
Global complex obstructions to a real Morse modification (Klein's enigma)

Oleg Viro
a joint work with Slava Kharlamov

July 5, 2017

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(German: "sind entwicklungsfähig nicht" - is not viable)

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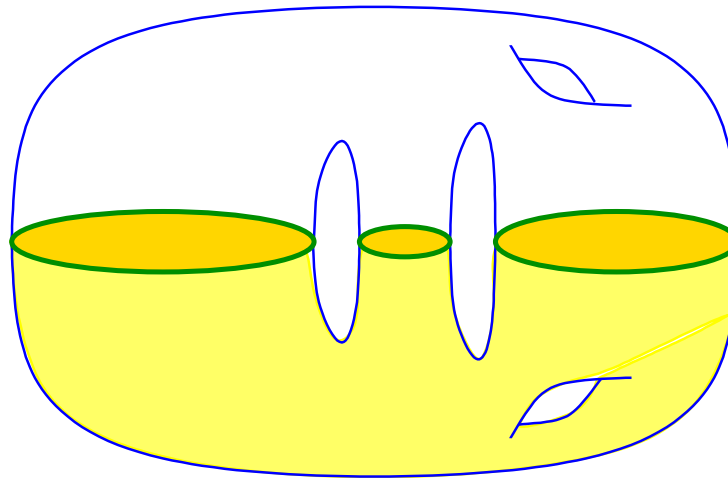
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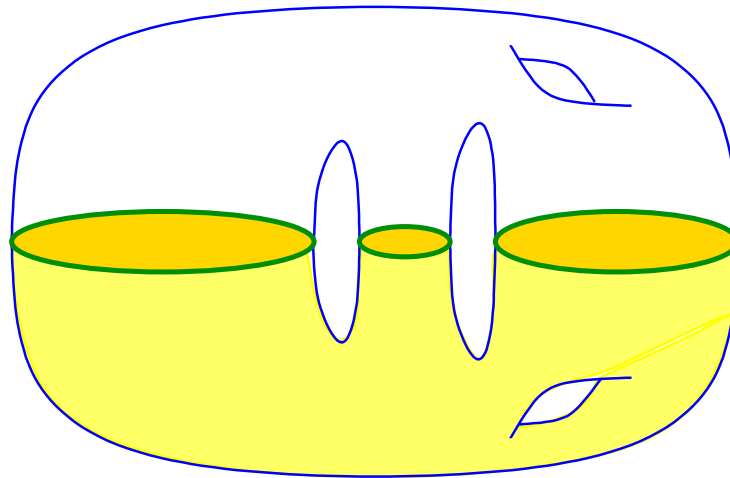
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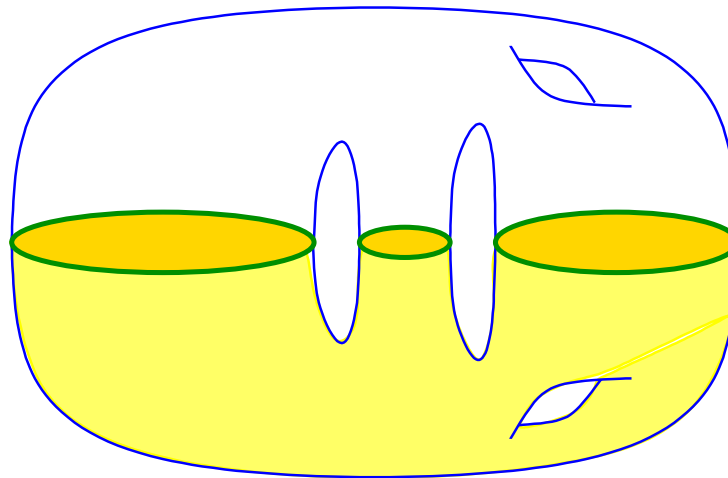
