

Degree of a map

$$\dim X = \dim Y$$

$$f: X \rightarrow Y \quad X, Y \text{ smooth mfd's,}$$

$$\partial X = \partial Y = \emptyset, \quad X \text{ compact,}$$

Mod 2 case

Y connected

We first define degree for smooth maps, then extend to continuous via smooth approx.

$$f: X \rightarrow Y \text{ smooth}$$

$$\text{Fix } y \in Y$$

$$\text{Define } \underline{\deg(f; y)}:$$

Use homotopy transversality

to find $g: X \rightarrow Y$ smooth,

s.t. y is a regular value of g

$g^{-1}(y)$. 0-dim submfd, cpct
 $\Rightarrow$ finitely many pts.

$$\text{Set } \deg(f; y) = \# \{ g^{-1}(y) \} \pmod{2}$$

$= 0 \text{ or } 1$

Claim 1 $\deg(f; y)$ is indep. of the choice of $g \sim f$ (y is fixed)

smooth homotopy

Suppose y is a reg. value for g_0, g_1 ,
 $f \sim g_0 \sim g_1$.

Consider $H: X \times I \rightarrow Y$ - homotopy between g_0 and g_1

By homotopy transversality applied to H , we can assume $H \pitchfork \{y\}$:

NOTE that for $g_0 = H|_{X \times \{0\}}$ and

$g_1 = H|_{X \times \{1\}}$, we already

have $g_0 \pitchfork \{y\}$ and $g_1 \pitchfork \{y\}$,

so we can achieve transversality
for H without changing

$$H|_{X \times \{0\}} \quad \text{and} \quad H|_{X \times \{1\}}.$$

[see G-P extension thm on p. 72,
also HW 6 Q4]

We will now apply

The Regular Preimage Thm for
manifolds w/ bdr:

$$\partial(X \times I) = X \times \{0\} \cup X \times \{1\},$$

and y is a regular value

both for H and $H|_{\partial(X \times I)}$.

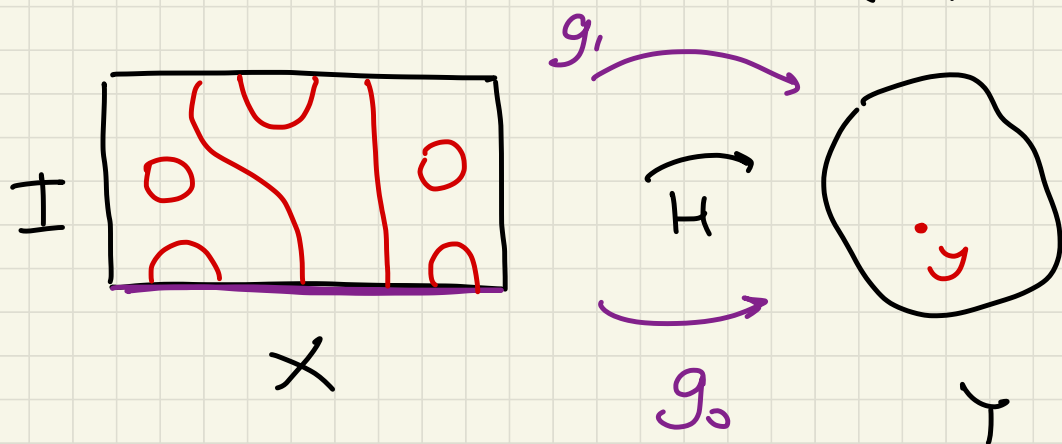
$$\text{Then } Z = H^{-1}(y) \subset X \times I$$

is a compact 1-dim submfld

$$\text{and } \partial Z = Z \cap \partial(X \times I).$$

Then Z = finite union of intervals & circles,

$$Z \cap (X \times \{0\} \cup X \times \{1\}) = \text{endpoints of intervals}$$



$$\text{Then } \# g_0^{-1}(y) \bmod 2$$

$$= \# \{ \text{points in } Z \cap X \times \{0\} \} \bmod 2$$

$$= \# \{ \text{endpoints of intervals on } X \times \{0\} \} \bmod 2$$

$$\# g_1^{-1}(y) \bmod 2 = \# \{ \text{endpoints on } X \times \{1\} \} \bmod 2$$

