MATH 311, FALL 2020 PRACTICE MIDTERM 1 SOLUTIONS

SEPTEMBER 23

Each problem is worth 10 points.

Problem 1. Find all pairs of integers (x, y) such that $23x + 81y = 1$ and explain why your list is complete.

Solution 1. Since $81 = 3^4$ is co-prime to 23, the solution may be found by the Euclidean algorithm.

$$\begin{aligned} 81 &= 3 \cdot 23 + 12 \\ 23 &= 1 \cdot 12 + 11 \\ 12 &= 1 \cdot 11 + 1. \end{aligned}$$

Working backward,

$$\begin{aligned} 1 &= 12 - 11 \\ &= 2 \cdot 12 - 23 \\ &= 2 \cdot (81 - 3 \cdot 23) - 23 \\ &= 2 \cdot 81 - 7 \cdot 23. \end{aligned}$$

The general solution is then

$$(2 + 23t)81 - (7 + 81t)23.$$

Problem 2. Find a reduced quadratic form $ax^2+bxy+cy^2$, satisfying either $-|a|<b\leq|a|<|c|$ or $0\leq b\leq|a|=|c|$, which is equivalent to $2x^2+5xy+y^2$.

Solution 2. First make the change $x\mapsto y$, $y\mapsto -x$ to obtain $2x^2+5xy+y^2\sim x^2-5xy+2y^2$. Next substitute x with $x+3y$ to obtain

$$2x^2+5xy+y^2\sim x^2+xy-4y^2,$$

which is reduced.

Problem 3.

a. State Hensel's Lemma.
b. Find a solution to $x^2 \equiv 5 \bmod 19^2$.

Solution 3.

a. Hensel's lemma states that if $f(x) \equiv 0 \bmod p^j$ and $f'(x) \not\equiv 0 \bmod p$, then there is a unique $y \equiv x \bmod p^j$ such that $f(y) \equiv 0 \bmod p^{j+1}$.
b. Check by examination that $9, 10$ are solutions to $x^2 \equiv 5 \bmod 19$. For $f(x) = x^2 - 5$, $f'(x) = 2x \not\equiv 0$ for both solutions. Expand

$$(9 + 19t)^2 \equiv 5 + (4 \cdot 19) + 19 \cdot 18 \cdot t \bmod 19^2$$

and hence $t = 4$ obtains a solution $x = 85$. The other solution is -85.

Problem 4.

a. State the Pigeonhole Principle.
b. Prove that there are integers a, b with $|a|, |b| \leq 1000$, not both zero, such that $|a + b\sqrt{2}| < \frac{1}{200}$.

Solution 4.

a. If $f : S \to T$ is a map of finite sets, with $|S| > |T|$, then $|f^{-1}(t)| > 1$ for some $t \in T$.
b. Divide the interval $[0, 1000(1 + \sqrt{2})]$ into 10^6 equal size pieces, each of length $\frac{1+\sqrt{2}}{1000}$. Consider the points $\{a + b\sqrt{2} : 0 \leq a, b \leq 1000\}$. Since there are more than 10^6 such points, two occupy the same interval, and hence differ by at most $\frac{1+\sqrt{2}}{1000} < \frac{1}{200}$.