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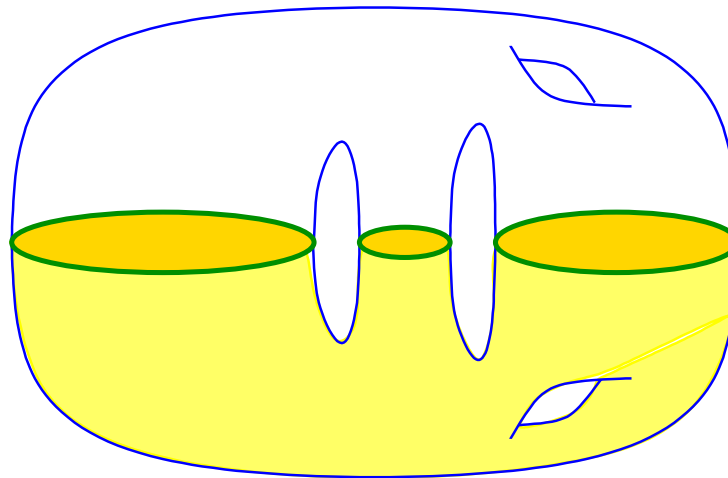
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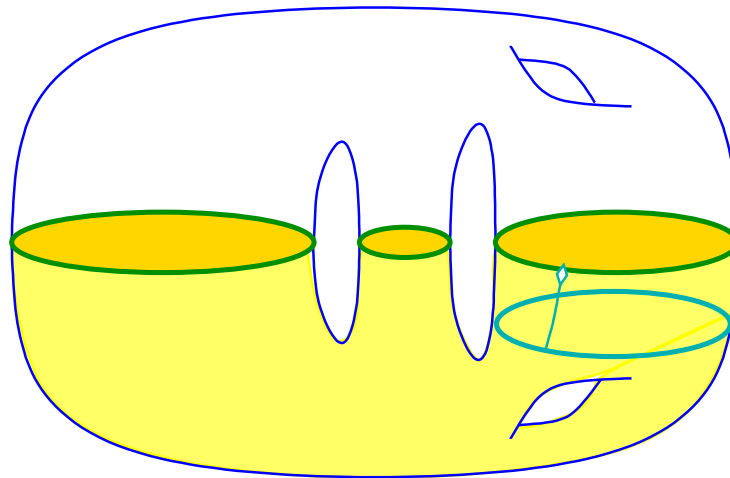
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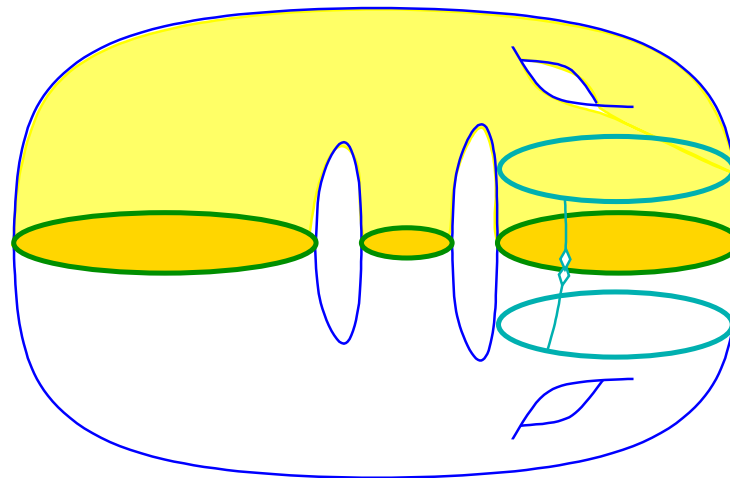
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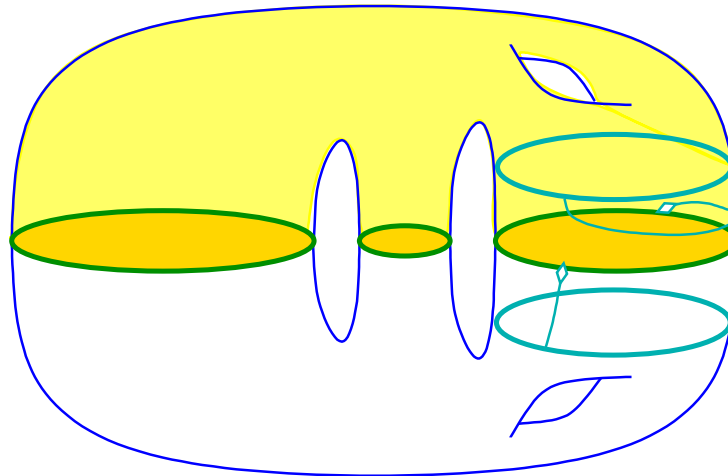
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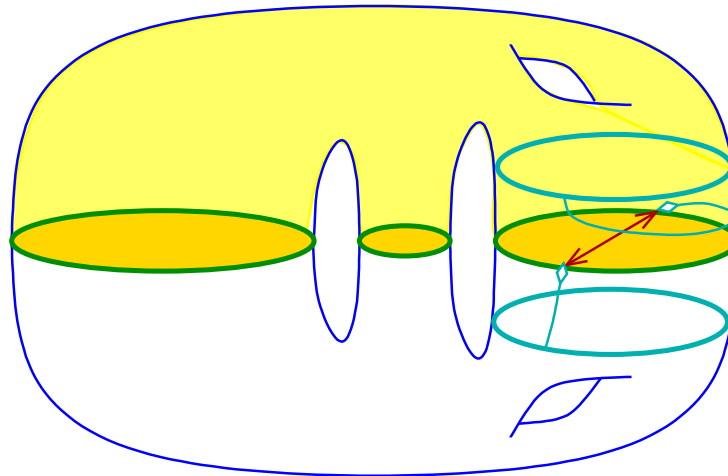
