MATH 311, FALL 2024 MIDTERM 2

OCTOBER 30

Each problem is worth 10 points.

Problem 1. Use the secant method starting with the point $(1, 0)$ to find all of the rational points on the ellipse $x^2 + 5y^2 = 1$.

Solution. Let m be the slope of a line through $(1, 0)$ and a second rational point, so that the line has equation $y = m(x - 1)$. This line intersects the curve where $x^2 + 5m^2(x - 1)^2 - 1 = 0$. Factoring out the $x - 1$, $(x + 1 + 5m^2(x - 1))(x - 1) = 0$ which gives the new solution $(5m^2 + 1)x = (5m^2 - 1)$ or $x = \frac{5m^2-1}{5m^2+1}$. At the intersection, $y = m(x - 1) = \frac{-2m}{5m^2+1}$.

Problem 2.

a. Define the Farey fractions of level n.

b. Assuming standard facts about the Farey fractions, prove that for any real number x, there is a rational $\frac{a}{b}$ such that $0 < b \le n$ and
$$\left|x - \frac{a}{b}\right| \le \frac{1}{b(n+1)}.$$

Solution.

a. The Farey fractions of level n are those fractions $\frac{a}{b}$ in lowest terms, written in increasing order, with $0 \le \frac{a}{b} \le 1$ and with $b \le n$.

b. Let $\frac{a}{b} \le x < \frac{c}{d}$ be consecutive Farey fractions. The fraction $\frac{a+c}{b+d}$ is the next Farey fraction to be added between $\frac{a}{b}$ and $\frac{c}{d}$, and has $b+d \ge n+1$ since it does not appear at level n. Then either $x \le \frac{a+c}{b+d}$ or $x > \frac{a+c}{b+d}$. In the first case, $\left|x - \frac{a}{b}\right| \le \frac{a+c}{b+d} - \frac{a}{b} = \frac{1}{b(b+d)} \le \frac{1}{b(n+1)}$, while in the second case, $\left|x - \frac{c}{d}\right| \le \frac{c}{d} - \frac{a+c}{b+d} = \frac{1}{d(b+d)} \le \frac{1}{d(n+1)}$.

Problem 3.

a. Give a proof that any rational $\frac{a}{b}$ has a unique continued fraction expansion whose last term is not equal to 1.
b. Find the continued fraction expansion of $\sqrt{3}$.

Solution.

a. Starting from $q = \frac{c}{d}$, we choose $q_0 = q$, $a_0 = \lfloor q \rfloor$, and in general $a_i = \lfloor q_i \rfloor$ and, while $a_i \neq q_i$, $q_{i+1} = \frac{1}{q_i - a_i}$. This process takes the same steps as running the Euclidean algorithm starting with c and d and hence terminates in a finite number of steps, giving a finite continued fraction representation for q. If $q = \langle a_0, a_1, ..., a_n \rangle$ with $a_n = 1$ then $q = \langle a_0, ..., a_{n-1} + 1 \rangle$ so we may assume the last digit is not equal to 1. Given two representations $q = \langle a_0, ..., a_n \rangle = \langle b_0, ..., b_j \rangle$ with $a_n, b_j > 1$ we have $q_0 = q = a_0 + \frac{1}{q_1}$ with $q_1 > 1$ and hence $a_0 = \lfloor q_0 \rfloor = b_0$. Subtracting off the first digit and inverting, then repeating this process we conclude all the digits are the same.
b. We know from lecture $\lfloor \sqrt{3} \rfloor + \sqrt{3} = 1 + \sqrt{3}$ has purely periodic continued fraction representation. Given $\xi_0 = 1 + \sqrt{3}$, $a_0 = 2$ and $\xi_1 = \frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{2}$. Thus $a_1 = 1$ and $\xi_2 = \frac{2}{\sqrt{3}-1} = \sqrt{3} + 1$. Thus $\sqrt{3} = \langle 1, \overline{1, 2} \rangle$.

Problem 4.

a. State Minkowski's convex body theorem from the geometry of numbers.
b. Show that any lattice Λ of covolume 1 in the plane $\mathbb{R}^2$ contains a non-zero point (x, y) with $x^2 + y^2 \leq \frac{4}{\pi}$.

Solution.

a. Let $\mathcal{C} \subset \mathbb{R}^n$ be convex, centrally symmetric and have volume greater than 2^n. Then $\mathcal{C}$ contains a lattice points of $\mathbb{Z}^n$ other than 0.
b. The theorem in part a remains valid if the inequality greater than 2^n is replaced with $\geq 2^n$ if the set is closed and bounded and if the lattice points $\mathbb{Z}^n$ are replaced with any lattice of covolume 1. Changing between the general lattice and the lattice $\mathbb{Z}^n$ amounts to multiplication by a matrix of determinant 1, which leaves volumes unchanged and preserves the properties of being closed, convex, bounded and centrally symmetric. To check the claim about being closed and bounded, let $\mathcal{C}$ be closed, bounded, centrally symmetric and convex with volume 2^n. Then $\mathcal{C}_k = (1 + 1/k)\mathcal{C}$ has volume $> 2^n$, so contains a non-zero lattice points x_k. Since $\mathcal{C}$ is bounded, some x_k occurs infinitely often, and hence is in the intersection $\mathcal{C} = \bigcap \mathcal{C}_k$. The problem is now solved by noticing the disc $x^2 + y^2 \leq \frac{4}{\pi}$ has volume 4, is closed, bounded, convex and centrally symmetric.