point of intersection, deform both paths by a small semi-circular loop on either side of the intersection point. The difference between the paths prior to intersection and post intersection is thus homotopic to a curve as in part c, hence contributes an integer change.

c. The surface

$$\gamma(s) - \delta(t) = \left(1 + \cos s - \frac{1}{2}\cos t, \sin s, -\frac{1}{2}\sin t\right)$$

is a torus containing 0. Slice the torus with two planes to cut away the part containing 0. The remaining surface is contractible in $\mathbb{R}^3 \setminus \{0\}$. The cut part is homotopic to a sphere centered at 0. Integrating θ_3 over a sphere centered at 0 gives 1.