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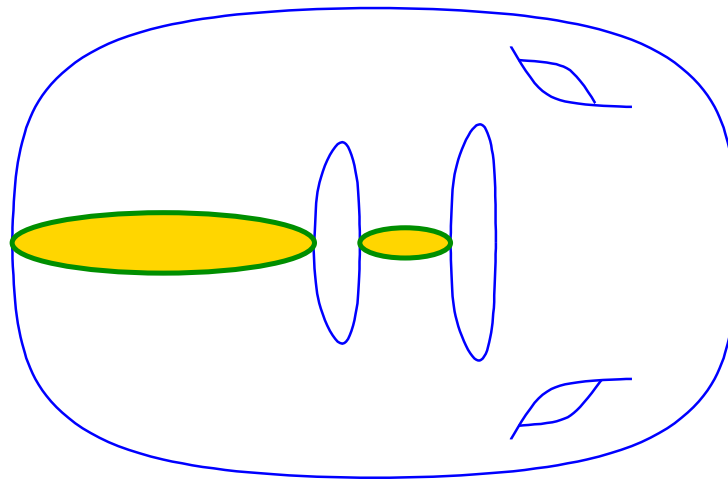
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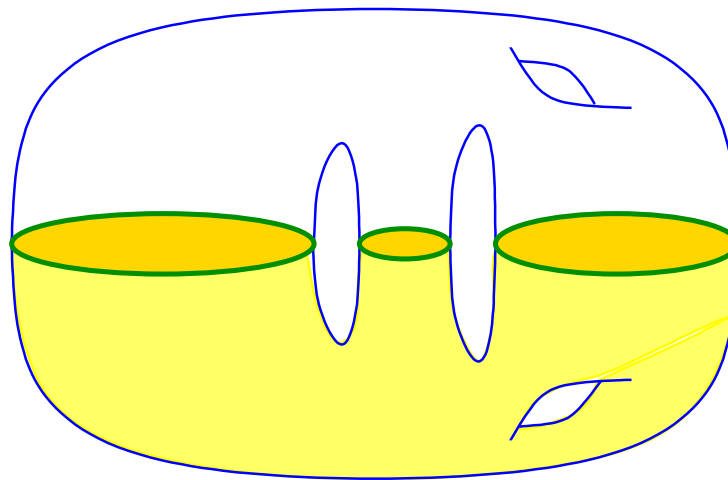
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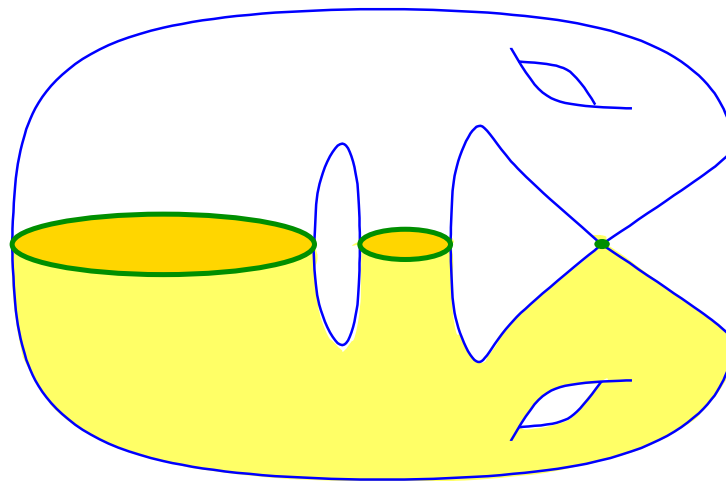
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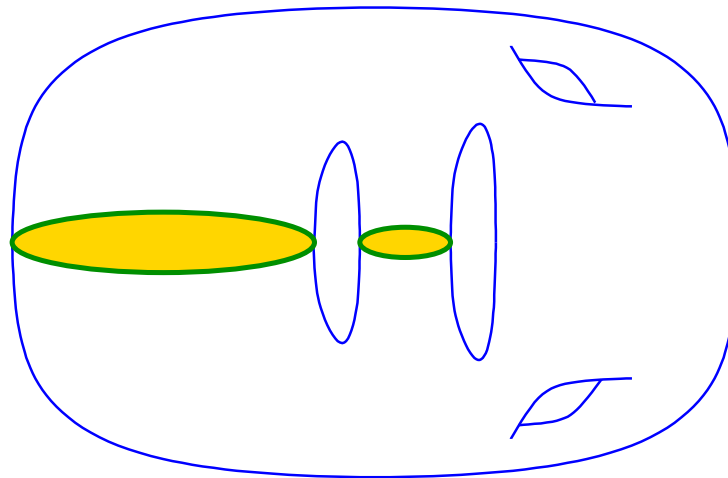
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The talk is about this theorem and its high-dimensional generalizations.