Now look at endpoints:

- intervals crossing from $X \times \{0\}$ to $X \times \{1\}$ contribute 1 to both $\deg g_0$ and $\deg g_1$
- circles don't have endpoints
- intervals from each side to itself ($X \times \{0\}$ to $X \times \{0\}$) or $X \times \{1\}$ to $X \times \{1\}$) contribute 2 to degree of one map but not the other, but $2 \equiv 0 \pmod{2}$!

Claim 2 $\deg(f; y)$ is independent of y .

Pf: we show that $\deg(f; y)$

is locally constant on Y ,
 then $\deg(f; y)$ must be
 the same for all y since Y
 is connected.

Suppose $f \sim g$, g smooth,
 y a regular value of g .

Then $g^{-1}(y) = \{x_1, \dots, x_k\} \subset X$,

and $\exists V \ni y$, $U_i \ni x_i$ - open
 nbhds
 $Y \cap X$
 $U_i \cap U_j = \emptyset$

 $g|_{U_i} : U_i \rightarrow V$ is a
 diffeo.

[See Lee 4.46 or HW4 Q1]

This means $g^{-1}(V) = \bigsqcup U_i$,

and every point has the same #
 preimages.

Claim 1 + Claim 2 show
that degree of a smooth map
is well-defined, and

Claim 1 also implies that
 $\deg f$ is a homotopy invariant
(under smooth homotopy).

Extension to continuous maps

First we show that

if g_0, g_1 are smooth,
y regular value for both,

and H is a continuous
homotopy between g_0, g_1 ,

Then $\deg g_0 = \deg g_1$.

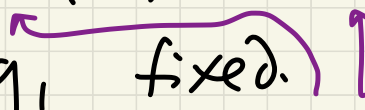
To see this, use smooth approximation for H :

since g_0, g_1 are already smooth, we can change H into a smooth function without changing g_0, g_1 , i.e. we get a smooth homotopy

G from g_0 to g_1 , and apply Claim 1. (see 6.26)

* Caution: Smooth approximation

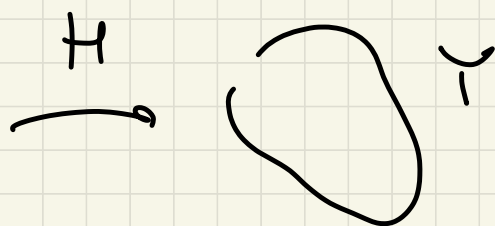
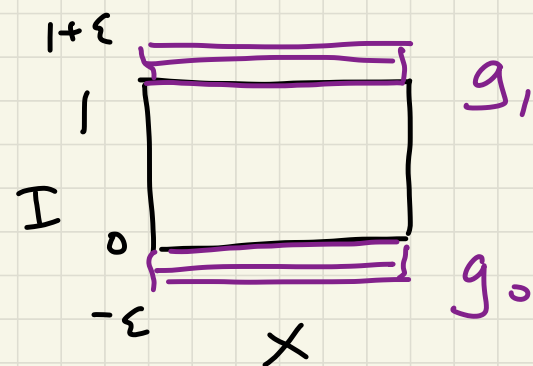
then needs the hypothesis that H is smooth on the closed set $X \times \{0\} \cup X \times \{1\}$

to keep g_0 & g_1 fixed. 

(i.e. smooth on open nbhd at each point)

This is stronger than
 just having $g_0 = H|_{x=0}$
 and $g_1 = H|_{x=1}$ smooth!

