MAT 548, Homework 7, due Friday, Oct 23

1. This question is about properties of the degree of a map. Note that we have first defined degree for smooth maps and proved that this is a smooth homotopy invariant. Then we used smooth approximation of continuous maps to drop the smoothness requirement: we extended the definition to continuous maps on smooth manifolds and showed that degree is a (continuous) homotopy invariant. In the oriented case, degree is integer-valued; without orientations, it is defined mod 2.

(a) Suppose that L, M, N are compact, connected oriented smooth manifolds of the same dimension (without boundary), $f : N \to M$ and $g : M \to L$ are continuous maps. Show that

$$\deg(g \circ f) = \deg g \cdot \deg f.$$

(b) Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere. Let $r : S^n \to S^n$ and $a : S^n \to S^n$ be a *reflection* and the *antipodal map*, defined by

$$r(x_1, x_2, \dots x_{n+1}) = (-x_1, x_2, \dots, x_{n+1}), \qquad a(x_1, x_2, \dots, x_{n+1}) = (-x_1, -x_2, \dots, -x_{n+1}).$$

Find $\deg r$ and $\deg a$. (Fix the orientation of the sphere as the boundary of the ball.)

(c) Show that if a continuous map $h : S^n \to S^n$ has no fixed points, then h is homotopic to the antipodal map.

(d) Prove the Hairy Ball Theorem (and more):
Show that S^n has a non-vanishing continuous tangent vector field if and only if n is odd.

Hint for (d): if $v(x)$ is a non-vanishing continuous vector field on S^n, normalize it so $|v(x)| = 1$. Construct an explicit continuous homotopy between the identity map $\mathrm{id} : S^n \to S^n$ and the antipodal map $a : S^n \to S^n$ by rotating each $x \in S^n$ to $-x$ in the plane spanned by x and $v(x)$.
For n odd, engineer a non-vanishing vector field directly: for $(x_1, \dots, x_{2k}) \in S^{2k-1} \subset \mathbb{R}^{2k}$, write an expression for a vector that is orthogonal to the radius vector and depends continuously on the coordinates.

Questions 2 and 3 prove some properties of *the winding number* and establish the Jordan–Brouwer separation theorem for smooth hypersurfaces in $\mathbb{R}^n$. These questions are adapted from Guillemin–Pollack Chapter 2 Section 5 (that whole section is a collection of exercises). We will work with the winding number mod 2; below, deg stands for the mod 2 degree of a map. (The integer-valued winding number can be similarly defined if one works with the integer-valued degree in the oriented case.)

Definition. Let X be a compact, connected smooth manifold (without boundary), $\dim X = n - 1$, and $F : X \to \mathbb{R}^n$ a smooth map. Fix $z \in \mathbb{R}^n$, $z \notin F(X)$, and consider the *direction map* $u : X \to S^{n-1}$ defined by

$$u(x) = \frac{F(x) - z}{|F(x) - z|}.$$

Define *the winding number of F around z* as $W(F, z) = \deg u \in \{0, 1\}$.
Intuitively, $W(F, z)$ counts how many times the vector from z to $F(x)$ points in the given direction. Draw some pictures for the case $F : S^1 \to \mathbb{R}^2$ to convince yourself that this describes "winding".
When $X \subset \mathbb{R}^n$ is a smoothly embedded compact connected hypersurface, $z \notin X$, we define $W(X, z) = W(i, z)$ by using the direction map for the embedding $i : X \to \mathbb{R}^n$, $i(x) = x$.

2. (a) Show that $\mathbb{R}^n - X$ has at most two connected components. Prove the following lemma first:

Lemma. Let $z \in \mathbb{R}^n - X$. Then for every $x \in X$ and a neighborhood $U \ni x$, there is a point $y \in U - X$ that can be connected to z by a path disjoint from X.

Hint: show that the set of points $x \in X$ for which the lemma is true is open, closed, and non-empty.

(b) Show that if z_0, z_1 are in the same connected component of $\mathbb{R}^n - X$, then $W(X, z_0) = W(X, z_1)$.

3. (a) Given a point $z \in \mathbb{R}^n - X$ and a direction vector $v \in S^{n-1}$, consider the ray r emanating from z in the direction of v, $r = \{z + tv, t \geq 0\}$.