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The distribution of isotopy classes of non-singular plane projective curves between the types in low degrees.

degree	type I	type II
1	1	0
2	1	1
3	1	1
4	2	4
5	3	6
6	14	50

Complex orientations

A curve of type I has a pair of distinguished [orientations](#) .

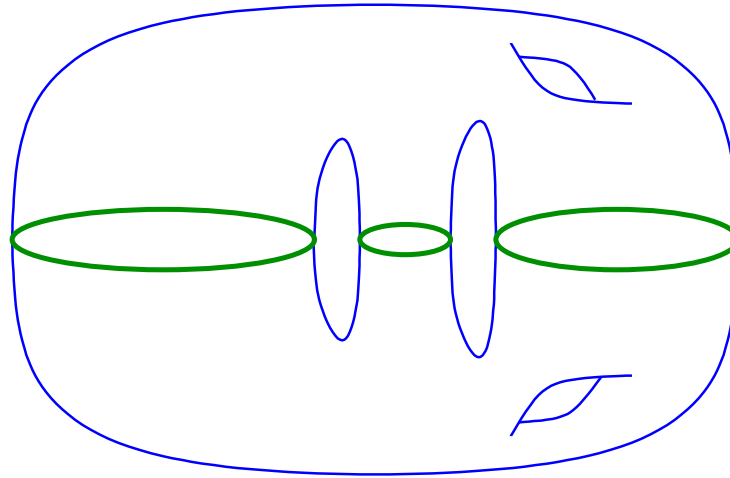
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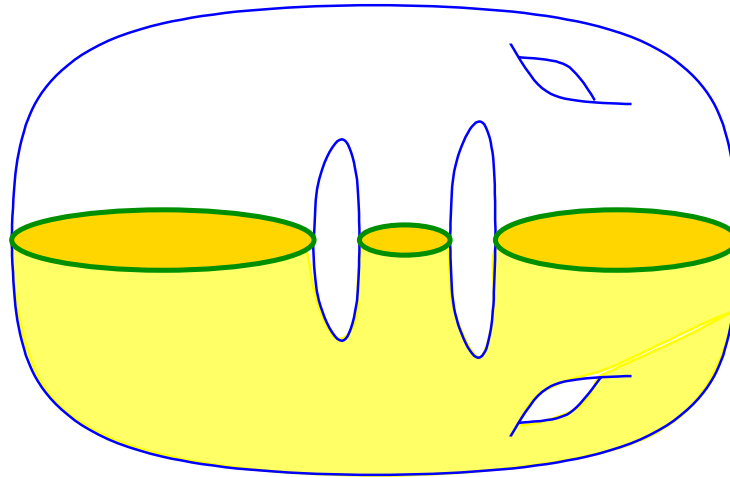
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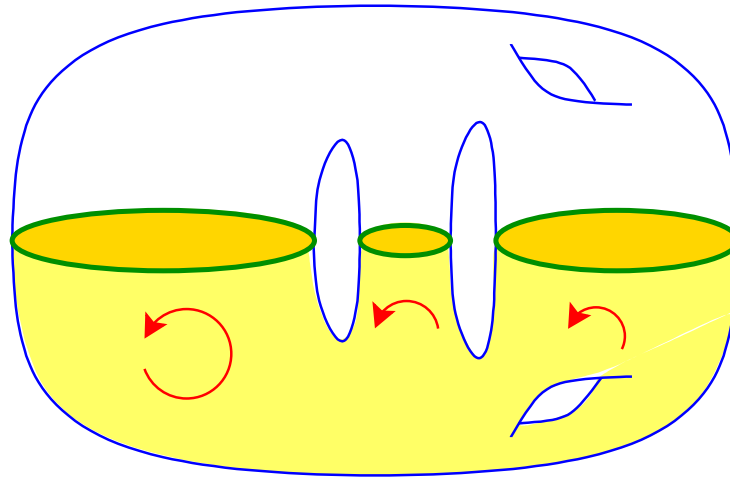
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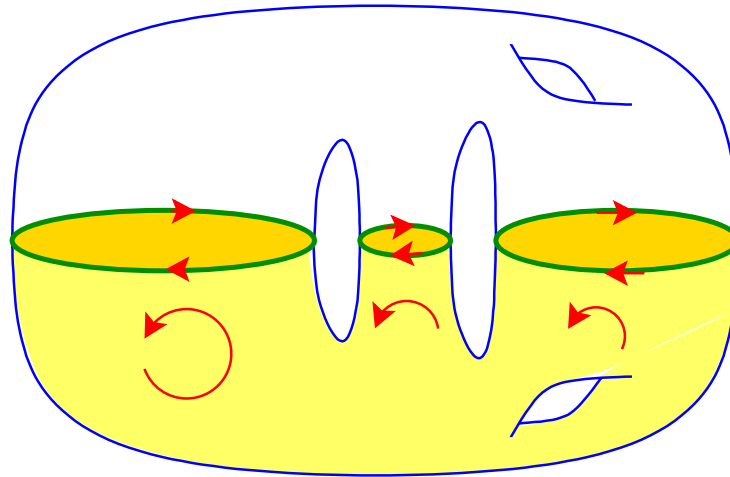
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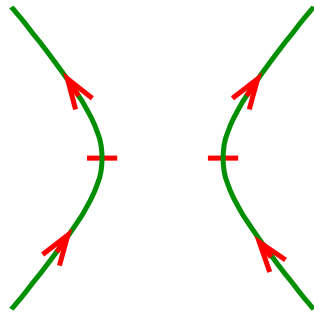
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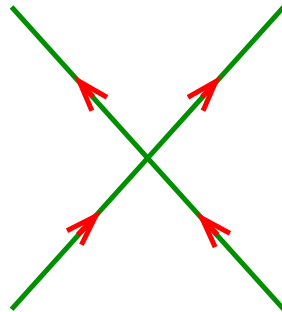
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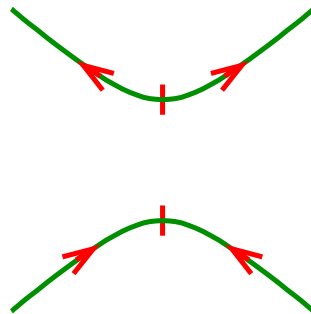
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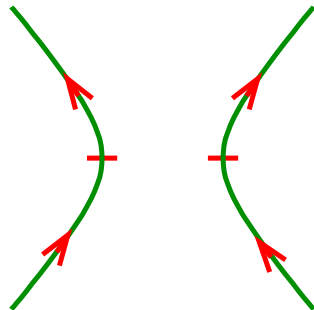