But we can extend H to
 $x \in [-\varepsilon, 1+\varepsilon]$ by g_-, g_+



and then
 smooth approx
 applies!

Set

$$H(x, t) \equiv g_0$$

$$-\varepsilon \leq t \leq 0$$

$$H(x, t) \equiv g_1$$

$$1 \leq t \leq 1+\varepsilon$$

Now we can define

$\deg(f; y)$ for continuous maps:

Fix y , find $g \sim f$ s.t. ↙ continuous homotopy

g is smooth & y -reg value for g .

Any two such g_0, g_1 are related by continuous homotopy,

then we can argue as above to show that

$$\deg(f; y) = \# g_0^{-1}(y) = \# g_1^{-1}(y)$$

is indep. of g & of y .

Oriented Case

$$f: X \rightarrow Y$$

When X and Y are oriented,
we can define $\deg f \in \mathbb{Z}$

by signed count of preimages
of a regular value.

As before: to define $\deg(f; y)$
find $g \sim f$, y reg value of f .
_{smooth}

$$g^{-1}(y) = \{x_1, \dots, x_n\} \subset X$$

g is a local diffeo on a nbhd
of each x_i :

$$\text{circle} \cdot x_i$$

$$\downarrow g$$

$$\text{circle} \cdot y$$

$$D_{x_i} g: T_{x_i} X \rightarrow T_y Y$$

invertible

$$T_{x_i} X, T_y Y$$

come with orientation:

$D_{x_i} g$ (basis in $T_{x_i} X$) is
a basis in $T_y Y$.

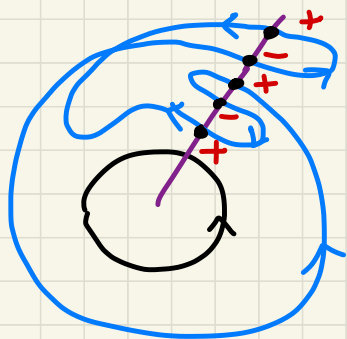
$D_{x_i} g$ is orientation-preserving
or orientation-reversing

depending on whether a basis
in the preferred orientation
class on $T_{x_i} X$ is sent to
a basis in the preferred class
in $T_y Y$ or not.

[Accordingly, we also say
" g is orientation-preserving"
or " g is orientation-reversing" at x_i]

Now count x_i 's in $g^{-1}(y)$

w/ signs: $+1$ if $D_{x_i} g$
is or-preserving, -1 if or-revers.



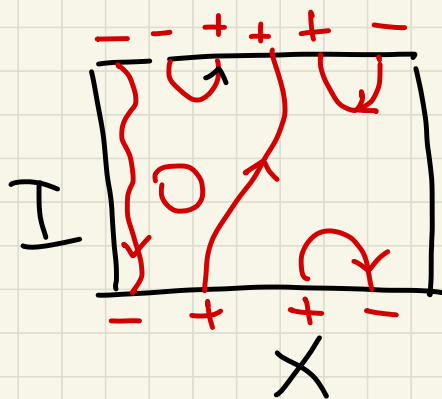
This gives
 $\deg(f; y) \in \mathbb{Z}$.

Proofs of Claim 1 & Claim 2
 in the oriented case:

Claim 1: all data comes

w/ orientations:

$X \times I$ $\supset \mathbb{Z}$
 oriented oriented



• endpoints of
 intervals from
 $X \times \{0\}$ to itself
 come in
canceled pairs
 w/ opposite signs!

- intervals crossing from $x = \{0\}$ to $x = \{1\}$ contribute the same sign $+1$ or -1 to both ends.

See pic.

Details of the proof (omitted) ^{here} consist of carefully working out all related orientations.

See G-P Chapter 3.

Claim 2 works w/ orientations:

if each U_i is oriented,
then gl_{U_i} is or-preserving
or or-reversing for all
 $x \in U_i$: that is,

each U_i in the "stack
of pancakes" picture can be
assigned a $+/-$ sign.