MAT 540, Homework 4, due Wednesday, Sept 25

From Lee's textbook: questions **5.2, 5.6.** In 5.6, show also that UM is a smooth fiber bundle.

1. (a) Check that the quotient map $q : S^n \to \mathbb{R}P^n$ that identifies the antipodal points of the sphere is a local diffeomorphism. (See Example 2.13f in the textbook).

(b) Define a map $F : S^2 \to \mathbb{R}^4$ by

$$F(x, y, z) = (x^2 - y^2, xy, xz, yz).$$

Here, $S^2 = \{x^2 + y^2 + z^2\} = 1 \subset \mathbb{R}^3$. Using part (a), show that the map F descends to a smooth embedding of $\mathbb{R}P^2$ into $\mathbb{R}^4$.

Note: $\mathbb{R}P^2$ cannot be embedded into $\mathbb{R}^3$.

2. (a) Let Γ be the graph of the function $g(t) = t^{1/3}$, so that

$$\Gamma = \{(t, t^{1/3}), t \in \mathbb{R}\} \subset \mathbb{R}^2.$$

Here, $\mathbb{R}^2$ has its standard smooth structure from the coordinates.

Consider a single-chart atlas on Γ given by the homeomorphism $\pi : \Gamma \to \mathbb{R}$, $\pi(t, t^{1/3}) = t$. With this smooth structure, is Γ a smooth submanifold of $\mathbb{R}^2$?

(b) Consider Γ as above, this time with a different single-chart atlas given by the chart $\phi(\Gamma) \to \mathbb{R}$, $\phi(t, t^{1/3}) = t^{1/9}$. With this smooth structure, is Γ a smooth submanifold of $\mathbb{R}^2$?

(c) Give a smooth structure on Γ so that Γ is a smooth submanifold in $\mathbb{R}^2$.

(d) Let

$$L = \{0 \leq x < 1, y = 0\} \cup \{x = 0, 0 \leq y < 1\} \subset \mathbb{R}^2.$$

Prove that L cannot be a smooth submanifold of the standard $\mathbb{R}^2$, with any smooth structure on L. (Hint: use the strategy described in Lee question 3-5.)

3. (a) Let X be a topological space, $\pi : E \to X$ a rank 1 vector bundle. (That is, each fiber $E_p = \pi^{-1}(p)$ is a 1-dimensional vector space over $\mathbb{R}$.) Show that $E \to X$ is orientable if and only if it is trivial. Conclude that the Möbius bundle (Lee Example 10.3) is not orientable.

(b) Show that the Möbius band M is a nonorientable manifold. (Caution: this is not the same as asking about orientability of the Möbius vector bundle!)

(c) Suppose S is a smooth surface (that is, a 2-dimensional manifold), and S contains the Möbius band M as an embedded submanifold. Show that S is nonorientable. Conclude that the projective plane $\mathbb{R}P^2$ is nonorientable.

If you are worried about the Möbius band being a manifold with boundary, feel free to work with the "open" Möbius band (without boundary), although the boundary doesn't change anything in this question.

4. Let M be a smooth n-manifold. Show that the smooth $2n$-manifold TM is always orientable. (Caution: this is not the same as asking about orientability of the vector bundle $TM \to M$.)