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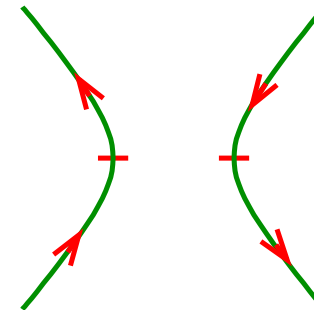
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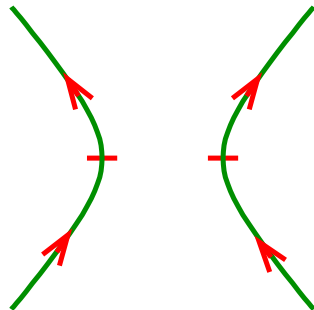
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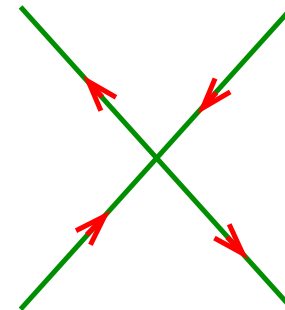
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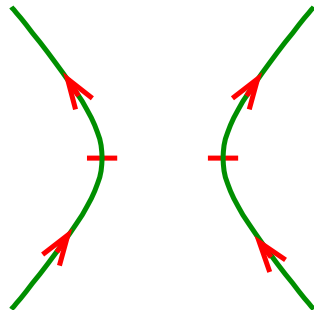
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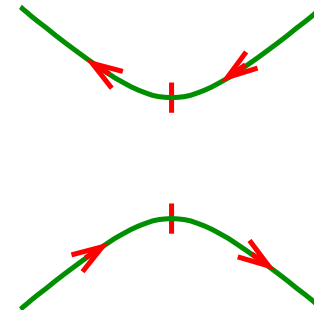
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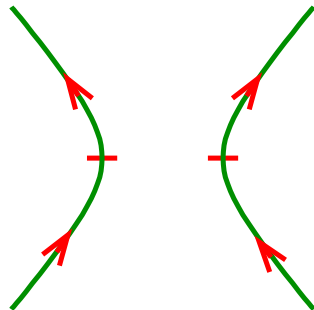
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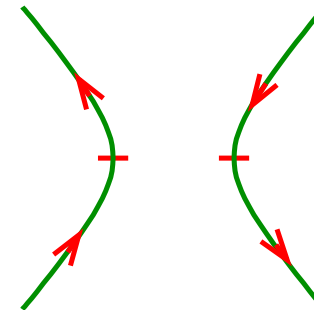
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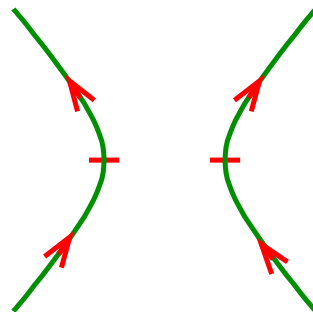
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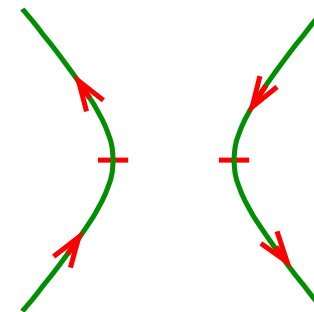
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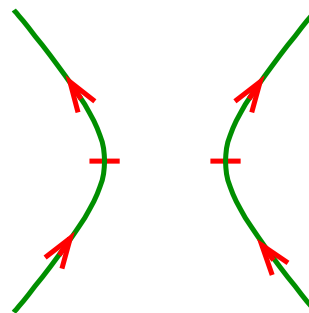


