MAT 540, Homework 5, due Wednesday, Oct 2

From Lee's textbook: questions **6-1, 6-2.** In 6-2, feel free to assume that M is compact.
Prove Lemma 5.34a; read the discussion in the book preceding the lemma. Optional: prove Lemma 5.34b as well. [This is problem **5-17** in the book.]
Please also do question **5-23**. This will require some work, and you will need to read the material on submanifolds with boundary at the end of Lee Chapter 5.

Questions 1 and 2e below will require Theorem 6.26 (to be discussed in class on Monday).

1. Let $f : S^m \to S^n$ be a continuous map between spheres, $m < n$. Prove that f is homotopic to a constant map.

2. Let X, Y be topological spaces. The *compact-open topology* on the space of continuous functions $C(X, Y)$ is generated by the pre-basis given by sets

$$\mathcal{O}(K, U) = \{f : X \to Y | f(K) \subset U\},$$

where $K \subset X$ is compact, $U \subset Y$ is open. The basis of this topology is given by the collection $\mathcal{B}$ of all finite intersections of such sets.

(a) Check that $\mathcal{B}$ satisfies the conditions that ensure that $\mathcal{B}$ is a basis for some topology on the space.

(b) Suppose that X is compact, $Y = \mathbb{R}$. Show that in this case, the compact-open topology coincides with the *topology of uniform convergence.* The latter is the topology on $C(X, Y)$ induced by the metric

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

(It's obvious that the formula above gives a metric; no need to write up checking the metric space axioms.)

Your proof should work for any metric space Y, with only notational changes.

(c) Let $X = \mathbb{R}$, $Y = [-1, 1]$. Show that on the space $C(X, Y) = \{f : \mathbb{R} \to [-1, 1]\}$, the metric topology as in (b) is *different* from the compact-open topology.

In general, when Y is a metric space and X may be non-compact, the compact-open topology is the *topology of uniform convergence on compact sets*, which is typically different from uniform convergence on the entire X for bounded functions. (Think about this, but you don't need to submit any explanations for this more general case.)

(d) Let M be a smooth manifold. With the above topology, show that the set of smooth maps $C^\infty(M, R^n)$ is dense in the space of continuous maps $C^0(M, \mathbb{R}^n)$.

(e) Let M, N be smooth manifolds. Show that the set of smooth maps $C^\infty(M, N)$ is dense in the space of continuous maps $C^0(M, N)$ with the compact-open topology.