MAT 548, Homework 3, due Friday, Sept 18

From Lee's textbook: please do **questions 4-2, 4-5** at the end of the Chapter 4. More questions:

1. Define a map $F : S^2 \to \mathbb{R}^4$ by

$$F(x, y, z) = (x^2 - y^2, xy, xz, yz).$$

Here, $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. Show that F descends to a smooth embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$.

Note: $\mathbb{RP}^2$ can be smoothly immersed into $\mathbb{R}^3$ but cannot be embedded.

2. In this question, $\mathbb{R}^2$ has its standard smooth structure. Let Γ be the graph of the function $g(t) = t^{1/3}$, so that

$$\Gamma = \{(t, t^{1/3}) : t \in \mathbb{R}\} \subset \mathbb{R}^2.$$

(a) Explain why Γ is a topological submanifold of $\mathbb{R}^2$.

(b) Consider a single-chart atlas on Γ given by the homeomorphism $\pi : \Gamma \to \mathbb{R}$, $\pi(t, t^{1/3}) = t$. With this smooth structure, is Γ a smooth submanifold of $\mathbb{R}^2$?

(c) Consider Γ as above, this time with a different single-chart atlas given by the chart $\phi : \Gamma \to \mathbb{R}$, $\phi(t, t^{1/3}) = t^{1/9}$. With this smooth structure, is Γ a smooth submanifold of $\mathbb{R}^2$?

(d) Give a smooth structure on Γ so that Γ is a smooth submanifold in $\mathbb{R}^2$.

Questions on vector bundles We stated the definitions and first examples of vector bundles in class. Please do the following exercises to get some experience with these notions. You can look up the definitions on the first few pages of Chapter 10.

A vector bundle $\pi : E \to M$ is called trivial if it is isomorphic to the product bundle, i.e. there is a *global trivialization*, that is, a homeomorphism $f : E \to M \times \mathbb{R}^n$ such that the diagram below commutes, and f is a linear isomorphism on every fiber:

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi"'] & & M \times \mathbb{R}^n \arrow[dl, "\pi_M"] \\ & M & \end{tikzcd}$$

For "smoothly trivial", the global trivialization f as above is required to be a diffeomorphism.

3. (a) Prove that the tangent bundle TS^1 to the circle is trivial.

(b) Prove that the tangent bundle TS^3 is trivial. It is helpful to think of the standard embedding $S^3 \subset \mathbb{R}^4$ to identify the tangent vectors.

Note: If $\pi : E \to M$ is smoothly trivial, then the total space E is diffeomorphic to $M \times \mathbb{R}^n$. But the converse is not true: is can happen that E is diffeomorphic to $M \times \mathbb{R}^n$, but the bundle is non-trivial. (Although it should be intuitively clear that triviality of the bundle must be a stronger condition, counterexamples require more knowledge.)

Note: the tangent bundle to S^2 is non-trivial! We don't have tools to prove it yet, but this is related to "the hairy ball theorem" that you may have heard about.

Note: for every compact oriented 3-manifold Y without boundary, the tangent bundle TY is trivial. (This is a non-trivial theorem.)

4. In class, we defined the *tautological* vector bundle γ_n over $\mathbb{RP}^n$, as follows. The base of the bundle is the projective space $\mathbb{RP}^n$, the space of lines through 0 in $\mathbb{R}^{n+1}$. The total space is

$$(E(\gamma_n) = \{([x], v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : v \in [x]\}.$$

The projection map $\gamma_n : E(\gamma_n) \to \mathbb{RP}^n$ is given by $\gamma_n([x], v) = [x]$; the fiber over a point $[x] \in \mathbb{RP}^n$ is the line in $\mathbb{R}^{n+1}$ representing that point.

(a) Construct local trivializations and show that γ_n is a smooth vector bundle.

(b) Show that γ_1 is non-trivial.

Recall that a vector bundle $\pi : E \to B$ is *oriented* if each fiber E_x is given an orientation, and there exists a family of local trivializations $\pi^{-1}(U) \to U \times \mathbb{R}^n$ covering B that restrict to orientation-preserving linear isomorphisms on each fiber. (A standard orientation is fixed on $\mathbb{R}^n$.) A vector bundle is orientable if it can be oriented.

5. Let X be a smooth manifold, $\pi : E \to X$ a smooth rank 1 vector bundle. (That is, each fiber $E_p = \pi^{-1}(p)$ is a 1-dimensional vector space over $\mathbb{R}$.) Show that $E \to X$ is orientable if and only if it is trivial.

6. (a) Show that the Möbius band M is a nonorientable manifold. (You can work with the open Möbius band if you prefer a manifold without boundary.)

(b) Suppose S is a smooth surface (that is, a 2-dimensional manifold), and S contains the open Möbius band M as an embedded submanifold. Show that S is nonorientable. Conclude that the projective plane $\mathbb{R}\mathrm{P}^2$ is nonorientable.

Reading assignment: (1) Please extract the proof of the Rank Theorem for submersions from the general proof of Theorem 4.12 in Lee's book. (We did immersions in class.) These special cases are easier and a bit more illumnating than the general case.

Please read Theorem 5.27 (easy!), Theorem 5.29 & Corollary 5.30 in Lee (focus on the embedded case only), and Theorem 5.31. The latter explains why there is only one smooth structure that can make $S \subset M$ a submanifold.