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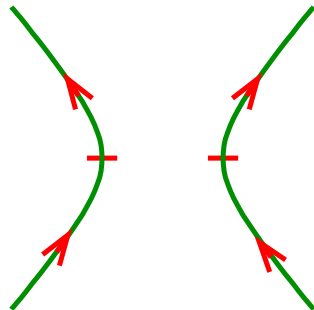
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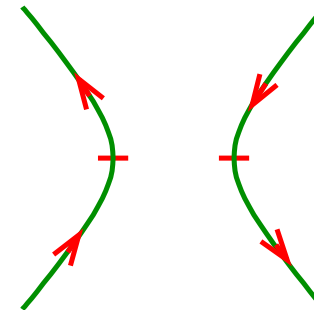
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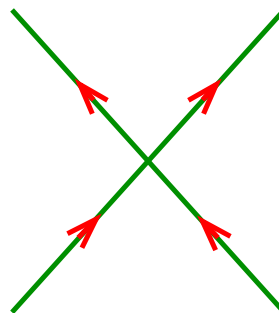


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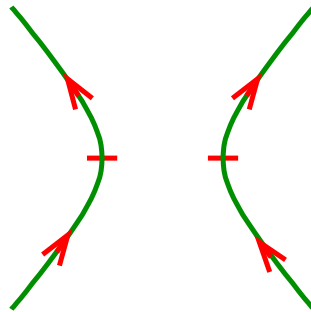
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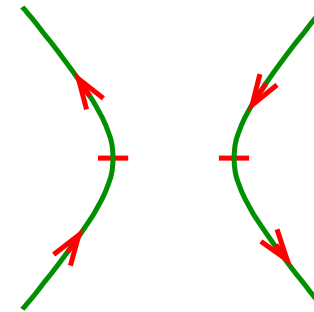
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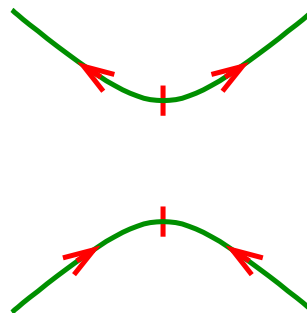


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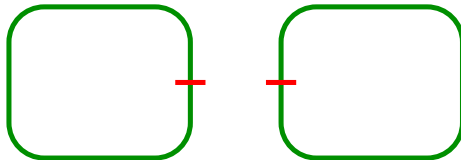
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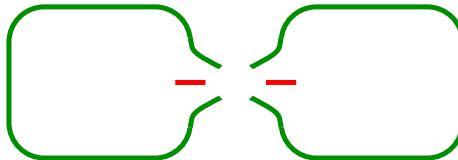
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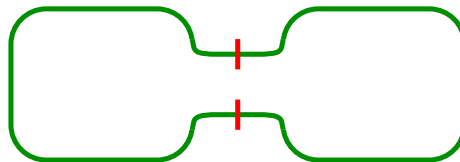
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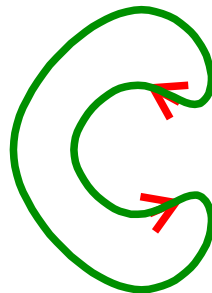
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Thus the $b_*(A_{\mathbb{R}})$ does not increase. \square

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Stiefel orientations

Stiefel orientations generalize both orientations and Spin-structures.

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A **Stiefel k -orientation** of an O_n -bundle ξ is $o \in \widetilde{H}^k(V_{n,n-k}(\xi); \mathbb{Z}/2)$ such that its restriction to $\widetilde{H}^k(V_{n,n-k}; \mathbb{Z}/2)$ is non-trivial for any fiber.

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Stiefel 1-orientation + orientation = Spin-structure.