Check that the ray r is transverse to X if and only if v is a regular value of the direction map $u : X \to S^{n-1}$ as above,

$$u(x) = \frac{x - z}{|x - z|}.$$

In particular, almost every ray from z intersects X transversally (and the intersection is a finite set).

(b) Suppose that r is a ray emanating from z_0 that intersects X transversally in a nonempty set. Suppose that $z_1 \notin X$ is any other point on r, and let l be the number of times r intersects X between z_0 and z_1. Check that $W(X, z_0) = W(X, z_1) + l \mod 2$.

(c) Show that if $|z|$ is very large, then $W(X, z) = 0$.

(d) Prove the following corollary:

Jordan–Brouwer Separation Theorem. The complement of a compact, connected smooth hypersurface X in $\mathbb{R}^n$ has two connected components, the "outside" D_0 and the "inside" D_1. Moreover, the closure $\bar{D}_1$ is a compact manifold with boundary, and $\partial \bar{D}_1 = X$.

4. Prove that $\mathbb{R}\mathrm{P}^2$ does not admit a smooth embedding into $\mathbb{R}^3$.

Hint: use the Separation theorem and Question 5b from Homework 5.

5. (a) Let $F : \mathbb{R}^m \to \mathbb{R}^n$ be a smooth map such that $F(0) = 0$. Define

$$H(x, t) = \begin{cases} \frac{1}{t} F(tx), & 0 < t \leq 1 \\ D_0 F(x), & t = 0. \end{cases}$$

Show that H is a homotopy between F and the linear map $D_0 F$. (Prove continuity only; H is smooth but you are not required to check this.)

(b) Let $B_r(0) \subset B_R(0)$ be the balls of radius r resp. R centered at 0, $R > r > 0$. Using bump functions or partition of unity, construct a homotopy $H(x, t)$ giving the map $F(x)$ at time $t = 1$ and such that:

(i) $H(x, t) = F(x)$ for all $0 \leq t \leq 1$, $x \in \mathbb{R}^m \setminus B_R(0)$;

(ii) $H(x, 0) = D_0 F(x)$, $x \in B_r(0)$.

This homotopy makes the given map linear in a small neighborhood of 0, while keeping F fixed outside of a larger given neighborhood.

(c) Suppose that $m = n$ and $D_0 F$ is non-singular. Suppose that $\det D_0 F > 0$. Using (a) and (b), construct a homotopy $H(x, t)$ *that uses a path in* $GL(n, \mathbb{R})$ *to connect* $D_0 F$ *to the identity matrix* I, so that

(i) $H(x, t) = F(x)$ for all $0 \leq t \leq 1$, for all x outside of a given neighborhood of 0;

(ii) $H(x, 0) = x$ for x in a small neighborhood of 0.

State and prove a similar result for the case $\det D_0 F < 0$, *connecting* $D_0 F$ *to a "standard" degree* -1 *linear map through invertible matrices.*

Note: working in a chart, you can use the result of this exercise to linearize a smooth map between manifolds in a neighborhood of a given point by a homotopy. This is a very useful technical trick (especially when $D_0 F$ is non-singular).

We used it in class to show that an arbitrary map from (S^n, s_0) *to itself is homotopic to a multiple of the identity map. By smooth approximation we can assume that the given map is smooth. Letting* $\{x_1, \ldots, x_k\}$ *be the preimage of a regular value* $y \neq 0$*, we can use the above construction to homotop the map in a neighborhood* U_i *of each* x_i *to the standard orientation-preserving or orientation-reversing map, so that after the appropriate identifications each* U_i *is sent to a neighborhood* V *of* y *by a map modeled on a standard degree 1 or -1 linear map (the identity or the reflection). The key point here is that we can also ensure, by construction, that the complement of* $\cup_i U_i$ *is mapped outside of* y *by the resulting map. Finally, we can apply another homotopy to send the complement of* $\cup_i U_i$ *to the basepoint in the target sphere and stretch the image of each* U_i *over the target sphere to represent our map as a sum of the standard degree* 1 *and degree* -1 *maps.*