MAT 540, Homework 1, due Wednesday, Sept 4

From Lee's textbook: please do **questions 1-7, 1-9, and 2-6** at the end of the chapters (read example 1.33 about real projective spaces).

Please also do **Exercise 2.7** on p.35 in the middle of the text (not question 2-7 at the end of the chapter!), proving Proposition 2.5, Proposition 2.6, and Corollary 2.8 (streamline your argument and write short self-contained proofs). Consider manifolds without boundary only.

More questions:

1. Let k be an odd positive integer. Consider the smooth atlas on $\mathbb{R}$ consisting of a single chart $(\mathbb{R}, \phi)$, $\phi(t) = t^{1/k}$.

(a) Show that this chart is not compatible with the chart $(\mathbb{R}, \mathrm{id})$ that defines the standard smooth structure on $\mathbb{R}$, and therefore the given smooth structure is *not the same* as the standard smooth structure on $\mathbb{R}$.

(b) Show, however, that the resulting smooth manifold is *diffeomorphic* to $\mathbb{R}$ with its standard smooth structure.

This tells you that you have to be careful with your equivalence relations: "diffeomorphic" is a different notion from "the same" smooth structure. Usually, one wants to study smooth manifolds up to diffeomorphism.

Note: exotic (i.e. non-diffeomorphic to the standard) smooth structures exist on $\mathbb{R}^n$ only for $n = 4$. Existence for $n = 4$ is a famous result of Donaldson.

2. Check that

$$SL(\mathbb{R}, n) = \{A \text{ is an } n \times n \text{ real matrix }, \det A = 1\}$$

is a smooth manifold. Indeed, for the determinant function

$$\det : M(n, \mathbb{R}) \to \mathbb{R}$$

on the Euclidean space $M(n, \mathbb{R})$ of all $n \times n$ real matrices, $SL(\mathbb{R}, n) = \det^{-1}(1)$ is a level set. It remains to check that this level set is *non-critical*, namely $D(det)(A)$ is non-zero whenever $\det A = 1$. For this, first show

$$\frac{d}{dt} \det(I + tX) = \operatorname{tr} X,$$

by expanding $\det(I + tX)$ in powers of t. [This is a very important relation between determinant and trace.] Then, compute $D(\det)(A)$ by multiplying by A^{-1} to reduce to the case $A = I$, and explain the conclusion.

3. The Grassmannian $G(k, n)$ is defined as the space of all k-dimensional linear subspaces of $\mathbb{R}^n$. Real projective spaces are a special case of Grassmannians ($k = 1$). Explain briefly what the natural topology on $G(k, n)$ is (you don't have to check any axioms if you describe it directly; a convenient description involves quotient topology).

Grassmannians are smooth manifolds, which can be seen as follows. The smooth charts consist of subspaces that project isomorphically onto a given k-dimensional subspace of $\mathbb{R}^n$ (coordinate maps can be given via identification with certain sets of matrices).

Work out the details of the definition of the smooth charts, and show that the transition maps are smooth. (You don't have to check other properties.)

This fact is explained in many standard textbooks, including Lee. If using literature, please state so (with a reference), but use your own notation and make a self-contained proof without copying from a book. For concreteness, you can work with $G(2, 4)$ if that's easier for notation and calculations.

Reading assignment: in Milnor's Differential Topology textbook, read Appendix on Classifying one-dimensional manifolds.