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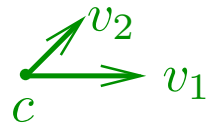
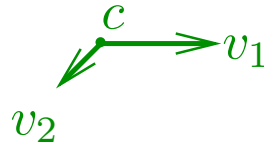
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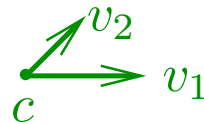
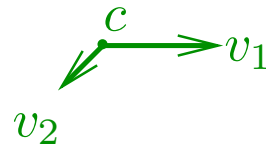
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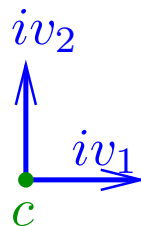
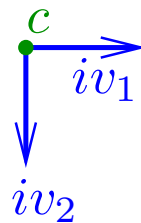
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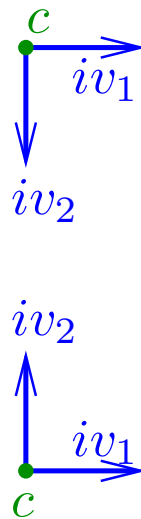
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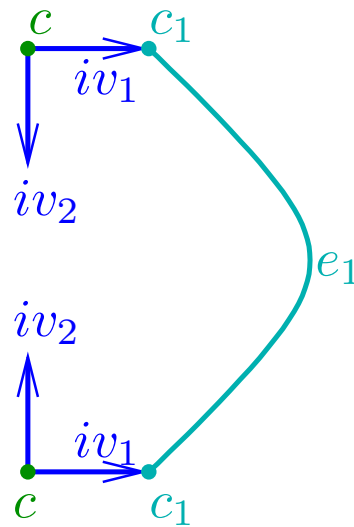
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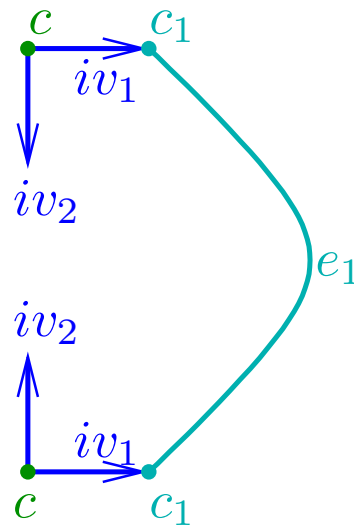
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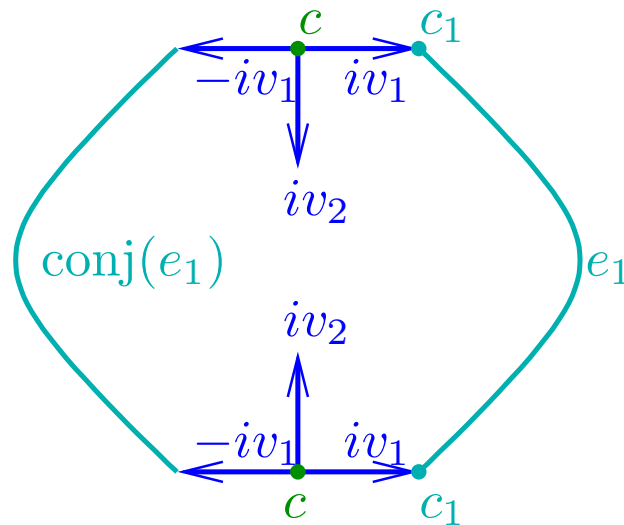
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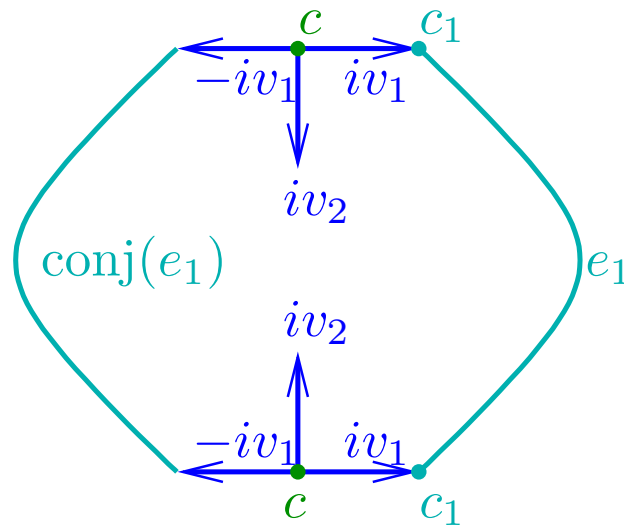
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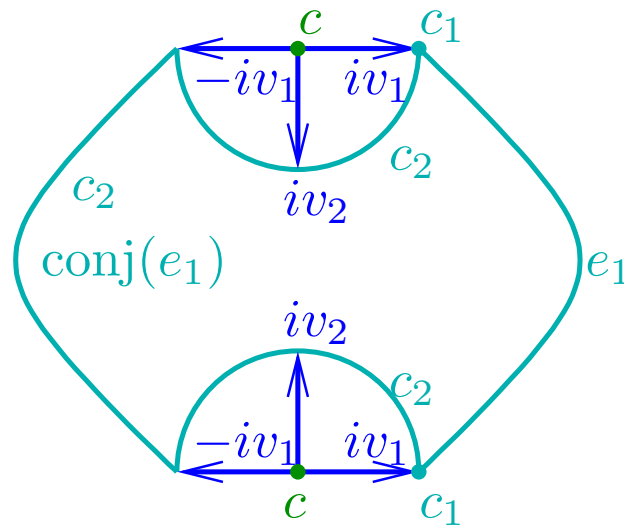
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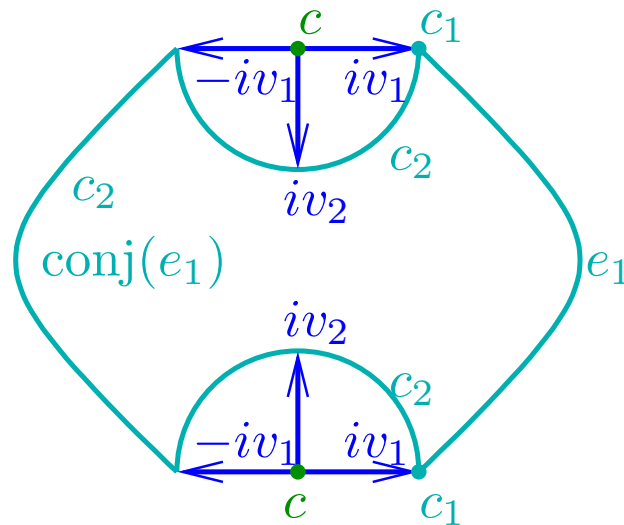
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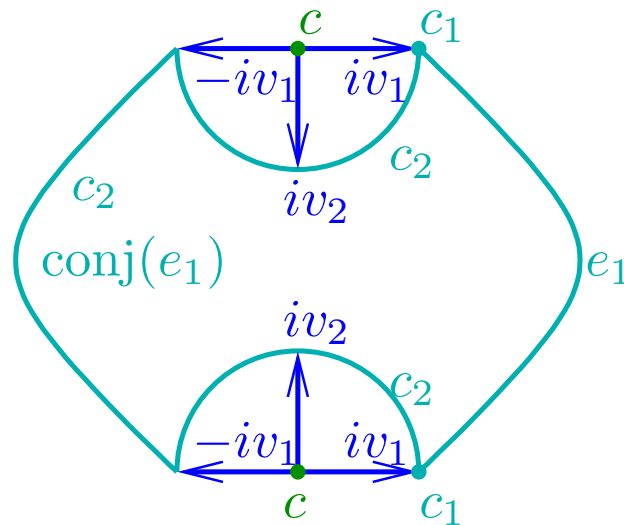
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 Construct similarly cycles c_3, \dots, c_{n-k} .

