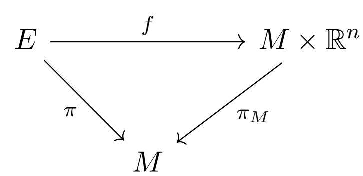
MAT 540, Homework 3, due Wednesday, Sept 18

From Lee's textbook: please do **questions 3-1, 3-2, 3-3, 4-1, 4-2, 4-5** at the end of the chapters. The purpose of 4.1, 4.2 is to be careful with manifolds with boundary in 4.1 and 4.2; please read the revelant material in the textbook as we discussed the boundary case only briefly.

If you haven't seen the proof of the inverse function theorem, please read it in Appendix C.

Questions on vector bundles We stated the definitions and first examples of vector bundles and fiber bundles in class. Please do the following exercises to get some experience with these notions. You can look up the definitions on the first few pages of Chapter 10.

A vector bundle $E \to^{\pi} M$ is called trivial if it is isomorphic to the product bundle, i.e. there is a *global trivialization*, that is, a homeomorphism $f : E \to M \times \mathbb{R}^n$ such that the diagram below commutes, and f is a linear isomorphism on every fiber:



For "smoothly trivial", the global trivialization f as above is required to be a diffeomorphism.

1. (a) Prove that the tangent bundle TS^1 to the circle is trivial.

(b) Prove that the tangent bundle TS^3 is trivial.

In (b), it is helpful to think of the standard embedding $S^3 \subset \mathbb{R}^4$ to identify the tangent vectors. We will discuss tangent spaces to submanifolds in class on Monday, but you can read this material in Lee Chapter 5.

Note: the tangent bundle to S^2 is non-trivial! We don't have tools to prove it yet, but this is related to "the hairy ball theorem" that you may have heard about.

Note: for every compact oriented 3-manifold Y without boundary, the tangent bundle TY^3 is trivial. (This is a non-trivial theorem.)

2. (a) Let $G(k, n)$ be the Grassmannian of the k-dimensional subspaces of $\mathbb{R}^n$ (see Homework 1 and Chapter 1 in Lee). Let T be the subset of $G(k, n) \times \mathbb{R}^n$ defined by

$$T = \{(S, v) \in G(k, n) \times \mathbb{R}^n : v \in S\}.$$

Explain why T is a (smooth) vector bundle over $G(k, n)$. It is called the *tautological vector bundle* because the fiber over the subspace $S \in G(k, n)$ is the vector space S itself.

(b) Show that the tautological vector bundle over $G(1, 2)$ is (smoothly) isomorphic to the Möbius bundle of Example 10.3 in Lee.

Checking smoothness shouldn't give you much trouble, but you can omit it if you don't feel motivated.