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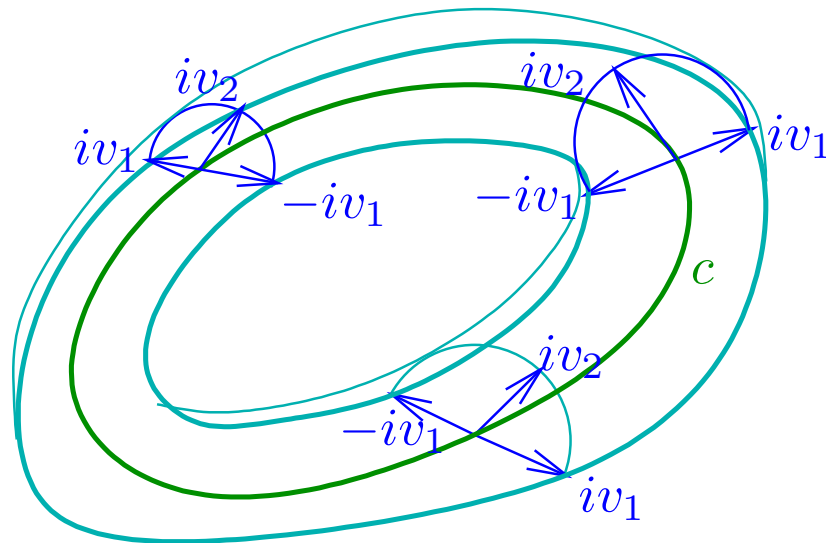
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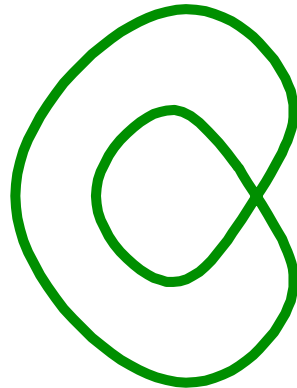
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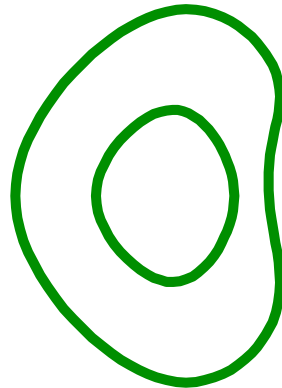
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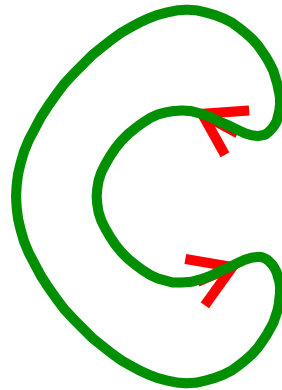
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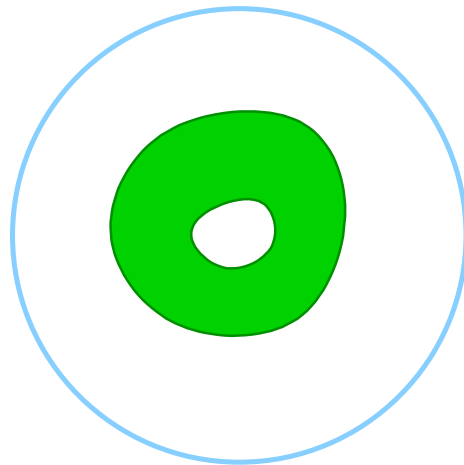
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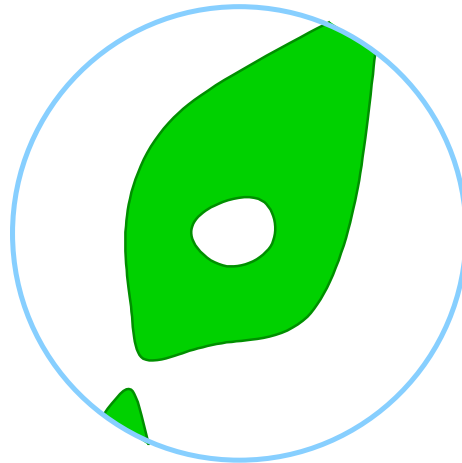
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This is a work in progress.

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